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On the distance spectra of the product of signed graphs

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Abstract: In this article, we study the signed distance matrix of the product of signed graphs such as the Cartesian product and the lexicographic product in terms of the signed distance matrices of the factor graphs. Also, we discuss the distance spectra of some special classes of product of signed graphs.

Keywords: Signed graph, Signed distance matrix, Signed distance spectrum, Product of signed graphs

AMS Subject classification: 05C12, 05C22, 05C50, 05C76

1. Introduction

Throughout this paper we consider only simple, finite and connected signed graphs. For standard terminology and notion in graph theory, the reader may refer to Harary [4]. A signed graph is a graph $G = (V, E)$ with a signature function $\sigma : E \to \{1, -1\}$. We call a signed graph Σ as balanced if every cycle in it has an even number of negative edges. The sign of a cycle and that of a path in a signed graph is the product of the sign of the edges in each of them. The notion of signed distances for signed graph and that of the distance compatibility in signed graphs are adopted as in [3]. For any two vertices u and v in a signed graph Σ, according to the sign of the uv-path, there are two types of signed distances $d_{\max}(u, v)$ and $d_{\min}(u, v)$. Using these signed distances, the signed distance matrices (see [3]) are defined as,
(D1) $D^{\max}(\Sigma) = (d_{\max}(u, v))_{n\times n}$.
(D2) $D^{\min}(\Sigma) = (d_{\min}(u, v))_{n\times n}$.
We adopt the construction of a signed complete graph described in [3], obtained from the signed distance matrices $D^{\max}(\Sigma)$ and $D^{\min}(\Sigma)$, as follows.

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The associated signed complete graph $K^{D^{\max}}(\Sigma)$ with respect to $D^{\max}(\Sigma)$ is obtained by joining the non-adjacent vertices of Σ with edges having signs

$$\sigma(uv) = \sigma_{\max}(uv).$$

The associated signed complete graph $K^{D^{\min}}(\Sigma)$ with respect to $D^{\min}(\Sigma)$ is obtained by joining the non-adjacent vertices of Σ with edges having signs

$$\sigma(uv) = \sigma_{\min}(uv).$$

Whenever, $D^{\max} = D^{\min}$, the distance matrix of Σ is denoted by $D^{\pm}(\Sigma)$ and the associated signed complete graph of Σ is denoted by $K^{D^{\pm}}(\Sigma)$.

A signed graph Σ is said to be *distance compatible* (briefly *compatible*) if for any two vertices u and v, $d_{\min}(u,v) = d_{\max}(u,v)$. For all notations and definitions related to signed distances in signed graph, that are not defined in this paper, the reader may refer to [3].

The Cartesian product and lexicographic product of signed graphs are defined as follows.

The *Cartesian product* $\Sigma_1 \times \Sigma_2$ of two signed graphs $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ is defined in [2] as the signed graph with vertex set and edge set as that of the Cartesian product of the underlying unsigned graphs and the signature function σ for the labeling of the edges is defined by

$$\sigma((u_i, v_j)(u_k, v_l)) = \begin{cases} \sigma_1(u_i u_k) & \text{if } j = l, \\ \sigma_2(v_j v_l) & \text{if } i = k. \end{cases}$$

The *lexicographic product* $\Sigma_1[\Sigma_2]$ (also called composition) of two signed graphs $\Sigma_1 = (V_1, E_1, \sigma_1)$ and $\Sigma_2 = (V_2, E_2, \sigma_2)$ is defined in [5] as the signed graph $(V_1 \times V_2, E, \sigma)$, where the edge set is that of the lexicographic product of underlying unsigned graphs and the signature function σ for the labeling of the edges is defined by

$$\sigma((u_i, v_j)(u_k, v_l)) = \begin{cases} \sigma_1(u_i u_k) & \text{if } i \neq k, \\ \sigma_2(v_j v_l) & \text{if } i = k. \end{cases}$$

The distance compatibility criterion for the Cartesian product and lexicographic product of signed graphs are discussed in [6].

