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Research Article

On the 2-independence subdivision number of graphs

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Abstract: A subset S of vertices in a graph $G = (V, E)$ is 2-independent if every vertex of S has at most one neighbor in S. The 2-independence number is the maximum cardinality of a 2-independent set of G. In this paper, we initiate the study of the 2-independence subdivision number $\mathrm{sd}_{\beta_2}(G)$ defined as the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the 2-independence number. We first show that for every connected graph G of order at least three, $1 \leq \mathrm{sd}_{\beta_2}(G) \leq 2$, and we give a necessary and sufficient condition for graphs G attaining each bound. Moreover, restricted to the class of trees, we provide a constructive characterization of all trees T with $\mathrm{sd}_{\beta_2}(T) = 2$, and we show that such a characterization suggests an algorithm that determines whether a tree T has $\mathrm{sd}_{\beta_2}(T) = 2$ or $\mathrm{sd}_{\beta_2}(T) = 1$ in polynomial time.

Keywords: Trees, 2-independence, subdivision numbers

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1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \in V \ : \ uv \in E\}$ and the *closed neighborhood* of v is the set $N(v) \cup \{v\}$. For a subset $S \subseteq V$, we denote by $\langle S \rangle$ the *subgraph induced* by the vertices of S. The *degree* of a vertex v is $\deg_G(v) = |N(v)|$. An *isolated vertex* is a vertex with degree zero. A vertex of degree one is called a *leaf* and its neighbor is called a *stem*. A stem is said to be *strong* if it is adjacent to at least two leaves. A *healthy spider* SS_p for $p \geq 2$ is obtained from a star $K_{1,p}$ by

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subdividing each edge by exactly one vertex. The center vertex of a healthy spider will be called a *head.* An *induced matching* in a graph G is a set of edges, no two of which meet a common vertex or are joined by an edge of G. In other words, an induced matching is a matching which forms an induced subgraph. The *diameter* of a connected G, denoted by $\mathrm{diam}(G)$, is the maximum value among minimum distances between all pairs of vertices of G.

In [5], Fink and Jacobson generalized the concept of independent sets as follows. Let k be a positive integer. A subset S of V is k-*independent* if the maximum degree of the subgraph induced by the vertices of S is less or equal to $k-1$. The k-*independence number* $\beta_k(G)$ is the maximum cardinality of a k-independent set of G. Clearly, for $k=1$, the 1-independent sets are the classical independent sets. In this paper, we are interested in the case $k=2$. A 2-independent set of a graph G with cardinality $\beta_2(G)$ is called a $\beta_2(G)$-set. For more details on the k-independence, we refer the reader to the survey by Chellali et al. [3].

The aim of this paper is to initiate the study of the 2-independence subdivision number $\mathrm{sd}_{\beta_2}(G)$ of a graph G that we define as the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the 2-independence number of G. It is worth noting that since the 2-independence number of the complete graph K_2 remains unchanged when its unique edge is subdivided, we will assume in our study that at least one component of the graph G has order at least 3. Moreover, if $G_1, G_2, \ldots, G_t$ are the components of G, then $\beta_2(G) = \sum_{i=1}^{t} \beta_2(G_i)$ and if $G_1, G_2, \ldots, G_m$ are the components of G of order at least 3, then $\mathrm{sd}_{\beta_2}(G) = \min\{\mathrm{sd}_{\beta_2}(G_i) \mid 1 \le i \le m\}$. Therefore we will only consider connected graphs of order at least three. It should be noted that if $k \ge 3$, then $\mathrm{sd}_{\beta_k}(G) = 1$ for every nontrivial connected graph G. Indeed, consider any $\beta_k(G)$-set S, and subdivide an arbitrary edge belonging to $\langle S\rangle$ or $\langle V-S\rangle$; if not any edge of G.

We close this section by mentioning that studies about the influence of edge subdivisions over a domination parameter was first defined in 2000 by Arumugam for the domination number. Afterwards, Haynes et al. [6] studied the 1-independence subdivision number $\mathrm{sd}_{\beta_1}(G)$, and so the concept was extended to total domination (see [7]), 2-domination (see [1]) and Roman domination (see [2]).

