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# Restrained double Italian domination in graphs

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**Abstract:** Let $G$ be a graph with vertex set $V(G)$. A double Italian dominating function (DIDF) is a function $f : V(G) \longrightarrow \{0, 1, 2, 3\}$ having the property that $f(N[u]) \geq 3$ for every vertex $u \in V(G)$ with $f(u) \in \{0, 1\}$, where $N[u]$ is the closed neighborhood of $u$. If $f$ is a DIDF on $G$, then let $V_0 = \{v \in V(G) : f(v) = 0\}$. A restrained double Italian dominating function (RDIDF) is a double Italian dominating function $f$ having the property that the subgraph induced by $V_0$ does not have an isolated vertex. The weight of an RDIDF $f$ is the sum $\sum_{v\in V(G)} f(v)$, and the minimum weight of an RDIDF on a graph $G$ is the restrained double Italian domination number. We present bounds and Nordhaus-Gaddum type results for the restrained double Italian domination number. In addition, we determine the restrained double Italian domination number for some families of graphs.

**Keywords:** Double Italian domination, restrained double Italian domination, restrained domination

**AMS Subject classification:** 05C69

## 1. Introduction

For definitions and notations not given here we refer to [12]. We consider simple graphs $G$ with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of $G$ is $n = n(G) = |V|$. The *open neighborhood* of a vertex $v$ is the set $N(v) = N_G(v) = \{u \in V(G) \mid uv \in E\}$ and its *closed neighborhood* is the set $N[v] = N_G[v] = N(v) \cup \{v\}$. The *degree* of vertex $v \in V$ is $d(v) = d_G(v) = |N(v)|$. The *maximum degree* and *minimum degree* of $G$ are denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. The *complement* of a graph $G$ is denoted by $\overline{G}$. For a subset $D$ of vertices in a graph $G$, we denote by $G[D]$ the subgraph of $G$ induced by $D$. The *diameter* of a graph $G$, denoted by $\mathrm{diam}(G)$, is the greatest distance between two vertices of $G$. A *leaf* is a vertex of degree one, and its neighbor is called a *support vertex.* A set $S \subseteq V(G)$ is called a *dominating set* if every vertex is either an element of $S$ or is adjacent to an element of $S$. The *domination number* $\gamma(G)$ of a graph $G$ is the minimum cardinality of a dominating set of $G$. A *restrained dominating set* is a set $S \subseteq V(G)$ where every vertex in $V(G) \setminus S$ is adjacent to a vertex in $S$ as well as to another vertex in $V(G) \setminus S$. The *restrained domination number* of $G$, denoted by $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set of $G$. Restrained domination was formally defined by Domke, Hattingh, Hedetniemi, Laskar and Markus in their 1999 paper [9]. For more information on this paramter we refer the reader to the survey paper [11]. We write $P_n$ for the path of order $n$, $C_n$ for the cycle of length $n$ and $K_n$ for the complete graph of order $n$. Also, let $K_{n_1,n_2,\ldots,n_p}$ denote the complete $p$-partite graph with vertex set $S_1 \cup S_2 \cup \ldots \cup S_p$ where $|S_i| = n_i$ for $1 \leq i \leq p$. For $n \geq 2$, the *star* $K_{1,n-1}$ has one vertex of degree $n-1$ and $n-1$ leaves. The *corona* $H \circ K_1$ is the graph constructed from a copy of $H$, where for each vertex $v \in V(H)$, a new vertex $v'$ and a pendant edge $vv'$ are added.

Cockayne, Dreyer, S.M. Hedetniemi and S.T. Hedetniemi [8] introduced the concept of *Roman domination* in graphs, and since then a lot of related variations and generalizations have been studied (see [4–7]).

In 2016, Chellali, Haynes, S.T. Hedetniemi and McRae [3] defined a new variant of Roman dominating functions, the so called Italian dominating functions.

Mojdeh and Volkmann [13] considered a variant of Italian domination which they called double Italian domination. A *double Italian dominating function* (DIDF) on a graph $G$ is a function $f : V(G) \longrightarrow \{0,1,2,3\}$ having the property that for every vertex $u \in V(G)$, if $f(u) \in \{0,1\}$, then $f(N[u]) \geq 3$. The weight of a DIDF $f$ is the sum $w(f) = \sum_{v \in V(G)} f(v)$, and the minimum weight of a DIDF in a graph $G$ is the *double Italian domination number*, denoted by $\gamma_{dI}(G)$. For a DIDF $f$, one can denote $f = (V_0, V_1, V_2, V_3)$, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0,1,2,3$. This concept was further studied in [1, 2, 16].

A *restrained double Italian dominating function* (RDIDF) is a DIDF $f$ having the property that the subgraph induced by $V_0$ does not have an isolated vertex. The weight of an RDIDF $f$ is the sum $\sum_{v \in V(G)} f(v)$, and the minimum weight of an RDIDF on a graph $G$ is the *restrained double Italian domination number*, denoted by $\gamma_{rdI}(G)$. Clearly, $\gamma_{dI}(G) \leq \gamma_{rdI}(G)$.