Theorem 1 ([6]). *The Cartesian product $\Sigma_1 \times \Sigma_2$ is compatible if and only if Σ_1 and Σ_2 are compatible.*

Theorem 2 ([6]). *Let Σ_1 and Σ_2 be two signed graphs. Then, $\Sigma_1[\Sigma_2]$ is compatible if and only if Σ_1 is compatible and Σ_2 is either all-positive or all-negative.*

For a compatible signed graph, $D^{\max}(\Sigma) = D^{\min}(\Sigma) = D^{\pm}(\Sigma)$. In this paper we simply denote the distance matrix of a compatible signed graph as $D(\Sigma)$ and the corresponding associated signed complete graph is denoted by $K^{D(\Sigma)}$.

In this paper, we derive explicit formulae for the distance matrices of the compatible signed graph products and we obtain the distance spectra of the Cartesian product of some special classes of signed graphs. Also, we compute the distance spectra of the lexicographic product $\Sigma_1[\Sigma_2]$, where Σ_1 is a compatible signed graph and $\Sigma_2 = (K_2, \sigma)$.

2. Distance matrices of compatible product of signed graphs

To deal with the distance matrices of compatible product of signed graphs, we use the Kronecker product of a $m \times n$ matrix $A = (a_{ij})$ and a $p \times q$ matrix B, which is defined to be the $mp \times nq$ matrix $A \otimes B = (a_{ij}B)$.
Let $\Sigma_1 = (G_1, \sigma)$ and $\Sigma_2 = (G_2, \sigma')$ be two compatible signed graphs with $|V(\Sigma_1)| = m$ and $|V(\Sigma_2)| = n$. Let σ_{ij} and σ'_{kl} be defined in Σ_1 and Σ_2 respectively, as follows.

$$\sigma_{ij} = \begin{cases} \sigma(P_{(u_i,u_j)}) & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

$$\sigma'_{kl} = \begin{cases} \sigma'(P_{(v_k,v_l)}) & \text{if } k \neq l, \\ 1 & \text{if } k = l. \end{cases}$$

Then,

$$K^{D(\Sigma_1)} + I_m = \begin{pmatrix} 1 & \sigma_{12} & \dots & \sigma_{1m} \\ \sigma_{21} & 1 & \dots & \sigma_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \dots & 1 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \dots & \sigma_{mm} \end{pmatrix}.$$

Similarly,

$$K^{D(\Sigma_2)} + I_n = \begin{pmatrix} 1 & \sigma'_{12} & \dots & \sigma'_{1n} \\ \sigma'_{21} & 1 & \dots & \sigma'_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \sigma'_{n1} & \sigma'_{n2} & \dots & 1 \end{pmatrix} = \begin{pmatrix} \sigma'_{11} & \sigma'_{12} & \dots & \sigma'_{1n} \\ \sigma'_{21} & \sigma'_{22} & \dots & \sigma'_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \sigma'_{n1} & \sigma'_{n2} & \dots & \sigma'_{nn} \end{pmatrix}.$$

Theorem 3. *Let $\Sigma_1 = (G_1, \sigma)$ and $\Sigma_2 = (G_2, \sigma')$ be two compatible signed graphs, where $|V(\Sigma_1)| = m$ and $|V(\Sigma_2)| = n$. Then, the distance matrix of the Cartesian product $\Sigma_1 \times \Sigma_2$ is given by,*

$$D(\Sigma_1 \times \Sigma_2) = D(\Sigma_1) \otimes (K^{D(\Sigma_2)} + I_n) + (K^{D(\Sigma_1)} + I_m) \otimes D(\Sigma_2).$$

Proof. Let $D(\Sigma_1)$ and $D(\Sigma_2)$ be the distance matrices of Σ_1 and Σ_2 respectively. Let $V(\Sigma_1) = \{u_1, u_2, \ldots, u_m\}$ and $V(\Sigma_2) = \{v_1, v_2, \ldots, v_n\}$. Suppose that $\Sigma_1 \times \Sigma_2$ is compatible. Let σ_{ij} and σ'_{kl} be defined in Σ_1 and Σ_2 respectively, as follows.