2. Tight bounds for sd_{β_2}

We begin by showing that the subdivision of an edge of G cannot decrease the 2-independence number but can increase it by at most one.

Proposition 1. *If G' is the graph obtained from a graph G by subdividing one edge of G, then $\beta_2(G) \le \beta_2(G') \le \beta_2(G)+1$.*

Proof. Let $G' = (V', E')$ be the graph obtained from the graph $G = (V, E)$ by subdividing an edge $uv \in E$ with a new vertex x. Note that $V' = V \cup \{x\}$ and

$E' = \{ux, xv\} \cup E - \{uv\}$. Let S be a $\beta_2(G)$-set. If $|S \cap \{u, v\}| \leq 1$, then clearly S remains a 2-independent set of G'. Hence assume that $|S \cap \{u, v\}| = 2$. Then $\{x\} \cup S - \{u\}$ is a 2-independent set of G', and either case we have $\beta_2(G') \geq \beta_2(G)$. To prove the upper bound, let S' be a $\beta_2(G')$-set. If $x \in S'$, then $S = S' - \{x\}$ is a 2-independent set of G. Hence let $x \notin S'$. Then $|S' \cap \{u, v\}| \geq 1$, for otherwise $S' \cup \{x\}$ is a 2-independent set of G' larger than S', a contradiction. Thus, without loss of generality, let $u \in S'$. It follows that $S' - \{u\}$ is a 2-independent set of G. In either case, $\beta_2(G) \geq \beta_2(G') - 1$. □

Trivially, $\mathrm{sd}_{\beta_2}(G) \geq 1$ for every connected graph of order at least three. The next result provides a necessary and sufficient condition for connected graphs G with $\mathrm{sd}_{\beta_2}(G) = 1$.

Theorem 1. *Let G be a connected graph of order $n \geq 3$. Then $\mathrm{sd}_{\beta_2}(G) = 1$ if and only if there exists a $\beta_2(G)$-set S such that one of the conditions below is fulfilled.*

(a) *$\langle V - S \rangle$ contains an edge.*

(b) *$\langle S \rangle$ contains an isolated vertex.*

Proof. Let G be a connected graph of order $n \geq 3$ with $\mathrm{sd}_{\beta_2}(G) = 1$. Suppose that no $\beta_2(G)$-set fulfills neither condition (a) nor (b). Hence for every $\beta_2(G)$-set S, $V - S$ is independent and $\langle S \rangle$ contains no isolated vertex. Since $\mathrm{sd}_{\beta_2}(G) = 1$, let uv be an edge of G whose subdivision into ux and xv increases $\beta_2(G)$. Let G' be the graph resulting from the subdivision of uv, and let S' be a $\beta_2(G')$-set. Clearly, $|\{u, x, v\} \cap S'| \geq 1$ for otherwise $S' \cup \{x\}$ is a 2-independent set of G' larger than S', a contradiction. Assume first that $u, v \notin S'$. Then $x \in S'$, and $S = S' - \{x\}$ is a 2-independent set of G. It follows that $\beta_2(G) \geq \beta_2(G') - 1 \geq \beta_2(G) + 1 - 1 = \beta_2(G)$, and therefore S is a $\beta_2(G)$-set. But since $u, v \in V - S$, $V - S$ is not independent, a contradiction. Hence $\{u, v\} \cap S' \neq \emptyset$. Without loss of generality, let $u \in S'$. If $x, v \notin S'$, then $\{x\} \cup S' - \{u\}$ is a $\beta_2(G')$-set that does not contain u, v and such a situation has been considered above. Hence either $u, x \in S'$ or $u, v \in S'$. Suppose that $u, x \in S'$. Thus $v \notin S'$, and as above $S' - \{x\}$ is $\beta_2(G)$-set in which u has no neighbor, a contradiction. Finally, suppose that $u, v \in S'$. Then $x \notin S'$. If neither u nor v has any neighbor in S', then S' is a 2-independent set of G, and thus $\beta_2(G) \geq |S'| = \beta_2(G) + 1$, which leads to a contradiction. Hence u or v has a neighbor in S'. Without loss of generality, let u' be a neighbor of u in S'. In this case, $S' - \{u\}$ is a $\beta_2(G)$-set in which u' has no neighbor, a contradiction.