In this paper, we present sharp bounds and Nordhaus-Gaddum type results for the restrained double Italian domination number. In addition, we determine the restrained double Italian domination number for some families of graphs.

We make use of the following results.

**Proposition 1.** *1. (Ore's Theorem) If a graph $G$ of order $n$ has no isolated vertices, then $\gamma(G) \leq n/2$.*

*2. [10, 15] For a graph $G$ of order $n$ with no isolated vertices, $\gamma(G) = n/2$ if and only if the components of $G$ are the cycle $C_4$ or the corona $H \circ K_1$ for any connected graph $H$.*

**Proposition 2.** *[9] If $n \geq 2$ is an integer, then $\gamma_r(K_{1,n-1}) = n$.*

**Proposition 3.** *[13] If $n \geq 3$ is an integer, then $\gamma_{dI}(C_n) = n$.*

**Proposition 4.** *[13] If $G$ is a graph of order $n \geq 2$, then $\gamma_{dI}(G) \geq 3$.*

**Proposition 5.** *If $G$ is a graph of order $n$, then $\gamma_{rdI}(G) \leq 2n$, with equality if and only if $G = \overline{K_n}$.*

*Proof.* Define the function $f$ on $G$ by $f(x) = 2$ for each vertex $x \in V(G)$. Since $f$ is an RDIDF on $G$ of weight $2n$, we deduce that $\gamma_{rdI}(G) \leq 2n$. If $G = \overline{K_n}$, then obviously $\gamma_{rdI}(G) = 2n$. If $G$ contains an edge $uv$, then define $g$ by $g(u) = 1$ and $g(x) = 2$ for $x \in V(G) \setminus \{u\}$. Then $g$ is an RDIDF on $G$ of weight $2n - 1$ and thus $\gamma_{rdI}(G) \leq 2n - 1$. This completes the proof. □

**Proposition 6.** *If $G$ is a graph of order $n$, with $\delta(G) \geq 2$, then $\gamma_{rdI}(G) \leq n$.*

*Proof.* Define the function $f$ by $f(x) = 1$ for each vertex $x \in V(G)$. Since $\delta(G \geq 2$, we observe that $f(N[x]) \geq 3$ for every vertex $x \in V(G)$ with $f(x) \in \{0, 1\}$. Therefore $f$ is an RDIDF on $G$ of weight $n$ and thus $\gamma_{rdI}(G) \leq n$. □

**Proposition 7.** *If $G$ is a connected graph of order $n \geq 2$, then $\gamma_{rdI}(G) \leq \frac{3n}{2}$ with equality if an only if $G \in \{P_2, P_4\}$.*

*Proof.* Let $S$ be a dominating set of $G$ and define the function $f$ by $f(x) = 2$ if $x \in S$ and $f(x) = 1$ otherwise. Then $f$ is an RDIDF of $G$ and by Ore's Theorem we have $\gamma_{rdI}(G) \leq n + \gamma(G) \leq \frac{3n}{2}$.
If $G \in \{P_2, P_4\}$, then clearly $\gamma_{rdI}(G) = \frac{3n}{2}$. Conversely, assume that $\gamma_{rdI}(G) = \frac{3n}{2}$. By Proposition 1-(2), $G = C_4$ or $G$ is the corona $H \circ K_1$ for some connected graph $H$. We deduce from Proposition 6 that $G \neq C_4$. Hence $G = H \circ K_1$ for some connected graph $H$. If $n(H) \geq 3$ and $u_1u_2u_3$ is a path in $H$ and $v_i$ is the leaf adjacent to $u_i$ in $G$ for $1 \leq i \leq 3$, then the function $f$ defined by $f(u_1) = f(u_2) = 0$, $f(v_1) = 3$, $f(v_2) = 2$, $f(x) = 2$ for $x \in V(H) - \{u_1, u_2\}$ and $f(x) = 1$ otherwise, is an RDIDF on $G$ of weight $\frac{3n}{2} - 1$, a contradiction. Thus $n(H) \leq 2$ and so $G \in \{P_2, P_4\}$. □

## 2. Special classes of graphs

In this section we determine the restrained double Italian domination number for complete graphs, complete $p$-partite graphs, paths and cycles. The proof of the first observation is easy and therefore omitted.