$$\sigma_{ij} = \begin{cases} \sigma(P_{(u_i,u_j)}) & \text{if } i \neq j, \\ 1 & \text{if } i = j \end{cases}$$

and

$$\sigma'_{kl} = \begin{cases} \sigma'(P_{(v_k,v_l)}) & \text{if } k \neq l, \\ 1 & \text{if } k = l. \end{cases}$$

Also, the shortest path between two vertices u_i and u_j in Σ_1 and v_k and v_l in Σ_2 are denoted by $d_{\Sigma_1}(i,j)$ and $d_{\Sigma_2}(k,l)$ respectively. Let $u = (u_i, u_j)$ and $v = (v_k, v_l)$ be two vertices in $\Sigma_1 \times \Sigma_2$. Then,

$$d_{\Sigma_1 \times \Sigma_2}(u,v) = \sigma_{ij}.\sigma'_{kl}(d(u_i,u_j) + d(v_k,v_l)) = \sigma'_{kl} d_{\Sigma_1}(i,j) + \sigma_{ij} d_{\Sigma_2}(k,l).$$

Then, the distance matrix of $\Sigma_1 \times \Sigma_2$ can be written in the form

$$D(\Sigma_1 \times \Sigma_2) = \begin{pmatrix} B_{1,1} & B_{1,2} & B_{1,3} & \ldots & B_{1,m} \\ B_{2,1} & B_{2,2} & B_{2,3} & \ldots & B_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B_{m,1} & B_{m,2} & B_{m,3} & \ldots & B_{m,m} \end{pmatrix}$$

where each block $B_{i,j}$ is given by,

$$B_{i,j} = \begin{pmatrix} \sigma'_{11} d_{\Sigma_1}(i,j) + \sigma_{ij} d_{\Sigma_2}(1,1) & \sigma'_{12} d_{\Sigma_1}(i,j) + \sigma_{ij} d_{\Sigma_2}(1,2) & \ldots & \sigma'_{1n} d_{\Sigma_1}(i,j) + \sigma_{ij} d_{\Sigma_2}(1,n) \\ \sigma'_{21} d_{\Sigma_1}(i,j) + \sigma_{ij} d_{\Sigma_2}(2,1) & \sigma'_{22} d_{\Sigma_1}(i,j) + \sigma_{ij} d_{\Sigma_2}(2,2) & \ldots & \sigma'_{2n} d_{\Sigma_1}(i,j) + \sigma_{ij} d_{\Sigma_2}(2,n) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \sigma'_{n1} d_{\Sigma_1}(i,j) + \sigma_{ij} d_{\Sigma_2}(n,1) & \sigma'_{n2} d_{\Sigma_1}(i,j) + \sigma_{ij} d_{\Sigma_2}(n,2) & \ldots & \sigma'_{nn} d_{\Sigma_1}(i,j) + \sigma_{ij} d_{\Sigma_2}(n,n) \end{pmatrix}.$$

$B_{i,j}$ can be split into two matrices as $B'_{i,j}$ and $B''_{i,j}$, given as

$$B'_{i,j} = \begin{pmatrix} \sigma'_{11} d_{\Sigma_1}(i,j) & \sigma'_{12} d_{\Sigma_1}(i,j) & \ldots & \sigma'_{1n} d_{\Sigma_1}(i,j) \\ \sigma'_{21} d_{\Sigma_1}(i,j) & \sigma'_{22} d_{\Sigma_1}(i,j) & \ldots & \sigma'_{2n} d_{\Sigma_1}(i,j) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \sigma'_{n1} d_{\Sigma_1}(i,j) & \sigma'_{n2} d_{\Sigma_1}(i,j) & \ldots & \sigma'_{nn} d_{\Sigma_1}(i,j), \end{pmatrix}$$

that is, $B'_{i,j} = d_{\Sigma_1}(i,j)(K^{D(\Sigma_2)} + I_n)$, and

$$B''_{i,j} = \begin{pmatrix} \sigma_{ij} d_{\Sigma_2}(1,1) & \sigma_{ij} d_{\Sigma_2}(1,2) & \dots & \sigma_{ij} d_{\Sigma_2}(1,n) \\ \sigma_{ij} d_{\Sigma_2}(2,1) & \sigma_{ij} d_{\Sigma_2}(2,2) & \dots & \sigma_{ij} d_{\Sigma_2}(2,n) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{ij} d_{\Sigma_2}(n,1) & \sigma_{ij} d_{\Sigma_2}(n,2) & \dots & \sigma_{ij} d_{\Sigma_2}(n,n), \end{pmatrix}$$

that is, $B''_{i,j} = \sigma_{ij}(D(\Sigma_2))$.

