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# Vertex-edge Roman domination in graphs: Complexity and algorithms

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**Abstract:** For a simple, undirected graph $G(V, E)$, a function $h : V(G) \to \{0, 1, 2\}$ such that each edge $(u, v)$ of $G$ is either incident with a vertex with weight at least one or there exists a vertex $w$ such that either $(u, w) \in E(G)$ or $(v, w) \in E(G)$ and $h(w) = 2$, is called a vertex-edge Roman dominating function (ve-RDF) of $G$. For a graph $G$, the smallest possible weight of a ve-RDF of $G$ which is denoted by $\gamma_{veR}(G)$, is known as the *vertex-edge Roman domination number* of $G$. The problem of determining $\gamma_{veR}(G)$ of a graph $G$ is called minimum vertex-edge Roman domination problem (MVERDP). In this article, we show that the problem of deciding if $G$ has a ve-RDF of weight at most $l$ for star convex bipartite graphs, comb convex bipartite graphs, chordal graphs and planar graphs is NP-complete. On the positive side, we show that MVERDP is linear time solvable for threshold graphs, chain graphs and bounded tree-width graphs. On the approximation point of view, a 2-approximation algorithm for MVERDP is presented. It is also shown that vertex cover and vertex-edge Roman domination problems are not equivalent in computational complexity aspects. Finally, an integer linear programming formulation for MVERDP is presented.

**Keywords:** Vertex-edge Roman-domination, Graph classes, NP-complete, Vertex cover, Integer linear programming

**AMS Subject classification:** 05C69, 68Q25

## 1. Introduction

Let $G(V,\ E)$ be a simple, undirected graph.For a vertex $u$ of $G$, the (*open*) *neighborhood* which is denoted by $N_G(u)$, is the set $\{v : (u, v) \in E(G)\}$ and its *degree*

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is $|N_G(u)|$. The *closed neighborhood* of $u$ is $N_G[u] = \{u\} \cup N_G(u)$. *Maximum degree* of $G$, denoted by $\Delta$, is $max_{u \in V(G)}|N_G(u)|$. A vertex $v$ is called an *isolated vertex* if $|N_G(v)| = 0$. A vertex of degree $n - 1$ is called a *universal vertex*, where $n = |V(G)|$. A graph formed with the vertex set $S \subseteq V(H)$ of graph $H$ and the edge set $\{(u, v) : u, v \in S\}$ is called an *induced subgraph* of $H$ and is denoted by $H[S]$. A subset $S$ of $V$ in a graph $G$ is said to be *independent* if no two vertices in $S$ are adjacent. A bipartite graph $G = (X, Y, E)$ is called *tree convex* if there exists a tree $T = (X, F)$ such that, for each $y$ in $Y$, the neighbors of $y$ induce a subtree in $T$. When $T$ is a star (comb), $G$ is called *star (comb) convex bipartite* graph [19]. In a graph $G$, a vertex $v$ is *simplicial* if its neighborhood $N_G(v)$ induces a clique. An ordering $\{v_1, v_2, \ldots, v_n\}$ of the vertices of $V$ is a *perfect elimination ordering* (PEO), if $v_i$ is a simplicial of the induced subgraph $G_i = G[\{v_i, v_{i+1}, \ldots, v_n\}]$ for every $i$, $1 \leq i \leq n$. A graph $G$ is *chordal* if and only if $G$ admits a PEO. For an integer $n \geq 1$, a *star graph* is a complete bipartite graph $K_{1,n}$. The maximum degree vertex of a star graph is called the *center vertex* of it. A *vertex cover* of a graph $G$ is a subset $C \subseteq V(G)$ such that if $(u, v) \in E(G)$, then $u \in C$ or $v \in C$ or both. A minimum vertex cover is a vertex cover of smallest possible size. The *vertex cover number* denoted $\tau(G)$ is the size of a minimum vertex cover in $G$. For undefined terminology and notations we refer to [28].
A *dominating set* (DS) of a graph $G$ is a set $D$ such that $D \subseteq V(G)$ and $\cup_{w \in D} N_G[w] = V(G)$. The *domination number* of $G$, which is denoted by $\gamma(G)$, is $min\{|T| : T \text{ is a DS of } G\}$. Given a graph $G$ and a positive integer $l$, the domination decision problem is to check whether $G$ has a DS of size at most $l$. Literature on the concept of domination has been surveyed in [11, 12].
The concept of Roman domination was introduced in 2004 by Cockayne et al. in [8]. A function $f : V \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on $G$ if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. We refer to [1–4] for the literature on the concept of new variants of Roman domination in graphs.