**Observation 1.** (i) $\gamma_{rdI}(K_n) = 3$ for $n \geq 2$,

(ii) $\gamma_{rdI}(K_{1,n-1}) = n + 1$ for $n \geq 2$,

(iii) $\gamma_{rdI}(K_{2,2}) = 4$, $\gamma_{rdI}(K_{2,3}) = 5$ and $\gamma_{rdI}(K_{p,q}) = 6$ for $p, q \geq 2$ and $p + p \geq 6$,

(iv) Let $K_{n_1,n_2,\ldots,n_p}$ be the complete $p$-partite graph such that $p \geq 3$ and $n_1 \leq n_2 \leq \ldots \leq n_p$. Then $\gamma_{rdI}(K_{1,n_2,\ldots,n_p}) = 3$, $\gamma_{rdI}(K_{2,n_2,\ldots,n_p}) = 4$, $\gamma_{rdI}(K_{n_1,n_2,n_3}) = 5$ for $n_1 \geq 3$ and $\gamma_{rdI}(K_{n_1,n_2,\ldots,n_p}) = 4$ for $n_1 \geq 3$ and $p \geq 4$.

**Observation 2.** If $n \geq 3$ is an integer, then $\gamma_{rdI}(C_n) = n$.

*Proof.* Proposition 3 implies $\gamma_{rdI}(C_n) \geq \gamma_{dI}(C_n) = n$. Since $\gamma_{rdI}(C_n) \leq n$ by Proposition 6, we obtain the desired result. □

**Observation 3.** If $n \geq 4$ is an integer, then $\gamma_{rdI}(P_n) = n + 2$.

*Proof.* Let $P_n = v_1v_2 \ldots v_n$. Define the function $f$ by $f(v_1) = f(v_n) = 2$ and $f(v_i) = 1$ for $2 \leq i \leq n-1$. Then $f$ is an RDIDF on $P_n$ of weight $n+2$ and therefore $\gamma_{rdI}(P_n) \leq n + 2$.
Now we show that $\gamma_{rdI}(P_n) \geq n + 2$. It is straightforward to verify that $\gamma_{rdI}(P_n) = n + 2$ for $4 \leq n \leq 6$. For $n \geq 7$ we proceed by induction on $n$. Let $n \geq 7$ and let the inverse inequality be valid for every path of order at least four and less than $n$. Assume that $f$ is a $\gamma_{rdI}(P_n)$-function. Clearly, $f(v_n) \geq 1$. Now we distinguish three cases.
If $f(v_n) = 1$, then $f(v_{n-1}) \geq 2$, and the function $g$ with $g(v_i) = f(v_i)$ for $1 \leq i \leq n-1$ is an RDIDF on $P_{n-1} = P_n - \{v_n\}$. Hence the induction hypothesis implies

$$\gamma_{rdI}(P_n) = \omega(f) = \omega(g) + 1 \geq \gamma_{rdI}(P_{n-1}) + 1 \geq (n-1) + 2 + 1 = n + 2.$$

If $f(v_n) = 2$, then $f(v_{n-1}) = 1$ and $f(v_{n-2}) \geq 1$. We note that the function $g$ with $g(v_{n-1}) = 2$ and $g(x) = f(x)$ for $1 \leq i \leq n-2$, is an RDIDF of $P_{n-1}$ and the result follows as above. Finally, let $f(v_n) = 3$. Then $f(v_{n-1}) = f(v_{n-2}) = 0$ and $f(v_{n-3}) = 3$. Clearly, the function $g$ with $g(v_i) = f(v_i)$ for $1 \leq i \leq n-3$ is an RDIDF on $P_{n-3} = P_n - \{v_n, v_{n-1}, v_{n-2}\}$. The induction hypothesis leads to

$$\gamma_{rdI}(P_n) = \omega(f) = \omega(g) + 3 \geq \gamma_{rdI}(P_{n-3}) + 3 \geq (n-3) + 2 + 3 = n + 2.$$

This completes the proof. □

## 3. Sharp bounds on $\gamma_{rdI}(G)$

**Theorem 4.** *If $G$ is a graph of order $n \geq 2$, then $\gamma_{rdI}(G) \geq 3$, with equality if and only if $\Delta(G) = n - 1$ and $G$ contains a vertex $w$ of maximum degree such that $\delta(G[N_G(w)]) \geq 1$.*

*Proof.* Using Proposition 4, we obtain $\gamma_{rdI}(G) \geq \gamma_{dI}(G) \geq 3$ immediately.
We next prove the equality part. Assume that $G$ contains a vertex $w$ with $d_G(w) =$

$n-1$ such that $\delta(G[N_G(w)]) \geq 1$. Define the function $f$ by $f(w) = 3$ and $f(x) = 0$ for $x \in V(G) \setminus \{w\}$. Since $G[N_G(w)]$ does not contain an isolated vertex, we observe that $f$ is an RDIDF on $G$ of weight 3 and so $\gamma_{rdI}(G) = 3$.

Conversely, assume that $\gamma_{rdI}(G) = 3$. If $f$ is a $\gamma_{rdI}(G)$-function, then there are three cases possible.

There is a vertex $w$ with $f(w) = 3$ such that the remaining $n-1$ vertices with value 0 are adjacent to $w$ and $\delta(G[N_G(w)]) \geq 1$.