Conversely, let S be a $\beta_2(G)$-set satisfying (a) or (b). Assume first that $\langle V - S \rangle$ contains an edge $uv \in E$, and let G' be the graph obtained from G by subdividing the edge uv with a new vertex x. In this case, it is clear that $S \cup \{x\}$ is a 2-independent set of G. Hence $\beta_2(G') \geq |S| + 1 = \beta_2(G) + 1$, and thus $\mathrm{sd}_{\beta_2}(G) = 1$. Assume now that $\langle S \rangle$ contains an isolated vertex u. Since G is a connected graph of order at least three, let $v \in V - S$ be a neighbor of u. Let G' be defined as before. Then $S \cup \{x\}$

is a 2-independent set of G', and thus $\beta_2(G') \geq |S| + 1 = \beta_2(G) + 1$. Therefore, $\mathrm{sd}_{\beta_2}(G) = 1$. □

Corollary 1. *If G is a connected graph having a complete subgraph of order at least four, then $\mathrm{sd}_{\beta_2}(G) = 1$.*

Proof. Let H be a complete subgraph of G of order at least four. Clearly, for every $\beta_2(G)$-set S, the subgraph induced by $V(G)-S$ contains at least two adjacent vertices belonging to H, fulfilling item (a) of Theorem 1. Therefore $\mathrm{sd}_{\beta_2}(G) = 1$. □

Our next result gives a framework for $\mathrm{sd}_{\beta_2}(G)$ for every connected graph of order $n \geq 3$.

Theorem 2. *For every connected graph G of order $n \geq 3$, $\mathrm{sd}_{\beta_2}(G) \in \{1, 2\}$.*

Proof. Assume first that G is a complete graph K_p. Clearly, if $p = 3$, then clearly $\mathrm{sd}_{\beta_2}(G) = 2$. Hence let $p \geq 4$. Then for every $\beta_2(G)$-set S, the subgraph induced by $V - S$ contains an edge, and hence by Theorem 1, $\mathrm{sd}_{\beta_2}(G) = 1$. In the sequel, we can assume that G is not a complete graph. Since $n \geq 3$, G contains an induced path $P_3 = uxv$. Let S be a $\beta_2(G)$-set. Clearly $1 \leq |\{u, x, v\} \cap S| \leq 2$. Consider the following cases.

Case 1. $|\{u, x, v\} \cap S| = 1$.
If $u \in S$ (the case $v \in S$ is similar), then $v, x \in V - S$, and by Theorem 1-(a), $\mathrm{sd}_{\beta_2}(G) = 1$. Hence let $x \in S$. If x has no neighbor in S, then by Theorem 1-(b), $\mathrm{sd}_{\beta_2}(G) = 1$. Thus assume that x has a neighbor in S, and consider the graph G' obtained from G by subdividing the edges ux and vx with new vertices u' and v', respectively. Then $\{u', v'\} \cup S - \{x\}$ is a 2-independent set of G' of cardinality $|S|+1$. Therefore $\beta_2(G') > \beta_2(G)$ and so $\mathrm{sd}_{\beta_2}(G) \leq 2$.
Case 2. $|\{u, x, v\} \cap S| = 2$.
If $u, x \in S$ (the case $v, x \in S$ is similar), then consider the graph G' obtained from G by subdividing the edges ux and vx with new vertices u' and v', respectively. Then $S \cup \{u', v'\} - \{x\}$ is a 2-independent set of G' of size $|S| + 1$. Therefore $\beta_2(G') \geq \beta_2(G) + 1$, and so $\mathrm{sd}_{\beta_2}(G) \leq 2$. Assume now that $u, v \in S$. Thus $x \in V - S$. If u is isolated in $\langle S \rangle$, then by Theorem 1-(a), $\mathrm{sd}_{\beta_2}(G) = 1$. Hence we can assume that u has a neighbor $w \in S$. Let G' be the graph obtained from G by subdividing the edges uw and xu with new vertices w' and u', respectively. Then $\{w', u'\} \cup S - \{u\}$ is a 2-independent set of G' of size $|S| + 1$. Therefore $\beta_2(G') \geq \beta_2(G) + 1$, and thus $\mathrm{sd}_{\beta_2}(G) \leq 2$. □

According to Theorem 1, we obtain a necessary and sufficient condition for connected graphs G with $\mathrm{sd}_{\beta_2}(G) = 2$.