A vertex $u \in V$ is said to *ve-dominate* an edge $(x, y) \in E$ if $x \in N[u]$ or $y \in N[u]$ or both. The concept of vertex-edge domination was introduced by Peters in 1986 [25]. A set $D \subseteq V$ is a *vertex-edge dominating set* (or simply, a ve-dominating set) of $G$ if, every edge of $G$ is ve-dominated by some vertex of $G$. We refer to [5, 6, 14, 17, 18, 24, 25, 30] for the literature on the concept of vertex-edge domination and its variants in graphs.
The concept of verex-edge Roman domination has been introduced in [26]. A function $f : V \to \{0, 1, 2\}$ is a *vertex-edge Roman dominating function* (ve-RDF) on $G$ if every edge $(u, v)$ is either incident with a vertex with weight at least one or there exists a vertex $w$ such that either $(u, w) \in E(G)$ or $(v, w) \in E(G)$ and $h(w) = 2$. The weight of a ve-RDF $f$ is the value $f(V) = \sum_{u \in V} f(u)$. The *vertex-edge Roman domination number* is the minimum weight of a ve-RDF on $G$ and is denoted by $\gamma_{veR}(G)$. The minimum vertex-edge Roman domination problem (MVERDP) is to find a ve-RDF of minimum weight in the input graph. The decision version of vertex-edge Roman

domination problem is defined as follows.
**Vertex-Edge Roman Domination Problem (VERDP)**
**Instance :** A simple, undirected graph $G$ and a positive integer $k$.
**Question :** Does $G$ have a ve-RDF of weight at most $k$?
In [15] it has been shown that VERDP is NP-complete for bipartite graphs. In this paper, we investigate the algorithmic aspects of VERDP in some subclasses of bipartite graphs, chordal graphs and planar graphs. We also investigate the complexity of MVERDP in approximation point of view. It is also shown that vertex cover and vertex-edge Roman domination problems are not equivalent in computational complexity aspects. Finally, an integer linear programming formulation for MVERDP is presented.

## 2. Complexity Results

In this section, first, we show that VERDP is NP-complete for star convex bipartite graphs, comb convex bipartite graphs and chordal graphs by giving a polynomial time reduction from a well-known NP-complete problem, Exact-3-Cover ($X3SC$)[10], which is defined as follows.
EXACT-3-COVER (X3SC)
**Instance :** A finite set $X$ with $\mid X \mid = 3q$ and a collection $C$ of 3-element subsets of $X$.
**Question :** Is there a subcollection $C'$ of $C$ such that every element of $X$ appears in exactly one member of $C'$?

**Theorem 1.** *VERDP is NP-complete for star convex bipartite graphs.*

*Proof.* Given a star convex bipartite graph $G$ and a function $f$, whether $f$ is a ve-RDF of size at most $k$ can be checked in polynomial time. Hence VERDP for star convex bipartite graphs is a member of NP.
Now we show that VERDP is NP-hard by transforming an instance $\langle X, C\rangle$ of X3SC, where $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$, to an instance $\langle G, k\rangle$ of VERDP for star convex bipartite graphs as follows. Create vertices $x_i, y_i$ for each $x_i \in X$ and $c_i$ for each $C_i \in C$ and also create vertices $a, \{a_1, a_2, \ldots, a_{2q+4}\}$ and $\{b_1, b_2, \ldots, b_{2q+4}\}$. Add edges $(a, a_i)$ for each $a_i$ and $(a_i, b_i)$ for each $i, 1 \leq i \leq 2q+4$. Also add edges $(a, c_i)$ for each $c_i$ and $(c_j, x_i)$ if $x_i \in C_j$. The graph constructed is shown in Figure 1. Let $A = \{a\} \cup \{x_i : 1 \leq i \leq 3q\} \cup \{b_i : 1 \leq i \leq 2q+4\}$ and $B = V \setminus A$. The set $A$ induces a star with vertex $a$ as central vertex, as shown in Figure 2, and the neighbors of each element in $B$ induce a subtree of star. Therefore $G$ is a star convex bipartite graph and can be constructed from the given instance $\langle X, C\rangle$ of X3SC in polynomial time. Next we show that $X3SC$ has a solution if and only if $G$ has a ve-RDF with weight at most $2q + 2$.
Suppose $C'$ is a solution for $X3SC$ with $|C'| = q$. We define a function $f : V \rightarrow$

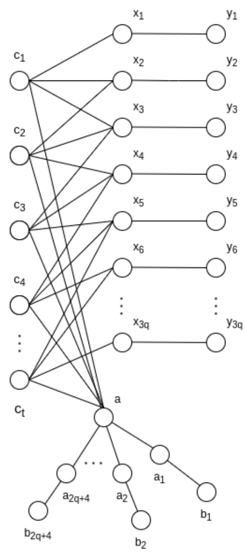


**Figure 1.** **Construction of a star convex bipartite graph from an instance of X3SC**

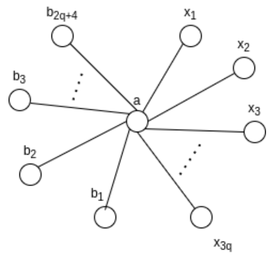


**Figure 2.** **Star Graph**

$\{0, 1, 2\}$ as follows.