There are two adjacent vertices $u$ and $v$ with $f(u) = 2$ and $f(v) = 1$ such that such that the remaining $n-2$ vertices with value 0 are adjacent to $u$ and $v$ and $G[V(G)\setminus\{u,v\}]$ has no isolated vertex. But then $d_G(u) = n-1$ and $\delta(G[N_G(u)]) \geq 1$.

There are three mutuality adjacent vertices $u, v, w$ with $f(u) = f(v) = f(w) = 1$ such that the remaining $n-3$ vertices with value 0 are adjacent to $u, v$ and $w$ and $G[V(G)\setminus\{u,v,w\}]$ has no isolated vertex. But then $d_G(u) = n-1$ and $\delta(G[N_G(u)]) \geq 1$.

In all three cases, we deduce that $\Delta(G) = n-1$ and $G$ contains a vertex $w$ of maximum degree such that $\delta(G[N_G(w)]) \geq 1$. This completes the proof. □

Using Proposition 2 and Observation 1 (ii), we observe that $\gamma_r(K_{1,n-1}) + 1 = n+1 = \gamma_{rdI}(K_{1,n-1})$ for $n \geq 2$. However, if $G$ is not a star, then we prove the following sharp inequality.

**Theorem 5.** *Let $G$ be a connected graph of order $n \geq 2$. If $G$ is not a star, then $\gamma_r(G) + 2 \leq \gamma_{rdI}(G)$.*

*Proof.* Let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{rdI}(G)$-function. We distinguish three cases.

**Case 1.** Let $|V_2| \geq 2$ or $|V_3| \geq 1$.
Then

$$\gamma_r(G) \leq |V_1| + |V_2| + |V_3| \leq |V_1| + 2|V_2| + 3|V_3| - 2 = \gamma_{rdI}(G) - 2.$$

**Case 2.** Let $|V_2| = |V_3| = 0$.
Then $|V_1| \geq 3$, and each vertex of $V_1$ is adjacent to two vertices of $V_1$ and each vertex of $V_0$ is adjacent to three vertices of $V_1$. Let $u \in V_1$ be adjacent to $v \in V_1$ and $w \in V_1$. Then $V_1 \setminus \{u, v\}$ is a restrained dominating set of $G$ of weight $|V_1| - 2$ and therefore $\gamma_r(G) + 2 \leq \gamma_{rdI}(G)$.

**Case 3.** Let $|V_3| = 0$ and $|V_2| = 1 = |\{w\}|$.

**Subcase 3.1.** Assume that $|V_0| \geq 1$. Then each vertex of $V_0$ is adjacent to $w$ and a vertex of $V_1$ or to at least three vertices in $V_1$. If there is a vertex $u \in V_0$ adjacent to $w$ and to a vertex $v \in V_1$, then $(V_1 \setminus \{v\}) \cup \{w\}$ is a restrained dominating of $G$ and so $\gamma_r(G) + 2 \leq \gamma_{rdI}(G)$. In the remaining case every vertex of $V_0$ is adjacent to three vertices of $V_1$. If the vertex $u \in V_0$ is adjacent to a vertex $v \in V_1$, then $(V_1 \setminus \{v\}) \cup \{w\}$ is a restrained dominating of $G$ and so $\gamma_r(G) + 2 \leq \gamma_{rdI}(G)$.

**Subcase 3.2.** Assume that $|V_0| = 0$. Let $V_1 = \{v_1, v_2, \ldots, v_{n-1}\}$ and let, without loss of generality, $v_1, v_2, \ldots, v_k$ adjacent to $w$ with $k \leq n-1$. Assume first that $k \leq n-2$. Since $G$ is connected, we assume, without loss of generality, that $v_{k+1}$ is adjacent to $v_k$. If $k \geq 2$, then $V(G)\setminus\{w, v_k\}$ is a restrained dominating of $G$ and hence $\gamma_r(G) + 2 \leq \gamma_{rdI}(G)$. Let next $k = 1$. Since $G$ is not a star, the vertex $v_2$ is adjacent to a further vertex, say $v_3$. Now $V(G) \setminus \{v_1, v_2\}$ is a restrained dominating of $G$ and thus $\gamma_r(G) + 2 \leq \gamma_{rdI}(G)$. Finally, assume that $k = n-1$. Since $G$ is not a star, $V_1$ contains two adjacent vertices. If, without loss of generality, $v_1$ and $v_2$ are adjacent, $V(G)\setminus\{v_1, v_2\}$ is a restrained dominating of $G$ and therefore $\gamma_r(G)+2 \leq \gamma_{rdI}(G)$. □

If $G$ is $C_4$ or $C_5$ or $G$ can be obtained from $C_3$ by attaching zero or more leaves to a vertex of $C_3$, then we observe that $\gamma_r(G) + 2 = \gamma_{rdI}(G) = n(G)$. These examples demonstrate that Theorem 5 is sharp.
If $S$ is an restrained dominating set of a graph $G$, then $(V(G)\setminus S, \emptyset, \emptyset, S)$ is an RDIDF on $G$. This implies the next observation immediately.