Theorem 3. *Let G be a connected graph of order $n \geq 3$. Then $\mathrm{sd}_{\beta_2}(G) = 2$ if and only if for every $\beta_2(G)$-set S, $V - S$ is an independent set and $\langle S \rangle$ is an induced matching.*

The following corollary follows from Theorem 3.

Corollary 2. *Let G be connected graph of order $n \geq 3$. If $\beta_2(G)$ is odd, then* $\mathrm{sd}_{\beta_2}(G) = 1$.

3. Trees T with $\mathrm{sd}_{\beta_2}(T) = 1$ or 2

In this section, we provide a constructive characterization of the family $\mathcal{O}$ of all trees T with $\mathrm{sd}_{\beta_2}(T) = 2$. Clearly since $\mathrm{sd}_{\beta_2}(T) \in \{1, 2\}$, every T of order at least three not in $\mathcal{O}$ satisfies $\mathrm{sd}_{\beta_2}(T) = 1$, yielding a complete characterization of all trees T with $\mathrm{sd}_{\beta_2}(G) = 1$ or 2. Each tree $T \in \mathcal{O}$ has a distinguished subset $A(T)$ of vertices. First, $\mathcal{O}$ contains any tree T_1 which is a healthy spider SS_t $(t \geq 2)$ with head y, and for such a tree we set $A(T_1) = V(T_1) - \{y\}$. Next, if T' is any tree in $\mathcal{O}$, then we put in $\mathcal{O}$ any tree T that can be obtained from T' by the following operation:

- **Operation $\mathcal{O}_1$**: Let H be either a path P_3 with a leaf x or a healthy spider SS_t $(t \geq 2)$ with head x. Then T is obtained from T' by adding an edge xw, where $w \in A(T')$. Let $A(T) = A(T') \cup (V(H) - \{x\})$.

Before proceeding further, we give the following useful Observation and Lemma.

Observation 4. For every graph G, there exists a $\beta_2(G)$-set which contains all leaves of G.

Lemma 1. *If G is a graph containing a strong stem or two adjacent stems, then* $\mathrm{sd}_{\beta_2}(G) = 1$.

Proof. By Observation 4, let S be a $\beta_2(G)$-set containing all leaves of G. Assume first that G contains a strong stem, say x. Clearly, $x \notin S$, since S contains all leaves adjacent to x. Hence $\langle S \rangle$ contains isolated vertices, and thus by Theorem 1-(a), $\mathrm{sd}_{\beta_2}(G) = 1$. Assume now that G contains two adjacent stems x and y. We may assume that neither x nor y is a strong stem. Let x' and y' be the leaf neighbors of x and y, respectively. Clearly, $x', y' \in S$, $|S \cap \{x, y\}| \leq 1$ and thus either x' or y' has no neighbor in S. Therefore $\langle S \rangle$ contains an isolated vertex and thus by Theorem 1-(a), $\mathrm{sd}_{\beta_2}(G) = 1$. □

The following Proposition follows from Observation 4, Lemma 1 and Theorem 1.

Proposition 2. *Let T be a tree such that $\mathrm{sd}_{\beta_2}(T) = 2$. Then all stems of T are weak and every $\beta_2(T)$-set contains all stems of T and their leaves.*

We state the following lemma.

