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Research Article

Line signed graph of a signed unit graph of commutative rings

Pranjali

Department of Mathematics, University of Rajasthan, JLN Marg, Jaipur-302004, India
pranjali48@gmail.com

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Abstract: In this paper, we have characterized the commutative rings with unity for which line signed graph of a signed unit graph is balanced and consistent. To do this, we first establish some sufficient conditions for balance and consistency of line signed graph of signed unit graphs.

Keywords: Finite commutative rings; Signed graph; Unit graph

AMS Subject classification: 05C22, 05C75, 16P10

1. Introduction

For standard terminology and notation in abstract algebra and graph theory, not specifically mentioned or defined, we refer the reader to the standard textbooks [2] and [6], respectively. Throughout the paper all rings are assumed to be finite commutative with $1 \neq 0$. First, we recall some basic notions about the line graph and line signed graph of a graph, which will further be studied in the realm of a unit graph and signed unit graph. For a commutative ring R with unity, let $Z(R)$ and $Z^0(R)$ be the set of all zero-divisors and the set of all nonzero zero-divisors of R, respectively.
Let R be a commutative ring with $1 \neq 0$ and $U(R)$ be its set of units. According to Ashrafi et al. [1], the *unit graph* of R, denoted by $G(R)$, is a simple graph whose vertices are the elements of R, and two distinct vertices x and y are adjacent if and only if $x + y$ is a unit of R.
The proof of the following result can be found in [7].

Lemma 1. [7] *Let* $R = \mathbb{Z}_2^t \times S (t \geq 0)$, *where* $S \in \{\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_6, \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}\}$. *Then* $G(R) = 2^t G(S)$.

For a given graph, there are several derived graphs in the literature, among which the line graph is one of them. The line graph $L(G)$ of a graph G, is a graph defined by $V(L(G)) := E(G)$ and $\{e_1, e_2\} \in E(L(G))$ if e_1 and e_2 are incident to a common vertex in G. If $(u, v) \in E(G)$ we will denote the corresponding vertex of $L(G)$ by $[u, v]$.

Lemma 2. [6, Page 71] *Let G be an arbitrary graph and $L(G)$ be its line graph. Then for any vertex $[u, v] \in V[L(G)]$ the degree is given by*

$$\deg_{L(G)}[u, v] = \deg_G(u) + \deg_G(v) - 2$$

Turning to signed graph: "Given a graph Γ, the graph Γ equipped with a signature σ is called a *signed graph*, denoted by $\Sigma := (\Gamma, \sigma)$, where $\Gamma = (V, E)$ is an *underlying graph* and $\sigma : E \to \{+, -\}$ is the signature that labels each edge of Γ either by '+' or '−'. The edge which receives the positive (negative) sign is called a positive (negative) edge. A signed graph is said to be an *all-positive* (*all-negative*) if all of its edges are positive (negative); it is also called *homogeneous* if it is either an all-positive or an all-negative and *heterogeneous* otherwise. The *negative degree* $d^-(v)$ of a vertex v is the number of negative edges incident at v in Σ".

One of the fundamental concepts in the theory of signed graphs is *balance*. Harary [5] introduced the fascinating concept of *balanced signed graphs* for the analysis of social networks, in which a positive edge stands for a positive relation and a negative edge represents a negative relation. "A signed graph is *balanced* if every *cycle* has an even number of negative edges, and a signed graph that is not balanced is called an unbalanced signed graph. Note that if a signed graph is disconnected, then it is balanced if each of its component is balanced."

Consistent marked graphs were introduced by Beineke and Harary [4]. The idea was earlier motivated by communication networks and later studied on social networks.

Definition 1. "A marked signed graph is an ordered pair $\Sigma_\mu = (\Sigma, \mu)$, where $\Sigma = (\Gamma, \sigma)$ is a signed graph and $\mu : V(\Sigma) \to \{+, -\}$ is a function from the vertex set $V(\Sigma)$ into the set $\{+, -\}$, called *marking* of Σ. In particular; σ induces a unique marking μ_σ defined by

$$\mu_\sigma(v) = \prod_{e \in E_v} \sigma(e),$$

where E_v is the set of edges incident at v in Σ, is called the *canonical marking* of Σ. If every vertex of a given signed graph Σ is canonically marked, then a cycle Z in Σ is said to be *canonically consistent* ($\mathcal{C}$-consistent) if it contains an even number of negative vertices and the given signed graph Σ is said to be $\mathcal{C}$-consistent if every cycle in it is $\mathcal{C}$-consistent."