**Observation 6.** If $G$ is a graph, then $\gamma_{rdI}(G) \leq 3\gamma_r(G)$.

## 4. Trees

By $S_{p,q}$ we denote the *double star*, where one center vertex is adjacent to $p$ leaves and the other one by $q$ leaves. Our first result on trees is easy to verify.

**Observation 7.** If $S_{p,q}$ is a double star, then $\gamma_{rdI}(S_{p,q}) = p + q + 4 = n(S_{p,q}) + 2$.

**Theorem 8.** *If $T$ is a tree of order $n$ with diameter 4, then $\gamma_{rdI}(T) \geq n + 2$.*

*Proof.* Let $v_0, v_1, v_2, \ldots, v_p$ be the non-leaves of $T$ such that $v_0$ is adjacent to the vertices $v_1, v_2, \ldots, v_p$. In addition, let $v_i^1, v_i^2, \ldots, v_i^{t_i}$ be the leaves adjacent to $v_i$ for $1 \leq i \leq p$ and $u_1, u_2, \ldots, u_k$ be the leaves adjacent to $v_0$. Since $T$ is of diameter 4, we note that $p \geq 2$. Now let $f$ be a $\gamma_{rdI}(T)$-function. We observe that $f(v_i) + f(v_i^1) + f(v_i^2) + \ldots + f(v_i^{t_i}) \geq t_i + 2$ and $f(v_0) + f(u_1) + f(u_2) + \ldots + f(u_k) \geq k + 1$ if $k \geq 1$. This implies $\omega(f) \geq n + p \geq n + 2$ if $k \geq 1$ and $\omega(f) \geq n - 1 + p \geq n + 2$ if $k = 0$ and $p \geq 3$. It remains the case that $k = 0$ and $p = 2$. If $f(v_0) \geq 1$, then we obtain the desired result. Let now $f(v_0) = 0$. Then, without loss of generality, $f(v_1) = 0$ and $f(v_2) = 3$. This leads to $f(v_1) + f(v_1^1) + f(v_1^2) + \ldots + f(v_1^{t_1}) \geq t_1 + 2$ and $f(v_2) + f(v_2^1) + f(v_2^2) + \ldots + f(v_2^{t_2}) \geq t_2 + 3$, and thus we obtain $\omega(f) \geq n + 2$ also in the last case. □

Let $T_1$ be the tree of diameter 4 consisting of the path $v_1v_2v_3v_4$ such that $v_4$ is adjacent to $t \geq 1$ leaves $w_1, w_2, \ldots, w_t$. Then $\gamma_{rdI}(T_1) = t + 6 = n(T_1) + 2$.

Let $T_2$ be the tree of diameter 4 consisting of the path $v_1v_2v_3$ such that $v_1$ is adjacent to two leaves $u_1$ and $u_2$ and $v_3$ is adjacent to $t \geq 1$ leaves $w_1, w_2, \ldots, w_t$. Then $\gamma_{rdI}(T_2) = t + 7 = n(T_2) + 2$.
These examples show that Theorem 8 is sharp.

**Theorem 9.** *Let $T$ be a tree of order $n \geq 4$. If $T$ is not a star, then $\gamma_{rdI}(T) \geq n + 2$.*

*Proof.* If $3 \leq \mathrm{diam}(T) \leq 4$, then Observation 7 and Theorem 8 lead to the desired result. Let now $\mathrm{diam}(T) \geq 5$. We proceed by induction on $n$. Assume that the result is valid for all trees which are not a star of order less than $n$. Let $v_1v_2 \ldots v_p$ be a diametrical path, and let $f$ be a $\gamma_{rdI}(T)$-function.
**Case 1.** Assume that there exists a leaf $v$ with $f(v) = 1$.
If $u$ is the neighbor of $v$, then $f(u) \geq 2$ and so the function $g$ with $g(x) = f(x)$ for $x \in V(T) \setminus \{v\}$ is an RDIDF on the tree $T - v$ of diameter at least 4. Hence the induction hypothesis implies

$$\gamma_{rdI}(T) = \omega(f) = \omega(g) + 1 \geq \gamma_{rdI}(T - v) + 1 \geq (n - 1) + 2 + 1 = n + 2.$$