$$f(u) = \begin{cases} 2, & \text{if } u \in C' \text{ or } u = a \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Clearly, $f$ is a ve-RDF and $f(V) = 2q + 2$.
Conversely, suppose that $G$ has a ve-RDF $g$ with weight $k = 2q+2$. Also, for $j = 0, 1$ or 2, let $V_j = \{v \mid g(v) = j\}$. We state two claims.
***Claim 1***: $g(a) = 2$ and $\{a_1, a_2, \ldots, a_{2q+4}\} \cup \{b_1, b_2, \ldots, b_{2q+4}\} \subset V_0$.
***Proof***: Proof follows from the definition of ve-RDF and the fact that $g(V) = 2q + 2$.
***Claim 2***: $\{x_1, x_2, \ldots, x_{3q}\} \cup \{y_1, y_2, \ldots, y_{3q}\} \subset V_0$.
***Proof***: (Proof by contradiction) Let $m = |\{i : g(x_i) \geq 1 \text{ or } g(y_i) \geq 1\}|$. Clearly, number of edges of the form $(x_i, y_i)$ with $g(x_i) = 0$ and $g(y_i) = 0$ is $3q - m$. Let $M = \{(x_i, y_i) : g(x_i) = 0 \text{ and } g(y_i) = 0\}$. Since $g$ is a ve-RDF, every edge of $M$ should

be ve-dominated by a vertex with weight 2. Therefore, number of $c_j$'s required with weight two is at least $\lceil\frac{3q-m}{3}\rceil$. Hence $g(V) = 2 + m + 2\lceil\frac{3q-m}{3}\rceil$, which is greater than $k$, a contradiction.
From the fact that $g$ is a ve-RDF and the above claim, it follows that $|\{c_1, c_2, \ldots, c_t\} \cap V_2| = q$. Now $\{c_1, c_2, \ldots, c_t\} \cap V_2$ is an exact cover for $X3SC$. □

Since every star convex bipartite graph is a tree convex bipartite, we have the the following.

**Corollary 1.** *VERDP is NP-complete for tree convex bipartite graphs.*

**Theorem 2.** *VERDP is NP-complete for comb convex bipartite graphs.*

*Proof.* Given a comb convex bipartite graph $G$ and a function $f$, whether $f$ is a ve-RDF of size at most $k$ can be checked in polynomial time. Hence VERDP for comb convex bipartite graphs is a member of NP.
Now we show that VERDP is NP-hard by transforming an instance $\langle X, C\rangle$ of X3SC, where $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$, to an instance $\langle G, k\rangle$ of VERDP for comb convex bipartite graphs as follows.
Create vertices $x_i, x_i', y_i$ for each $x_i \in X$ and $c_i$ for each $C_i \in C$. Add edges $(c_j, x_i)$ if $x_i \in C_j$. Also add edges by joining each $c_j$ to every $x_i'$ and $(x_i, y_i)$ for all $i, 1 \leq i \leq 3q$. The graph constructed is shown in Figure 3. Let $A = \{x_i, x_i' : 1 \leq i \leq 3q\}$ and $B = V \setminus A$. The set $A$ induces a comb with elements $\{x_i' : 1 \leq i \leq 3q\}$ as backbone and $\{x_i : 1 \leq i \leq 3q\}$ as teeth, as shown in Figure 4, and the neighbors of each element in $B$ induce a subtree of the comb. Therefore $G$ is a comb convex bipartite graph and can be constructed from the given instance $\langle X, C\rangle$ of X3SC in polynomial time. Next we show that $X3SC$ has a solution if and only if $G$ has a ve-RDF with weight at most $2q$.
Suppose $C'$ is a solution for $X3SC$ with $|C'| = q$. We define a function $f : V \to \{0, 1, 2\}$ as follows.

$$f(u) = \begin{cases} 2, & \text{if } u \in C' \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Clearly, $f$ is a ve-RDF and $f(V) = 2q$.
Conversely, suppose that $G$ has a ve-RDF $g$ with weight $k = 2q$. Also, for $j = 0, 1$ or 2, let $V_j = \{v \mid g(v) = j\}$. Converse proof is similar to the proof given in Theorem 1. We state the following claim without proof.
***Claim 3***: $\{x_1, x_2, \ldots, x_{3q}\} \cup \{y_1, y_2, \ldots, y_{3q}\} \subset V_0$.
From the fact that $g$ is a ve-RDF and the above claim, it follows that $|\{c_1, c_2, \ldots, c_t\} \cap V_2| = q$. Now $\{c_1, c_2, \ldots, c_t\} \cap V_2$ is an exact cover for $X3SC$. □