Lemma 2. *If $T \in \mathcal{O}$, then*

a) *$\langle A(T)\rangle$ is an induced matching.*

b) *$V(T) - A(T)$ is an independent set.*

c) *$A(T)$ is a unique $\beta_2(T)$-set.*

Proof. Parts (a) and (b) follow directly from the way a tree $T \in \mathcal{O}$ is constructed. To prove part (c), let $T \in \mathcal{O}$. Then T is obtained from a sequence $T_1, T_2, \ldots, T_k$ ($k \geq 1$) of trees, where T_1 is a healthy spider SS_t ($t \geq 2$) with head x, $T = T_k$, and, if $k \geq 2$, then T_{i+1} is obtained recursively from T_i by Operation $\mathcal{O}_1$ defined above. We proceed by induction on the total number of operation $\mathcal{O}_1$ performed to construct T. If $k = 1$, then SS_t ($t \geq 2$), and clearly $A(T_1) = V(T_1) - \{x\}$ is the unique $\beta_2(T)$-set. This establishes the basis case.
Assume now that $k \geq 2$ and that the result holds for all trees $T \in \mathcal{O}$ that can be constructed from a sequence of length at most $k-1$, and let $T' = T_{k-1}$. Applying our inductive hypothesis to $T' \in \mathcal{O}$ shows that $A(T')$ is the unique $\beta_2(T')$-set. Let T be a tree obtained from T' by using operation $\mathcal{O}_1$. We examine the following two situations.
Assume first that a path $P_3 = xyz$ has been added and attached to T' by the edge xw at a vertex $w \in A(T')$. It is easy to see that $\beta_2(T) = \beta_2(T') + 2$ and $A(T) = A(T') \cup \{y, z\}$ is a $\beta_2(T)$-set. Moreover, using the facts that every $\beta_2(T)$-set contains at most two vertices of $\{x, y, z\}$ and $A(T')$ is the unique $\beta_2(T')$-set, we conclude that $A(T)$ is the unique $\beta_2(T)$-set.
Assume now that a healthy spider SS_t ($t \geq 2$) with head x has been added and attached to T' by the edge xw at a vertex $w \in A(T')$. As above, it is easy to see that $\beta_2(T) = \beta_2(T') + 2t$ and $A(T) = A(T') \cup (V(SS_{1,t}) - \{x\})$ is a $\beta_2(T)$-set. Also, since every $\beta_2(T)$-set contains at most $2t$ vertices of $SS_{1,t}$, and the unicity of $\beta_2(T')$-set we deduce that $A(T)$ is the unique $\beta_2(T)$-set. □

According to Lemma 2 and Theorem 3, the following corollary is immediate.

Corollary 3. *If $T \in \mathcal{O}$, then $\mathrm{sd}_{\beta_2}(T) = 2$.*

Now, we are ready to state the main result of this section.

Theorem 5. *Let T be a tree of order at least three. Then $\mathrm{sd}_{\beta_2}(T) = 2$ if and only if $T \in \mathcal{O}$.*

Proof. If $T \in \mathcal{O}$, then by Corollary 3, $\mathrm{sd}_{\beta_2}(T) = 2$. To prove the necessity, we use an induction on the order n of T. Clearly, $\mathrm{diam}(T) \geq 4$, since T cannot have neither strong stems nor adjacent stems (by Lemma 1). Moreover, the smallest tree T of diameter 4 with $\mathrm{sd}_{\beta_2}(T) = 2$ is the path $P_5 = SS_2$ that belongs to $\mathcal{O}$, establishing the basis case. Let $n \geq 6$ and assume that every tree T' of order $5 \leq n' < n$ with $\mathrm{sd}_{\beta_2}(T') = 2$ belongs to $\mathcal{O}$. Let T be a tree of order n with $\mathrm{sd}_{\beta_2}(T) = 2$.

If $\mathrm{diam}(T) = 4$, then by Lemma 1, T is a healthy spider SS_t $(t \geq 3)$ and $T \in \mathcal{O}$. Hence we may assume that $\mathrm{diam}(T) \geq 5$.

We root T at a leaf r of a maximum eccentricity. Among all vertices at distance $\mathrm{diam}(T) - 2$ from r on a longest path starting at r, let v be one of maximum degree. Since $\mathrm{diam}(T) \geq 5$, let u be the parent of v in the rooted tree. Also, by Lemma 1, T_v is either a path P_3 $(\deg_T(v) = 2)$ or a healthy spider SS_t $(t \geq 2)$ with head v $(\deg_T(v) \geq 3)$. According to Proposition 2, let S be a $\beta_2(T)$-set containing all stems and their leaves. Hence $v \notin S$, and by Theorem 3, $u \in S$. Let $T' = T - T_v$ and $t = \deg_T(v) - 1$. Note that T' has order $n' \geq 3$ (since we assumed that $\mathrm{diam}(T) \geq 5$). It is easy to see that $\beta_2(T') = \beta_2(T) - 2t$. Moreover, assume that $\mathrm{sd}_{\beta_2}(T') = 1$. Let e be an edge of T' whose subdivision increases $\beta_2(T')$. Let T'_e (resp. T_e) be the resulting tree obtained from T' (resp. T) by subdividing the edge e. If D is a $\beta_2(T'_e)$-set, then clearly $D \cup (V(T_v) - \{v\})$ is a 2-independent set of T_e. Therefore