Then, $D(\Sigma_1 \times \Sigma_2) = \begin{pmatrix} d_{\Sigma_1}(1,1)(K^{D(\Sigma_2)} + I_n) & \dots & d_{\Sigma_1}(1,m)(K^{D(\Sigma_2)} + I_n) \\ d_{\Sigma_1}(2,1)(K^{D(\Sigma_2)} + I_n) & \dots & d_{\Sigma_1}(2,m)(K^{D(\Sigma_2)} + I_n) \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ d_{\Sigma_1}(m,1)(K^{D(\Sigma_2)} + I_n) & \dots & d_{\Sigma_1}(m,m)(K^{D(\Sigma_2)} + I_n) \end{pmatrix} +$

$$\begin{pmatrix} \sigma_{11} D(\Sigma_2) & \sigma_{12} D(\Sigma_2) & \dots & \sigma_{1n} D(\Sigma_2) \\ \sigma_{21} D(\Sigma_2) & \sigma_{22} D(\Sigma_2) & \dots & \sigma_{2n} D(\Sigma_2) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{n1} D(\Sigma_2) & \sigma_{n2} D(\Sigma_2) & \dots & \sigma_{nn} D(\Sigma_2) \end{pmatrix}.$$

Thus, $D(\Sigma_1 \times \Sigma_2) = D(\Sigma_1) \otimes (K^{D(\Sigma_2)} + I_n) + (K^{D(\Sigma_1)} + I_m) \otimes D(\Sigma_2)$. □

Theorem 4. *The distance matrix of the compatible lexicographic product of two signed graphs Σ_1 and Σ_2 is,*

$$D(\Sigma_1[\Sigma_2]) = D(\Sigma_1) \otimes J_n + I_m \otimes (2K^{D(\Sigma_2)} - A(\Sigma_2)).$$

Proof. Let Σ_1 and Σ_2 be two signed graphs with $|V(\Sigma_1)| = m$ and $|V(\Sigma_2)| = n$. Suppose that $\Sigma_1[\Sigma_2]$ is compatible. Then, the distance between two vertices $u = (u_i, v_k)$ and $v = (u_j, v_l)$ in $\Sigma_1[\Sigma_2]$ will be as follows.

$$d_{\Sigma_1[\Sigma_2]}(u,v) = \begin{cases} d_{\Sigma_1}(u_i,u_j) & \text{if } u_i \neq u_j, \\ 1\sigma(v_k v_l) & \text{if } u_i = u_j \text{ and } v_k \sim v_l, \\ 2\sigma(P_{(v_k,v_l)}) & \text{if } u_i = u_j \text{ and } v_k \nsim v_l. \end{cases}$$

The distance matrix of $\Sigma_1[\Sigma_2]$ can be written in the form

$$D(\Sigma_1[\Sigma_2]) = \begin{pmatrix} B_{1,1} & B_{1,2} & B_{1,3} & \dots & B_{1,m} \\ B_{2,1} & B_{2,2} & B_{2,3} & \dots & B_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B_{m,1} & B_{m,2} & B_{m,3} & \dots & B_{m,m} \end{pmatrix}$$

with,

$$B_{i,j} = \begin{pmatrix} d((u_i,v_1),(u_j,v_1)) & d((u_i,v_1),(u_j,v_2)) & \dots & d((u_i,v_1),(u_j,v_n)) \\ d((u_i,v_2),(u_j,v_1)) & d((u_i,v_2),(u_j,v_2)) & \dots & d((u_i,v_2),(u_j,v_n)) \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \\ d((u_i,v_n),(u_j,v_1)) & d((u_i,v_n),(u_j,v_2)) & \dots & d((u_i,v_n),(u_j,v_n)) \end{pmatrix}.$$