**Theorem 3.** *VERDP is NP-complete for chordal graphs.*

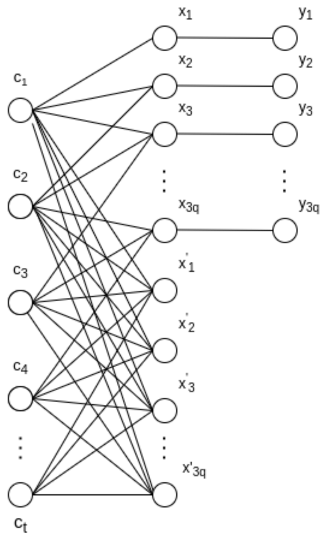


**Figure 3.** **Construction of a comb convex bipartite graph from an instance of X3SC**

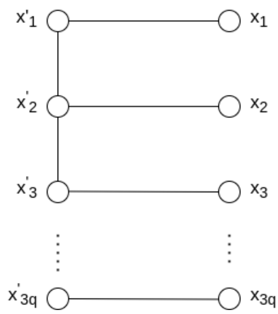


**Figure 4.** **Comb Graph**

*Proof.* Clearly, VERDP for chordal graphs is in NP. Now we show that VERDP is NP-hard by transforming an instance $\langle X, C\rangle$ of X3SC, where $X = \{x_1, x_2, \dots, x_{3q}\}$ and $C = \{C_1, C_2, \dots, C_t\}$, to an instance $\langle G, k\rangle$ of VERDP for chordal graphs as follows. Create vertices $x_i, y_i$ for each $x_i \in X$ and $c_i$ for each $C_i \in C$. Add edges $(x_i, y_i)$ for all $i, 1 \leq i \leq 3q$ and $(c_j, x_i)$ if $x_i \in C_j$. Also add edges between every pair of distinct vertices $c_i$ and $c_j$. The graph $G$ can be constructed in polynomial time. It is easy to verify that $(y_1, y_2, \dots, y_{3q}, x_1, x_2, \dots, x_{3q}, c_1, c_2, \dots, c_t)$ is a PEO of $G$. Hence $G$ is a chordal graph. Next, we show that $X3SC$ has a solution if and only if $G$ has a ve-RDF with weight at most $2q$.
Suppose $C'$ is a solution for $X3SC$ with $|C'| = q$. We define a function $f : V \to$

$\{0,1,2\}$ as follows.

$$f(u) = \begin{cases} 2, & \text{if } u \in C' \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Clearly, $f$ is a ve-RDF and $f(V) = 2q$.
Conversely, suppose that $G$ has a ve-RDF $g$ with weight $k = 2q$. Converse proof is similar to the proof given in Theorem 1. □

Next, we show that VERDP is NP-complete for planar graphs by giving a polynomial time reduction from Planar Exact Cover by 3-Sets (Planar $X3C$) [21], which is a NP-complete problem and is defined as follows.
**Planar Exact Cover by 3 Sets (Planar $X3C$)**
**Instance :** A finite set $X = \{x_1, x_2, \ldots, x_{3q}\}$ and a collection $C = \{c_1, c_2, \ldots, c_t\}$ of 3-element subsets of $X$ such that (i) every element of $X$ occurs in at most three subsets and (ii) the induced graph is planar. (This induced graph $H(V, E)$ is defined as the graph such that $V = X \cup C$ and $E = \{(x_i, c_j) \text{ if } x_i \in c_j\}$).
**Question :** Is there a subcollection $C'$ of $C$ such that every element of $X$ appears in exactly one member of $C'$ ?

**Theorem 4.** *VERDP is NP-complete for planar graphs.*

*Proof.* Clearly, VERDP is a member of *NP*. We transform an instance $\langle X, C\rangle$ of Planar $X3C$, where $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{c_1, c_2, \ldots, c_t\}$, to an instance $\langle G, k\rangle$ of VERDP same as in Theorem 3. Clearly, $G$ is a planar graph and can be constructed from the given instance $\langle X, C\rangle$ of Planar $X3C$ in polynomial time. Next we show that, Planar $X3C$ has a solution if and only if $G$ has a ve-RDF with weight at most $2q$.
Suppose $C'$ is a solution for Planar $X3C$ with $|C'| = q$. We construct a ve-RDF $f$, on $G$, same as in Equation 3. Clearly, $f(V) = 2q = k$.
The proof of the converse is similar to the proof given in Theorem 1. □

## 3. Threshold Graphs

In this section, we determine the vertex-edge Roman domination number of threshold graphs.
**Definition 1.** *A graph $G = (V, E)$ is called a threshold graph if there is a real number $T$ and a real number $w(v)$ for every $v \in V$ such that a set $S \subseteq V$ is independent if and only if $\sum_{v \in S} w(S) \leq T$.*
Although several characterizations are defined for threshold graphs, we use the following characterization of threshold graphs given in [20] to prove that vertex-edge Roman domination number can be computed in linear time for threshold graphs.
A graph $G$ is a threshold graph if and only if it is a split graph and, for split partition $(C, I)$ of $V$ where $C$ is a clique and $I$ is an independent set, there is an ordering

$(x_1, x_2, \ldots, x_p)$ of vertices of $C$ such that $N_G[x_1] \subseteq N_G[x_2] \subseteq N_G[x_3] \subseteq \ldots \subseteq N_G[x_p]$, and there is an ordering $(y_1, y_2, \ldots, y_q)$ of the vertices of $I$ such that $N_G(y_1) \supseteq N_G(y_2) \supseteq N_G(y_3) \supseteq \ldots \supseteq N_G(y_q)$.