Hence we assume in the following that $h(v) \geq 2$ for each $\gamma_{rdI}(T)$-function $h$ and each leaf $v$ of $T$.
**Case 2.** Assume that $f(v_1) = 2$.
Then $f(v_2) \leq 1$. If $f(v_2) = 1$, then the function $g$ with $g(v_1) = 1$, $g(v_2) = 2$ and $g(x) = f(x)$ otherwise is also a $\gamma_{rdI}(T)$-function, a contradiction.
Assume next that $f(v_2) = 0$. It follows that $f(v_3) = 0$ and there exists a further leaf $z$ adjacent to $v_2$ with $f(z) = 2$. If there exists a further neighbor $w$ of $v_3$ with $f(w) = 0$, then the function $g$ with $g(v_1) = g(z) = 1$, $g(v_2) = 2$ and $g(x) = f(x)$ otherwise is also a $\gamma_{rdI}(T)$-function, a contradiction. Therefore $f(x) \geq 1$ for each neighbor $x$ of $v_3$. If there is a further leaf $z_1$ adjacent to $v_2$, then the function $g$ with $g(v_1) = g(z) = g(z_1) = g(v_3) = 1$, $g(v_2) = 2$ and $g(x) = f(x)$ otherwise is also a $\gamma_{rdI}(T)$-function, a contradiction. Let now $u_1, u_2, \ldots, u_k$ be the leaves adjacent to $v_3$ and $w_1, w_2, \ldots, w_t$ be the support vertices adjacent to $v_3$. If $T_3$ is the component of $T - v_3v_4$ containing the vertex $v_3$, then we observe that $\sum_{x \in V(T_3)} f(x) \geq n(T_3) + 1$ if $k + t \geq 1$. Note that by Observation 1 (ii) $\gamma_{rdI}(K_{1,q-1}) = q + 1$. Thus this fact and the induction hypothesis implies

$$\gamma_{rdI}(T) = \omega(f) = \sum_{x \in V(T_3)} f(x) + \sum_{x \in V(T - T_3)} f(x) \geq n(T_3) + 1 + n(T - T_3) + 1 = n + 2.$$

If $d(v_3) = 2$, then $f(v_4) = 3$. Let again $T_3$ be the component of $T - v_3v_4$ containing the vertex $v_3$. If $T - T_3$ is a star, then the fact that $f(v_4) = 3$ leads to $\sum_{x \in V(T - T_3)} f(x) \geq n(T - T_3) + 3$, and thus

$$\gamma_{rdI}(T) = \omega(f) = \sum_{x \in V(T_3)} f(x) + \sum_{x \in V(T - T_3)} f(x) \geq 4 + n(T - T_3) + 3 = n + 3.$$

If $T - T_3$ is not a star, then the induction hypothesis yields to

$$\gamma_{rdI}(T) = \omega(f) = \sum_{x \in V(T_3)} f(x) + \sum_{x \in V(T-T_3)} f(x) \geq 4 + n(T - T_3) + 2 = n + 2.$$

**Case 3.** Assume that $f(v_1) = 3$.
Then $f(v_2) = f(v_3) = 0$. If there exists a further leaf $z$ adjacent to $v_2$ with $f(z) \geq 2$, then the function $g$ with $g(v_1) = 2$ and $g(x) = f(x)$ otherwise is also an RDIDF on $G$ with $\omega(g) < \omega(f)$, a contradiction. Thus $d(v_2) = 2$. If there exists a further neighbor $w$ of $v_3$ with $f(w) = 0$, then the function $g$ with $g(v_1) = 1$, $g(v_2) = 2$ and $g(x) = f(x)$ otherwise is also a $\gamma_{rdI}(T)$-function, a contradiction. Therefore $f(x) \geq 1$ for each neighbor $x$ of $v_3$. Let again $u_1, u_2, \ldots, u_k$ be the leaves adjacent to $v_3$ and $w_1, w_2, \ldots, w_t$ be the support vertices adjacent to $v_3$. If $T_3$ is the component of $T - v_3v_4$ containing the vertex $v_3$, then we observe that $\sum_{x \in V(T_3)} f(x) \geq n(T_3) + 1$ if $k + t \geq 1$. Thus the induction hypothesis implies

$$\gamma_{rdI}(T) = \omega(f) = \sum_{x \in V(T_3)} f(x) + \sum_{x \in V(T-T_3)} f(x) \geq n(T_3) + 1 + n(T - T_3) + 1 = n + 2.$$

If $d(v_3) = 2$, then $f(v_4) = 3$. Now the desired result follows as in Case 2. □

If $P_n$ is a path of order $n \geq 4$, then $\gamma_{rdI}(P_n) = n + 2$ by Observation 3. Thus Theorem 9 is sharp. However, there are many further trees with equality in Theorem 9.