Whenever $u_i = u_j$,

$$d((u_i,v_k),(u_i,v_l)) = \begin{cases} 1\sigma(v_k v_l) & \text{if } v_k \sim v_l, \\ 2\sigma(P_{(v_k,v_l)}) & \text{if } v_k \nsim v_l, \\ 0 & \text{if } v_k = v_l. \end{cases}$$

That implies, when $u_i = u_j$, $B_{i,j}$ is nothing but $2K^{D(\Sigma_2)} - A(\Sigma_2)$. Thus, the diagonal blocks of D will be $2K^{D(\Sigma_2)} - A(\Sigma_2)$.
Also, whenever $u_i \neq u_j$, $d((u_i,v_k),(u_j,v_l)) = d_{\Sigma_1}(u_i,u_j)$. Then,

$$B_{i,j} = \begin{pmatrix} d_{\Sigma_1}(u_i,u_j) & d_{\Sigma_1}(u_i,u_j) & \dots & d_{\Sigma_1}(u_i,u_j) \\ d_{\Sigma_1}(u_i,u_j) & d_{\Sigma_1}(u_i,u_j) & \dots & d_{\Sigma_1}(u_i,u_j) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ d_{\Sigma_1}(u_i,u_j) & d_{\Sigma_1}(u_i,u_j) & \dots & d_{\Sigma_1}(u_i,u_j) \end{pmatrix}.$$

Thus, the distance matrix of the compatible lexicographic product $\Sigma_1[\Sigma_2]$ is $D(\Sigma_1[\Sigma_2]) = D(\Sigma_1) \otimes J_n + I_m \otimes (2K^{D(\Sigma_2)} - A(\Sigma_2))$. □

3. Distance spectra of some compatible signed graphs and their products

In this section, we discuss the distance spectra of signed Petersen graphs and some product of signed graphs.

Definition 1. Let Σ be a compatible signed graph and $D(\Sigma) = (d_{ij})_{n\times n}$ be the distance matrix of Σ, then the distance characteristic polynomial of Σ is defined as, $f(D(\Sigma),\lambda) = \det(\lambda I - D(\Sigma))$, where I is the identity matrix of order n.
The roots of the characteristic equation $f(D(\Sigma),\lambda) = 0$, denoted by $\lambda_1, \lambda_2, \dots, \lambda_n$ are called the distance eigenvalues of Σ. Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ be the distinct eigenvalues of $D(\Sigma)$ with multiplicities $m_1, m_2, \dots, m_k$, respectively, then the distance spectrum of Σ is denoted by $\begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_k \\ m_1 & m_2 & \dots & m_k \end{pmatrix}$.

The collection of all distance eigenvalues of Σ is denoted by $spec_D(\Sigma)$ and $spec_D(\Sigma) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$.

The net-degree of a vertex v in Σ is $d_\Sigma^{\pm}(v) = d_\Sigma^{+}(v) - d_\Sigma^{-}(v)$, where $d_\Sigma^{+}(v)$ is the number of positive edges and $d_\Sigma^{-}(v)$ is the number of negative edges incident with the vertex v. A signed graph Σ is said to be net-regular, if every vertex has constant net-degree. The Petersen graph $+P$ is net-regular with net-degree 3.

The distance-regular graphs with diameter 2 are very special and form a subject of their own. The well-known Petersen graph is a distance-regular graph with diameter 2. While studying the balance on signed Petersen graphs, T. Zaslavsky [7] proved that though there are 2^{15} ways to put signs on the edges of the Petersen graph P, in many respects only six of them are essentially different. He proved the following.

Theorem 5 ([7]). *There are precisely six isomorphism types of minimal signed Petersen graph: $+P, P_1, P_{2,2}, P_{3,2}, P_{2,3}$, and $P_{3,3}$. Each one is the unique minimal isomorphism type in its switching isomorphism class.*

Since, the shortest path between any two pair of vertices is unique (such graphs are called geodetic), signed Petersen graphs are always compatible. As Petersen graph is an important object in graph theory, we study the distance spectrum of these six signed Petersen graphs. Listed below are the distance characteristic polynomial of the six isomorphism types of minimal signed Petersen graphs.