**Theorem 5.** *Let $G$ be a connected threshold graph. Then*

$$\gamma_{veR}(G) = \begin{cases} 0, & \textit{if } |V(G)| = 1 \\ 1, & \textit{if } G = K_{1,n} \ (n \geq 1) \\ 2, & \textit{otherwise} \end{cases}$$

*Proof.* If $|V(G)| = 1$ or $G = K_{1,n}$ $(n \geq 1)$ then by the definition of ve-RDF the result follows. Otherwise, let $G$ be a connected threshold graph with $p$ clique vertices $(x_1, x_2, \ldots, x_p)$ such that $N_G[x_1] \subseteq N_G[x_2] \subseteq N_G[x_3] \subseteq \ldots \subseteq N_G[x_p]$. Now, define a function $f : V \to \{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 2, & \text{if } v = x_p \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Clearly, $f$ is a ve-RDF and $\gamma_{veR}(G) \leq 2$. From the definition of ve-RDF, it follows that $\gamma_{veR}(G) \geq 2$. Therefore $\gamma_{veR}(G) = 2$. □

Now, the following result is immediate from Theorem 5.
If $G$ is disconnected with $k$ connected components $G_1, G_2, \ldots, G_k$ then
$\gamma_{veR}(G) = \sum_{i=1}^{k} \gamma_{veR}(G_i)$.

**Theorem 6.** *MVERDP can be solvable in linear time for threshold graphs.*

*Proof.* Since the ordering of the vertices of the clique and the number of connected components in a threshold graph can be determined in linear time [16, 20], the result follows. □

## 4. Chain Graphs

In this section, we determine the vertex-edge Roman domination number of chain graphs. A bipartite graph $G = (X, Y, E)$ is called a *chain graph* if the neighborhoods of the vertices of $X$ form a *chain*, that is, the vertices of $X$ can be linearly ordered, say $(x_1, x_2, ..., x_p)$, such that $N_G(x_1) \subseteq N_G(x_2) \subseteq \ldots \subseteq N_G(x_p)$. If $G = (X, Y, E)$ is a chain graph, then the neighborhoods of the vertices of $Y$ also form a chain. An ordering $\alpha = (x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)$ of $X \cup Y$ is called a *chain ordering* if $N_G(x_1) \subseteq N_G(x_2) \subseteq \ldots \subseteq N_G(x_p)$ and $N_G(y_1) \supseteq N_G(y_2) \supseteq \ldots \supseteq N_G(y_q)$. Every chain graph admits a chain ordering [29].

**Theorem 7.** *Let $G(X, Y, E)(\neq K_1)$ be a chain graph. Then,*

$$\gamma_{veR}(G) = \begin{cases} 1, & \textit{if } \ |X| = 1 \textit{ or } |Y| = 1 \\ 2, & \textit{otherwise} \end{cases} \tag{5}$$

*Proof.* Let $G(X, Y, E)$ be a connected chain graph with the chain ordering $\alpha = (x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)$. Now, define a function $f : V \to \{0, 1, 2\}$

*Case* (1) : $|X| = 1$ then $f(v) = \begin{cases} 1, & \text{if } v = x_1 \\ 0, & \text{otherwise} \end{cases}$

*Case* (2) : $|Y| = 1$ then $f(v) = \begin{cases} 1, & \text{if } v = y_1 \\ 0, & \text{otherwise} \end{cases}$

*Case* (3) : $|X| \geq 2$ and $|Y| \geq 2$ then $f(v) = \begin{cases} 2, & \text{if } v = x_p \\ 0, & \text{otherwise} \end{cases}$

Clearly, $f$ is a ve-RDF in each of the three cases. From the definition of ve-RDF it is clear that $\gamma_{veR}(G) \geq 1$ in cases 1, 2 and $\gamma_{veR}(G) \geq 2$ in case 3. □

If the chain graph $G$ is disconnected with $k$ connected components $G_1, G_2, \ldots, G_k$ then it is easy to verify that $\gamma_{veR}(G) = \sum_{i=1}^{k} \gamma_{veR}(G_i)$. Now, the following result is immediate from Theorem 7.

**Theorem 8.** *MVERDP can be solvable in linear time for chain graphs.*

*Proof.* Since the chain ordering and the connected components can be computed in linear time [16, 27], the result follows. □

## 5. Bounded Treewidth Graphs

A *tree decomposition* of a graph $H$ is a tree $T_1$ with the vertex set $V(T_1) = \{Z_1, Z_2, \ldots, \}$, a subset of the power set of $V(H)$ with the following requirements.
*i*). $V(H) = \bigcup_{Z_v \in V(T_1)} Z_v$
*ii*). $\forall (u, v) \in E(H)$, there exists a vertex $Z_t \in V(T_1)$ such that $u, v \in Z_t$, and
*iii*). $\forall v \in V(H)$, the induced subgraph $\{Z_t : v \in Z_t \textit{ and } Z_t \in V(T_1)\}$ is a subtree of $T$.
Then the tree decomposition $T_1$ of $H$ is said to have *width* equals to $max\{|Z_t| - 1 : Z_t \in V(T_1)\}$. The *treewidth* is the smallest width of a tree decomposition of a graph. A graph problem for bounded treewidth graphs is linear time solvable if there exists a counting monadic second-order logic (CMSOL) formula for it. We show that VERDP can be expressed in CMSOL [7].

**Theorem 9.** *Given a graph $G$ and a positive integer $k$, VERDP can be expressed in CMSOL.*

*Proof.* Let $f : V(G) \rightarrow \{0, 1, 2\}$ be a function. Also, for $j = 0, 1$ or 2, let $V_j = \{v \mid f(v) = j\}$. A CMSOL formula for the VERDP is expressed as follows.

$$veRom\_Dom(V) = (f(V) \leq k) \wedge \exists V_0, V_1, V_2, \forall u, v[[adj(u,v) \wedge [\neg(u \in V_0 \wedge v \in V_0) \vee \exists w((adj(u,w) \vee adj(v,w)) \wedge w \in V_2)]] \vee \neg adj(u,v)],$$

where $adj(u, v)$ holds true iff $(u, v) \in E(G)$. □

Now, the theorem below follows from Courcelle's result [9] and Theorem 9.