$$\begin{aligned} \beta_2(T_e) &\geq |D \cup (V(T_v) - \{v\})| = \beta_2(T'_e) + 2t \\ &> \beta_2(T') + 2t = \beta_2(T), \end{aligned}$$

implying that $\mathrm{sd}_{\beta_2}(T) = 1$, a contradiction. Hence $\mathrm{sd}_{\beta_2}(T') = 2$, and by the induction hypothesis we have $T' \in \mathcal{O}$. Note that $S \cap V(T')$ is a $\beta_2(T')$-set containing vertex u. By Lemma 2, $A(T')$ is the unique $\beta_2(T')$-set and thus $u \in A(T')$. It follows that $T \in \mathcal{O}$ because it is obtained from T' by using operation $\mathcal{O}_1$. □

The next result is an immediate consequence of Theorems 5 and 2.

Corollary 4. *If $T \notin \mathcal{O}$ is a tree of order at least three, then* $\mathrm{sd}_{\beta_2}(T) = 1$.

The proof of Theorem 5 suggests a polynomial-time algorithm which, given a tree T with n vertices, decides whether T is in $\mathcal{O}$ and thus has $\mathrm{sd}_{\beta_2}(T) = 2$ or $\mathrm{sd}_{\beta_2}(T) = 1$. Here is an outline of the algorithm. If $\mathrm{diam}(T) \leq 3$, then answer $T \notin \mathcal{O}$ and stop. Now let $\mathrm{diam}(T) = 4$. If T has neither a strong stem no adjacent stems, then answer $T \in \mathcal{O}$ and stop. Else answer $T \notin \mathcal{O}$ and stop. In the sequel, suppose $\mathrm{diam}(T) \geq 5$. Pick a vertex r, root the tree T at r, and pick a vertex b_1 at maximum distance from r. Let b_2 be the parent of b_1 in the rooted tree and b_3 be the parent of b_2. If either b_2 has at least two children, or b_3 has a child with degree one, then return the answer $T \notin \mathcal{O}$ and stop. Else, let b_4 be the parent of b_3. Call the algorithm recursively on the tree $T' = T - T_{b_3}$; if the answer to the recursive call is $T' \in \mathcal{O}$ and $b_4 \in A(T')$, then answer $T \in \mathcal{O}$, return $A(T) = A(T') \cup (V(T_{b_3}) - \{b_3\})$, and stop, else answer $T \notin \mathcal{O}$ and stop.

Recall that a subset S of $V(G)$ is a *double dominating set* of G if for every vertex $v \in V(G)$, $|N[v] \cap S| \geqslant 2$, that is, v is in S and has at least one neighbor in S or v is in $V(G) - S$ and has at least two neighbors in S. The *double domination number* $\gamma_{\times 2}(G)$ is the minimum cardinality among all double dominating sets of G. It is worth mentioning that from the way a tree $T \in \mathcal{O}$ is constructed, one can easily observe that set $A(T)$ is a double dominating set of T. Hence if $T \in \mathcal{O}$, then $\gamma_{\times 2}(T) \leq |A(T)| = \beta_2(T)$. The equality is obtained from a result given in [4] where the authors showed that for every nontrivial tree T, $\gamma_{\times 2}(T) \geq \beta_2(T)$. However, the converse is not true, as can be seen by the tree T^* obtained from two paths P_5 by adding an edge between their centers. Clearly, $\gamma_{\times 2}(T^*) = \beta_2(T^*) = 8$ but $T^* \notin \mathcal{O}$.

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