## 5. Nordhaus-Gaddum type results

Results of Nordhaus-Gaddum type study the extreme values of the sum or product of a parameter on a graph and its complement. In their classical paper [14], Nordhaus and Gaddum discussed this problem for the chromatic number. We present such inequalities for the restrained double Italian domination number

**Theorem 10.** *If $G$ is a graph of order $n \geq 2$, then $\gamma_{rdI}(G) + \gamma_{rdI}(\overline{G}) \geq 7$, with equality if and only if $n = 2$.*

*Proof.* If $n = 2$, then it is easy to see that $\gamma_{rdI}(G) + \gamma_{rdI}(\overline{G}) = 7$. Let now $n \geq 3$. According to Theorem 4 we only need to show that if $\gamma_{rdI}(G) = 3$, then $\gamma_{rdI}(\overline{G}) \geq 5$. Assume that $\gamma_{rdI}(G) = 3$. It follows from Theorem 4 that $\Delta(G) = n - 1$. Therefore $\overline{G} = H \cup \{w\}$, where $w$ is an isolated vertex of $\overline{G}$. Since $n(H) \geq 2$, Theorem 4 leads to $\gamma_{rdI}(\overline{G}) \geq \gamma_{rdI}(H) + 2 \geq 5$. □

**Theorem 11.** *If $G$ is a graph $G$ of order $n \geq 1$ such that $G \neq P_4$ and $\overline{G} \neq P_4$, then*

$$\gamma_{rdI}(G) + \gamma_{rdI}(\overline{G}) \leq 2n + 3.$$

*Proof.* This bound is easy to verify for $1 \le n \le 3$. Let now $n \ge 4$, and assume, without loss of generality, that $\delta(G) \le \delta(\overline{G})$. We distinguish three cases.

**Case 1.** Assume that $\delta(G) = 0$.
Let $u$ be a vertex such that $d_G(u) = 0$. Assume that there exists a second vertex $v$ with $d_G(v) = 0$. Then Theorem 4 implies $\gamma_{rdI}(\overline{G}) = 3$, and thus it follows from Proposition 5 that $\gamma_{rdI}(G) + \gamma_{rdI}(\overline{G}) \le 2n + 3$. Now assume that $d_G(x) \ge 1$ for $x \in V(G) \setminus \{u\}$. Assume next that $d_G(v) = 1$ for a vertex $v \in V(G) \setminus \{u\}$, and let $w$ be adjacent to $v$ in $G$. Define on $\overline{G}$ the function $f$ by $f(u) = 3$, $f(w) = 1$ and $f(x) = 0$ for $x \in V(G) \setminus \{u, w\}$. Then $f$ is an RDIDF on $\overline{G}$ of weight 4. Hence Proposition 7 yields to

$$\gamma_{rdI}(G) + \gamma_{rdI}(\overline{G}) \le 2 + \left\lfloor \frac{3(n-1)}{2} \right\rfloor + 4 \le 2n + 3.$$

Now assume that $d_G(x) \ge 2$ for $x \in V(G) \setminus \{u\}$. Then Proposition 6 implies $\gamma_{rdI}(G) \le 2 + (n-1) = n + 1$. If we define on $\overline{G}$ the function $g$ with $g(u) = 2$ and $g(x) = 1$ for $x \in V(G) \setminus \{u\}$, then $g$ is an RDIDF on $\overline{G}$ of weight $n + 1$. Consequently, $\gamma_{rdI}(G) + \gamma_{rdI}(\overline{G}) \le 2n + 2$ in this case.

**Case 2.** Assume that $\delta(G) = 1$.
Let $u$ be a vertex such that $d_G(u) = 1$, and let $v$ be adjacent to $u$ in $G$. If $d_G(v) = 1$, then let $w \in V(G) \setminus \{u, v\}$. If $n = 4$, then it is easy to verify that $\gamma_{rdI}(G) + \gamma_{rdI}(\overline{G}) = 6 + 4 = 10 < 2n + 3$. If $n \ge 5$, then define $f$ on $\overline{G}$ by $f(v) = f(w) = 3$ and $f(x) = 0$ for $x \in V(G) \setminus \{v, w\}$. Then $f$ is an RDIDF on $\overline{G}$ of weight 6. Therefore we deduce from Proposition 7 that

$$\gamma_{rdI}(G) + \gamma_{rdI}(\overline{G}) \le \left\lfloor \frac{3n}{2} \right\rfloor + 6 \le 2n + 3.$$

Now assume that there exists a vertex $w \ne u, v$ with $d_G(w) = 1$. Let $w$ be adjacent to $v$ in $G$. Define $f$ on $\overline{G}$ by $f(u) = 3$, $f(v) = 2$ and $f(x) = 0$ for $x \in V(G) \setminus \{u, v\}$. Then $f$ is an RDIDF on $\overline{G}$ of weight 5. Proposition 7 implies

$$\gamma_{rdI}(G) + \gamma_{rdI}(\overline{G}) \le \left\lfloor \frac{3n}{2} \right\rfloor + 5 \le 2n + 3.$$