1. $f(D(+P),\lambda) = \lambda^{10} - 135\lambda^8 - 1080\lambda^7 - 3645\lambda^6 - 5832\lambda^5 - 3645\lambda^4$.
2. $f(D(P_1),\lambda) = \lambda^{10} - 135\lambda^8 - 504\lambda^7 + 2851\lambda^6 + 15688\lambda^5 - 5229\lambda^4 - 122256\lambda^3 - 157680\lambda^2$.
3. $f(D(P_{2,2}),\lambda) = \lambda^{10} - 135\lambda^8 - 216\lambda^7 + 5587\lambda^6 + 13648\lambda^5 - 77957\lambda^4 - 220888\lambda^3 + 243912\lambda^2 + 645984\lambda - 308880$.
4. $f(D(P_{2,3}),\lambda) = \lambda^{10} - 135\lambda^8 - 184\lambda^7 + 6211\lambda^6 + 13720\lambda^5 - 111981\lambda^4 - 295840\lambda^3 + 690800\lambda^2 + 196800\lambda$.
5. $f(D(P_{3,2}),\lambda) = \lambda^{10} - 135\lambda^8 + 40\lambda^7 + 6675\lambda^6 - 4848\lambda^5 - 140725\lambda^4 + 195240\lambda^3 + 986040\lambda^2 - 2613600\lambda + 1724976$.
6. $f(D(P_{3,3}),\lambda) = \lambda^{10} - 135\lambda^8 - 120\lambda^7 + 6435\lambda^6 + 6696\lambda^5 - 145725\lambda^4 - 126000\lambda^3 + 1620000\lambda^2 + 800000\lambda - 7200000$.

The graph with integral spectrum is of special interest in literature, as such, it is noticed that among the six signed Petersen graphs, only the all-positive Petersen graph $+P$ and $P_{3,3} \simeq -P$ have integral distance spectrum. Also, the eigenspace of P and $-P$ corresponding to its distance eigenvalues are the same. The spectral values of these two signed Petersen graphs are discussed below.

The distance matrix of the Petersen graph $+P$ can be represented as $D(+P) = 2J_{10} - 2I_{10} - A(+P)$, where the adjacency spectrum of $+P$ is $\begin{pmatrix} 3 & 1 & -2 \\ 1 & 5 & 4 \end{pmatrix}$. Hence, the distance spectrum of $+P$ is $\begin{pmatrix} 15 & 0 & -3 \\ 1 & 4 & 5 \end{pmatrix}$.

Also, the distance matrix of the Petersen graph $-P$ can be represented as $D(-P) = 2J_{10} - 2I_{10} + 3A(-P)$, where the adjacency spectrum of $-P$ is $\begin{pmatrix} 2 & -1 & -3 \\ 4 & 5 & 1 \end{pmatrix}$. The distance spectrum of $-P$ is $\begin{pmatrix} 9 & 4 & -5 \\ 1 & 4 & 5 \end{pmatrix}$.

Now, we compute the distance spectrum of the Cartesian product of some classes of signed graphs. First we require a preliminary lemma which is given below.

Lemma 1 ([8]). *If A and B are square matrices of order m and n respectively, then $A \otimes B$ is a square matrix of order mn. Also, $(A \otimes B)(C \otimes D) = AC \otimes BD$, if the products AC and BD exists.*

Theorem 6. *Let $\Sigma_1 = (K_m, \sigma_1)$ and $\Sigma_2 = (K_n, \sigma_2)$ be two signed complete graphs. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$ be the distance eigenvalues of Σ_1 and Σ_2 respectively. Then,*