**Theorem 10.** *MVERDP for graphs with treewidth at most a constant is solvable in linear time.*

### 5.1. Approximation Algorithm

In this subsection, we design an approximation algorithm for optimization version of vertex-edge Roman domination problem based on the approximation result known for VERTEX COVER problem, which is given below.

**VERTEX COVER**
**Instance:** A simple, undirected graph $G = (V, E)$.
**Solution:** Minimum cardinality vertex cover $C$ of $G$.
**Measure:** Cardinality of $C$.

Now, we propose a 2-approximation algorithm for MVERDP.
The following approximation result has been obtained in [16] for VERTEX COVER problem.

**Theorem 11 ([16]).** *VERTEX COVER problem has an approximation algorithm with approximation ratio of* 2*.*

By Theorem 11, let APP-VERTEX-COVER be an approximation algorithm that gives a vertex cover $C$ of a graph $G$ such that $|C| \leq |C^*|$, where $C^*$ is an optimal vertex cover of the graph $G$.
Next, we propose an algorithm APP-VERD to compute an approximate solution of MVERDP. In our algorithm, first we compute a vertex cover $C$ of the input graph $G$ using the approximation algorithm APP-VERTEX-COVER. Next, we construct a triple $T_r$ in which every vertex in $C$ will be assigned with weight 1 and the remaining vertices will be assigned with weight 0.
Now, let $T_r = (C', \emptyset, C)$ be the triple obtained by using the APP-VERTEX-COVER algorithm. It can be easily seen that every vertex $v \in V$ is assigned with weight either 0 or 1. Since $C$ is a vertex cover of $G$, every edge $(u, v) \in E(G)$ is incident with a vertex with weight 1. Thus, $T_r$ gives a ve-RDF of $G$.

**Algorithm 1** APP-VERD($G$)

**Input:** A simple, undirected graph $G$.
**Output:** A vertex-edge Roman dominating triple $T_r$ of $G$.
1: $C \leftarrow$ APP-VERTEX-COVER($G$)
2: $T_r \leftarrow (V \setminus C, C, \emptyset)$
3: return $T_r$.

We note that the algorithm APP-VERD computes a vertex-edge Roman dominating triple $T_r$ of the given graph $G$ in polynomial time. Hence, we have the following result.

**Theorem 12.** *The MVERDP in a graph can be approximated with an approximation ratio of* 2.

*Proof.* Let $C$ be the vertex cover produced by the algorithm APP-VERTEX-COVER, $T_r$ be the vertex-edge Roman dominating triple produced by the algorithm APP-VERD and $W_r$ be the weight of $T_r$.
It can be observed that $W_r = |C|$. It is known that $|C| \leq 2\tau(G)$. Therefore, $W_r \leq 2\tau(G)$. Since $\tau(G) \leq \gamma_{veR}(G)$, it follows that $W_r \leq 2\gamma_{veR}(G)$. □

We have the following corollary of Theorem 12.

**Corollary 2.** *MVERDP problem is in APX.*

## 6. Complexity Contrast between Vertex Cover and Vertex-Edge Roman Domination Problems

Although cardinality of an optimal vertex cover is a lower bound on the minimum weight of a vertex-edge Roman dominating function of a graph, respective problems differ in computational complexity. In particular, we show that there exists a graph class for which VERDP is polynomial-time solvable, whereas the VERTEX COVER problem is NP-complete. Similar study has been made between domination and other domination parameters in [13, 22, 23].

We construct a new class of graphs in which the MVERDP can be solved trivially, whereas the decision version of the VERTEX COVER problem is NP-complete, which is defined as follows.

**VERTEX COVER DECISION**

**Instance :** A simple, undirected graph $G$ and a positive integer $k$.

**Question :** Does there exist a vertex cover of size at most $k$ in $G$?

**Definition 3.** (GP3 graph). A graph is *GP*3 *graph* if it can be constructed from a connected graph $G = (V, E)$ where $V(G) = \{v_1, v_2, \ldots, v_n\}$, in the following way :

1. Create three copies of $P_2$ graphs such as $a_i - b_i, c_i - d_i$ and $e_i - f_i$, for each $i$.

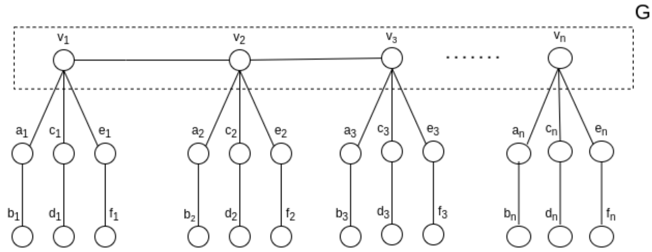


**Figure 5.** **An illustration to the construction of $GP3$ graph from $G$**

2. Add edges $\{(v_i, a_i), (v_i, c_i), (v_i, e_i) : 1 \le i \le n\}$.

General $GP3$ graph construction is shown in Figure 5.