If $w$ is not adjacent to $v$, then let $z$ be adjacent to $w$ in $G$. If $n = 4$ and $v$ and $z$ are adjacent in $G$, then $G = P_4$, a contradiction. If $n = 4$ and $v$ and $z$ are adjacent in $\overline{G}$, then we observe that $\gamma_{rdI}(G) + \gamma_{rdI}(\overline{G}) \le 6 + 4 = 10 < 11 = 2n + 3$. If $n \ge 5$, then define $f$ by $f(u) = 3$, $f(v) = 2$, $f(z) = 1$ and $f(x) = 0$ for $x \in V(G) \setminus \{u, v, z\}$. Then $f$ is an RDIDF on $\overline{G}$ of weight 6, and Proposition 7 leads to

$$\gamma_{rdI}(G) + \gamma_{rdI}(\overline{G}) \le \left\lfloor \frac{3n}{2} \right\rfloor + 6 \le 2n + 3.$$

Now assume that $d_G(x) \geq 2$ for $x \in V(G) \setminus \{u\}$. Define $f$ on $G$ by $f(u) = 2$ and $f(x) = 1$ for $x \in V(G) \setminus \{u\}$. Then $f$ is an RDIDF on $G$ of weight $n + 1$ and thus $\gamma_{rdI}(G) \leq n+1$. Define $g$ on $\overline{G}$ by $g(u) = g(v) = 2$ and $g(x) = 1$ for $x \in V(G) \setminus \{u, v\}$. Then $g$ is an RDIDF on $\overline{G}$ of weight $n + 2$ and thus $\gamma_{rdI}(\overline{G}) \leq n + 2$. Consequently, $\gamma_{rdI}(G) + \gamma_{rdI}(\overline{G}) \leq 2n + 3$ in this case.

**Case 3.** Assume that $\delta(G) \geq 2$.
Then $\delta(\overline{G}) \geq 2$ and so Proposition 6 yields to $\gamma_{rdI}(G) + \gamma_{rdI}(\overline{G}) \leq 2n$. □

If $n \geq 2$, then it follows from Proposition 5 and Observation 1 (i) that $\gamma_{rdI}(K_n) + \gamma_{rdI}(\overline{K_n}) = 2n + 3$. Thus Theorem 11 is sharp.

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## References

[1] F. Azvin and N. Jafari Rad, *Bounds on the double Italian domination number of a graph*, Discuss. Math. Graph Theory (in press).

[2] F. Azvin, N. Jafari Rad, and L. Volkmann, *Bounds on the outer-independent double Italian domination number*, Commun. Comb. Optim. **6** (2021), no. 1, 123–136.

[3] M. Chellali, T.W. Haynes, S.T. Hedetniemi, and A. MacRae, *Roman {2}-domination*, Discrete Appl. Math. **204** (2016), 22–28.

[4] M. Chellali, N. Jafari Rad, S.M. Sheikholeslami, and L. Volkmann, *Roman domination in graphs*, Topics in Domination in Graphs (T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, eds.), Springer, Berlin/Heidelberg, 2020, pp. 365–409.

[5] ______, *A survey on Roman domination parameters in directed graphs*, J. Combin. Math. Combin. Comput. **115** (2020), 141–171.

[6] ______, *Varieties of Roman domination II*, AKCE Int. J. Graphs Comb. **17** (2020), no. 3, 966–984.

[7] ______, *Varieties of Roman domination*, Structures of Domination in Graphs (T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, eds.), Springer, Berlin/Heidelberg, 2021, pp. 273–307.

[8] E.J. Cockayne, P.A. Dreyer Jr, S.M. Hedetniemi, and S.T. Hedetniemi, *Roman domination in graphs*, Discrete Math. **278** (2004), no. 1-3, 11–22.

[9] G.S. Domke, J.H. Hattingh, S.T. Hedetniemi, R.C. Laskar, and L.R. Markus, *Restrained domination in graphs*, Discrete Math. **203** (1999), no. 1-3, 61–69.

[10] J.F. Fink, M.S. Jacobson, L.F. Kinch, and J. Roberts, *On graphs having domination number half their order*, Period. Math. Hungar. **16** (1985), no. 4, 287–293.

[11] J.H. Hattingh and E.F. Joubert, *Restrained and total restrained domination in graphs*, Topics in Domination in Graphs (T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, eds.), Springer, 2020, pp. 129–150.

[12] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.

[13] D.A. Mojdeh and L. Volkmann, *Roman* $\{3\}$*-domination (double Italian domination)*, Discrete Appl. Math. **283** (2020), 555–564.

[14] E.A. Nordhaus and J.W. Gaddum, *On complementary graphs*, Amer. Math. Monthly **63** (1956), no. 3, 175–177.

[15] C. Payan and N.H. Xuong, *Domination-balanced graphs*, J. Graph Theory **6** (1982), no. 1, 23–32.

[16] Z. Shao, D.A. Mojdeh, and L. Volkmann, *Total Roman* $\{3\}$*-domination in graphs*, Symmetry **12** (2020), no. 2, Article ID: 268.