$$spec_D(\Sigma_1 \times \Sigma_2) = \{2\lambda_i\beta_j + \lambda_i + \beta_j;\ 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Proof. Let $\mathbf{P}_i$ and $\mathbf{Q}_j$ be the eigenvectors corresponding to the eigenvalues λ_i and β_j of $D(\Sigma_1)$ and $D(\Sigma_2)$, for $1 \leq i \leq m, 1 \leq j \leq n$. By Theorem 3, the distance matrix of $D(\Sigma_1 \times \Sigma_2)$ can be expressed as $D(\Sigma_1 \times \Sigma_2) = D(\Sigma_1) \otimes (K^{D(\Sigma_2)} + I_n) + (K^{D(\Sigma_1)} + I_m) \otimes D(\Sigma_2)$. Since, $\Sigma_1 = (K_m, \sigma_1)$ and $\Sigma_2 = (K_n, \sigma_2)$ are signed complete graphs, implies $D(\Sigma_1) = K^{D(\Sigma_1)}$ and $D(\Sigma_2) = K^{D(\Sigma_2)}$. Then,
$D(\Sigma_1 \times \Sigma_2)(\mathbf{P}_i \otimes \mathbf{Q_j}) = (D(\Sigma_1) \otimes (D(\Sigma_2) + I_n) + (D(\Sigma_1) + I_m) \otimes D(\Sigma_2))(\mathbf{P}_i \otimes \mathbf{Q_j})$
$= (2D(\Sigma_1) \otimes D(\Sigma_2) + D(\Sigma_1) \otimes I_n + I_m \otimes D(\Sigma_2))(\mathbf{P}_i \otimes \mathbf{Q_j})$
$= 2D(\Sigma_1)\mathbf{P}_i \otimes D(\Sigma_2)\mathbf{Q_j} + D(\Sigma_1)\mathbf{P}_i \otimes I_n\mathbf{Q_j} + I_m\mathbf{P}_i \otimes D(\Sigma_2)\mathbf{Q_j}$
$=2\lambda_i\mathbf{P}_i \otimes \beta_j\mathbf{Q_j} + \lambda_i\mathbf{P}_i \otimes \mathbf{Q_j} + \mathbf{P}_i \otimes \beta_j\mathbf{Q_j}$
$=(2\lambda_i\beta_j + \lambda_i + \beta_j)(\mathbf{P_i} \otimes \mathbf{Q_j})$
That is, $2\lambda_i\beta_j + \lambda_i + \beta_j$ are the eigenvalues of $D(\Sigma_1 \times \Sigma_2)$, for $1 \leq i \leq m, 1 \leq j \leq n$. □

Lemma 2 ([1]). *Let*

$$A = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$$

be a 2×2 block symmetric matrix. Then, the eigenvalues of A are those of $A_0 + A_1$ together with those of $A_0 - A_1$.

Theorem 7. *Let $\Sigma_1 = (G, \sigma)$ be an all negative signed graph of order n, where G is distance regular with $d(G) \leq 2$ and $\Sigma_2 = (K_2, \sigma')$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ be the distance eigenvalues of Σ_1 and $K^{D(\Sigma_1)}$ respectively. Then, the distance spectra of $\Sigma_1 \times \Sigma_2$ is*

$$spec_D(\Sigma_1 \times \Sigma_2) = \begin{pmatrix} -(\beta_i + 1) & 2\lambda_i + \beta_i + 1 \\ 1 & 1 \end{pmatrix}, i = 1, 2, \ldots, n.$$

Proof. Let $\Sigma_1 = (G,\sigma)$ be an all negative signed graph of order n. If $d(G) = 1$, then by Theorem 6 we done. Assume that $d(G) = 2$ and $\Sigma_2 = (K_2,\sigma')$ is negative. Then, the distance matrix of $\Sigma_1 \times \Sigma_2$ has the form,

$$D(\Sigma_1 \times \Sigma_2) = \begin{pmatrix} D(\Sigma_1) & -(D(\Sigma_1) + K^{D(\Sigma_1)} + I) \\ -(D(\Sigma_1) + K^{D(\Sigma_1)} + I) & D(\Sigma_1) \end{pmatrix}.$$