**Theorem 13.** *If $G'$ is a GP3 graph obtained from a graph $G = (V, E)$ $(|V| = n)$, then $\gamma_{veR}(G') = 2n$.*

*Proof.* Let $G' = (V', E')$ be a GP3 graph constructed from $G$. Also let $f : V' \to \{0, 1, 2\}$ be a function on graph $G'$, which is defined as follows

$$f(v) = \begin{cases} 2, & \text{if } v \in \{v_i : 1 \le i \le n\} \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

Clearly, $f$ is a ve-RDF and $\gamma_{veR}(G') \le 2n$. Next, we show that $\gamma_{veR}(G') \ge 2n$. From the definition of ve-RDF it follows that, $g(v_i) + g(a_i) + g(b_i) + g(c_i) + g(d_i) + g(e_i) + g(f_i) \ge 2$, where $1 \le i \le n$. Hence $g(V) \ge 2n$. Therefore $g(V) = 2n$. □

**Lemma 1.** *Let $G'$ be a GP3 graph constructed from a graph $G = (V, E)$. Then $G$ has a vertex cover of size at most $k$ if and only if $G'$ has a vertex cover of size at most $k + 3n$.*

*Proof.* Suppose $C$ be vertex cover of $G$ of size at most $k$, then it is clear that $C \cup \{b_i, d_i, f_i : 1 \le i \le n\}$ is a vertex cover of $G'$ of size at most $k + 3n$. Conversely, suppose $C'$ is a vertex cover of $G'$ of size at most $k + 3n$. Then at least one vertex from each pair of the vertices $\{a_i, b_i\}$, $\{c_i, d_i\}$, $\{e_i, f_i\}$ must be included in $C'$. Clearly, $C' \cap V$ is a vertex cover of $G$ of size at most $k$. □

The following result is well known for the VERTEX COVER DECISION problem.

**Theorem 14.** *([10]) The VERTEX COVER DECISION problem is NP-complete for general graphs.*

From Lemma 1 and Theorem 14, it follows that VERTEX COVER DECISION problem is NP-hard for GP3 graphs. Hence the following theorem.

**Theorem 15.** *The VERTEX COVER DECISION problem is NP-complete for GP3 graphs.*

## 7. Integer Linear Programming Formulation

Let $G = (V, E)$ be an undirected graph, with $|V| = n$, $|E| = m$ and $f : V(G) \to \{0, 1, 2\}$ be a ve-RDF of $G$. Here we present an Integer Linear Program (ILP) model for MVERDP. This model uses two sets of binary variables. Specifically, for each vertex $v \in V$, we define

$$x_v = \begin{cases} 1, & f(v) = 1 \\ 0, & \text{otherwise} \end{cases} \qquad y_v = \begin{cases} 1, & f(v) = 2 \\ 0, & \text{otherwise} \end{cases}$$

The ILP model of the MVERDP can now be formulated as
Determine :

$$min\{\sum_{v \in V}(x_v + 2y_v)\} \tag{7}$$

subject to constraints:

$$x_u + y_u + x_v + y_v + \sum_{u' \in N(u)} y_{u'} + \sum_{v' \in N(v)} y_{v'} \geq 1, \ (u, v) \in E \tag{8}$$

$$x_v + y_v \leq 1, \ v \in V \tag{9}$$

$$x_v, y_v \in \{0, 1\} \tag{10}$$

The objective function given in Equation 7 minimizes the weight of ve-RDF. Constraint 8 ensures vertex-edge Roman domination condition i.e., either at least one end vertex of every edge $(u, v)$ is assigned label 1 or 2, or if the labels assigned to the end vertices $u$ and $v$ are zero then either $u$ or $v$ is adjacent to a vertex with label 2. Condition 9, guarantees that exactly one label is assigned to every vertex and the condition 10 ensures that the variables are binary in nature.
In the proposed ILP model, the number of variables is $2n$ and the constraints is $m+n$.

## 8. Conclusion

In this paper, we have shown that the VERDP is NP-complete for star convex bipartite graphs, comb convex bipartite graphs, chordal graphs and planar graphs. Investigating the algorithmic complexity of these problems for other subclasses of bipartite and split graphs remains open. Next, it is also shown that MVERDP is solvable in linear time for threshold graphs, chain graphs and bounded tree-width graphs. From the approximation point of view, an approximation algorithm is proposed for MVERDP. Approximation hardness of MVERDP problem is still open. We have shown that the vertex cover and vertex-edge Roman domination problems are not equivalent in computational complexity aspects by constructing a new class of graphs called GP3 graphs. Thus, there is a scope to study each of these problems on its own for particular graph classes. An ILP formulation for MVERDP with $2n$ variables and $m+n$ constraints is presented. Finding a better ILP formulation(s) for MVERDP with fewer variables or constraints remains open.

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## References

[1] H. Abdollahzadeh Ahangar, *Maximal Roman domination numbers in graphs*, Util. Math. **103** (2017), 245–258.