Consider the product $D(\Sigma_1)K^{D(\Sigma_1)}$, where $D(\Sigma_1) = (d_{ij})_{n\times n}$ and $K^{D(\Sigma_1)} = (\frac{d_{ij}}{|d_{ij}|})_{n\times n}$. Then, $D(\Sigma_1)K^{D(\Sigma_1)} = (a_{ij})_{n\times n}$, where $a_{ij} = d_{i1}\frac{d_{1j}}{|d_{1j}|} + d_{i2}\frac{d_{2j}}{|d_{2j}|} + \cdots + d_{in}\frac{d_{nj}}{|d_{nj}|}$. Since, $D(\Sigma_1)$ is symmetric and for all $i,j = 1,2,\ldots,n$, the $(i,j)^{th}$ entry of $D(\Sigma_1)$ and $K^{D(\Sigma_1)}$ are having the same sign, we get $d_{ik}\frac{d_{kj}}{|d_{kj}|}$ and $d_{kj}\frac{d_{ik}}{|d_{ik}|}, k = 1,2,\ldots,n$ will be of same sign.
Therefore, if $d_{ik}\frac{d_{kj}}{|d_{kj}|}$ is positive then, $d_{ik} = 2$ (or -1) and $d_{kj} = 2$ (or -1). If $d_{ik}\frac{d_{kj}}{|d_{kj}|}$ is negative then, $d_{ik} = 2$ (or -1) and $d_{kj} = -1$ (or 2). The same property holds for all $d_{ik}\frac{d_{kj}}{|d_{kj}|}, i,j,k = 1,2,\ldots,n$. Since, G is distance regular with $d(G) \leq 2$ and Σ_1 is an all negative signed graph, 2 and -1 are the only entries of $D(\Sigma_1)$. Thus, $a_{ij} = a_{ji}$ for $i,j = 1,2,\ldots,n$. That is, $D(\Sigma_1)K^{D(\Sigma_1)}$ is symmetric, which implies $D(\Sigma_1)K^{D(\Sigma_1)} = K^{D(\Sigma_1)}D(\Sigma_1)$. Now, by Lemma 2 we get the proof.
In a similar way, we can prove the case when $\Sigma_2 = (K_2,\sigma')$ is positive. In this case the distance matrix of $\Sigma_1 \times \Sigma_2$ has the form,

$$D(\Sigma_1 \times \Sigma_2) = \begin{pmatrix} D(\Sigma_1) & D(\Sigma_1) + K^{D(\Sigma_1)} + I \\ D(\Sigma_1) + K^{D(\Sigma_1)} + I & D(\Sigma_1) \end{pmatrix}.$$

□

We end our discussion with a special case of lexicographic product $\Sigma[K_2^{\pm}]$ and compute its distance eigenvalues.

Theorem 8. *Let $\Sigma_1 = (G,\sigma_1)$ be a compatible signed graph and $\Sigma_2 = (K_2,\sigma_2)$. If the distance eigenvalues of Σ_1 are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. Then, $D(\Sigma_1[\Sigma_2])$ has eigenvalues,*
(1) $2\lambda_i + 1$, for $1 \leq i \leq m$ (each of multiplicity one) and -1 (of multiplicity m), if K_2 is positive.
(2) $2\lambda_i - 1$, for $1 \leq i \leq m$ (each of multiplicity one) and 1 (of multiplicity m), if K_2 is negative.

Proof. Suppose that K_2 is positive. Then, the distance matrix of $\Sigma_1[\Sigma_2]$ has the form,

$$D(\Sigma_1[\Sigma_2]) = \begin{pmatrix} D(\Sigma_1) & D(\Sigma_1) + I \\ D(\Sigma_1) + I & D(\Sigma_1) \end{pmatrix}.$$

Then, by using Lemma 2 we get, the spectrum of $D(\Sigma_1[\Sigma_2])$ are those of $2D(\Sigma_1) + I$ together with those of $-I$.

If K_2 is negative, the distance matrix of $\Sigma_1[\Sigma_2]$ has the form,

$$D(\Sigma_1[\Sigma_2]) = \begin{pmatrix} D(\Sigma_1) & D(\Sigma_1) - I \\ D(\Sigma_1) - I & D(\Sigma_1) \end{pmatrix}.$$

Again by using Lemma 2, the spectrum of $D(\Sigma_1[\Sigma_2])$ are those of $2D(\Sigma_1) - I$ together with those of I. $\square$

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