[2] H. Abdollahzadeh Ahangar, M.P. Álvarez, M. Chellali, S.M. Sheikholeslami, and J.C. Valenzuela-Tripodoro, *Triple Roman domination in graphs*, Applied Math. Comput. **391** (2021), Article ID: 125444.

[3] H. Abdollahzadeh Ahangar, J. Amjadi, M. Chellali, S. Nazari-Moghaddam, and S.M. Sheikholeslami, *Total Roman reinforcement in graphs.*, Discuss. Mathe. Graph Theory **39** (2019), no. 4, 787–803.

[4] H. Abdollahzadeh Ahangar, M. Chellali, D. Kuziak, and V. Samodivkin, *On maximal Roman domination in graphs*, Int. J. Comput. Math. **93** (2016), no. 7, 1093–1102.

[5] H. Abdollahzadeh Ahangar, M. Chellali, S.M. Sheikholeslami, M. Soroudi, and L. Volkmann, *Total vertex-edge domination in trees*, Acta Math. Univ. Comenianae **90** (2021), no. 2, 127–143.

[6] R. Boutrig, M. Chellali, T.W. Haynes, and S.T. Hedetniemi, *Vertex-edge domination in graphs*, Aequationes Math. **90** (2016), no. 2, 355–366.

[7] A. Brandstädt, V.B. Le, and J.P. Spinrad, *Graph classes: A survey*, SIAM, Philadelphia, 2004.

[8] E.J. Cockayne, P.A. Dreyer Jr, S.M. Hedetniemi, and S.T. Hedetniemi, *Roman domination in graphs*, Discrete Math. **278** (2004), no. 1-3, 11–22.

[9] B. Courcelle, *The monadic second-order logic of graphs. I. Recognizable sets of finite graphs*, Information and Computation **85** (1990), no. 1, 12–75.

[10] M.R. Garey and D.S. Johnson, *Computers and Interactability : A Guide to the Theory of NP-completeness*, Freeman, New York, 1979.

[11] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker Inc. New York, 1998.

[12] ______, *Fundamentals of Domination in Graphs*, Marcel Dekker Inc. New York, 1998.

[13] M.A. Henning and A. Pandey, *Algorithmic aspects of semitotal domination in graphs*, Theor. Comput. Sci. **766** (2019), 46–57.

[14] B. Krishnakumari, Y. B. Venkatakrishnan, and M. Krzywkowski, *Bounds on the vertex–edge domination number of a tree*, Comptes Rendus Mathematique **352** (2014), no. 5, 363–366.

[15] H.N. Kumar and Y.B. Venkata Krishna, *Vertex-edge Roman domination in graphs*, Kragujevac J. Math. **45** (2021), no. 5, 685–698.

[16] C.E. Leiserson, R.L. Rivest, T.H. Cormen, and C. Stein, *Introduction to Algorithms*, MIT press Cambridge, MA, 2001.

[17] J. Lewis, S.T. Hedetniemi, T.W. Haynes, and G.H. Fricke, *Vertex-edge domination*, Util. Math. **81** (2010), 193–213.

[18] J.R. Lewis, *Vertex-Edge and Edge-Vertex Domination in Graphs*, Ph.D. thesis, Clemson University, Clemson, 2007.

[19] M.-S. Lin and C.-M. Chen, *Counting independent sets in tree convex bipartite graphs*, Discrete Appl. Math. **218** (2017), 113–122.

[20] N. Mahadev and U. Peled, *Threshold Graphs and Related Topics*, Elsevier, 1995.

[21] A. Oganian and J. Domingo-Ferrer, *On the complexity of optimal microaggregation for statistical disclosure control*, Statistical Journal of the United Nations Economic Commission for Europe **18** (2001), no. 4, 345–353.

[22] B.S. Panda and A. Pandey, *Algorithm and hardness results for outer-connected dominating set in graphs*, J. Graph Algorithms Appl. **18** (2014), 493–513.

[23] B.S. Panda, A. Pandey, and S. Paul, *Algorithmic aspects of b-disjunctive domination in graphs*, J. Comb. Optim. **36** (2018), no. 2, 572–590.

[24] S. Paul and K. Ranjan, *On vertex-edge and independent vertex-edge domination*, International Conference on Combinatorial Optimization and Applications, Springer, 2019, pp. 437–448.

[25] K.W. Peters, *Theoretical and Algorithmic Results on Domination and Connectivity*, Ph.D. thesis, Clemson University, Clemson, 1986.

[26] E.N. Satheesh, *Some Variations of Domination and Applications*, Ph.D. thesis, Mahatma Gandhi University, 2014.

[27] R. Uehara and Y. Uno, *Efficient algorithms for the longest path problem*, International Symposium on Algorithms and Computation, Springer, 2004, pp. 871–883.

[28] D.B. West, *Introduction to Graph Theory*, Upper Saddle River: Prentice Hall, 2001.

[29] M. Yannakakis, *Node-and edge-deletion NP-complete problems*, Proceedings of the tenth annual ACM symposium on Theory of computing, 1978, pp. 253–264.

[30] R. Ziemann and P. Żyliński, *Vertex-edge domination in cubic graphs*, Discrete Math. **343** (2020), no. 11, Article ID: 112075.