For a signed graph Σ, Behzad and Chartrand [3] defined its *line signed graph* $L(\Sigma)$ as the signed graph in which the edges of Σ are represented as vertices, two of these vertices are defined to be adjacent whenever the corresponding edges in Σ have a

vertex in common, any such edge ef is negative whenever both e and f are negative edges in Σ and positive otherwise.
Pranjali and Amit Kumar [8] initiated the study of *signed unit graph* of a commutative ring. The formal definition of the *signed unit graph* defined in [8] is as follows:

Definition 2. "A signed unit graph is an ordered pair $G_\Sigma(R) := (G(R), \sigma)$, where $G(R)$ is the unit graph of a commutative ring R and for an edge (a, b) of $G_\Sigma(R)$, σ is defined as"

$$\sigma(a,b) = \begin{cases} + \,, & \text{if } \; a \in U(R) \text{ or } b \in U(R); \\ - \,, & \text{otherwise.} \end{cases}$$

The goal of that study was to associate the concept of unit graph to signed graph. Moreover, they have established the necessary and sufficient conditions on commutative rings for which the signed unit graph $G_\Sigma(R)$ is balanced.
The following significant result brought from [8] will be useful later in our work:

Lemma 3. *Let R be a finite commutative ring $1 \neq 0$. The signed unit graph $G_\Sigma(R)$ is homogeneous (all-positive) if and only if R is a local ring.*

This work is intended to extend the idea of line graph of a unit graph into a line signed graph of a signed unit graph with key focus on the fundamental concepts of balance and consistency. Although, local rings are ubiquitous, but to avoid trivialities, non-local rings have become the central point of the work. To address this issue, we attempt to characterize finite commutative rings with unity for which $L(G_\Sigma(R)$ is balanced and $\mathcal{C}$-consistent. In the sequel, we established sufficient conditions that will work as a stepping stone to derive the main results.

2. Rings for which $L_\Sigma(G_\Sigma(R))$ is balanced

This section is devoted to determining the commutative rings for which $L_\Sigma(G_\Sigma(R))$ of given $G_\Sigma(R)$ is balanced. More emphasis is placed on the non-local rings as for local rings $G_\Sigma(R)$ is an all-positive, and thus many things turn out trivially.

Theorem 1. *Let R be a local ring, and $L_\Sigma(G_\Sigma(R))$ be line signed graph of signed unit graph $G_\Sigma(R)$. Then $L_\Sigma(G_\Sigma(R))$ is balanced.*

Proof. Let R be a local ring, and $G_\Sigma(R)$ be its signed unit graph. Then, by Lemma 3 $G_\Sigma(R)$ is an all-positive, and hence $L_\Sigma(G_\Sigma(R))$ is an all-positive signed graph. Therefore, $L_\Sigma(G_\Sigma(R))$ is trivially balanced. □

Here, it is obvious to ask the following problem:

Problem 1. Characterize non-local rings for which $L_\Sigma(G_\Sigma(R))$ is balanced.

Towards attempting the above problem, we shall start with small ordered line signed graphs of a signed unit graph associated with R.

Theorem 2. *If R is a finite commutative ring with unity such that $|R| < 6$, then $L_\Sigma(G_\Sigma(R))$ of $G_\Sigma(R)$ is balanced.*

Proof. If $|R| = 2$, then $R \cong \mathbb{Z}_2$ and $G_\Sigma(R)$ is an all-positive and therefore $L_\Sigma(G_\Sigma(R))$ is also all-positive, so balanced trivially. Next, if $|R| = 3$, or $|R| = 5$, then $R \cong \mathbb{Z}_3$ or $R \cong \mathbb{Z}_5$, respectively. Since both the listed rings are local, $L_\Sigma(G_\Sigma(R))$ is balanced.
If $|R| = 4$, then R is either isomorphic to $\mathbb{Z}_4$ or $\mathbb{F}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Note that first three rings are local therefore, $L_\Sigma(G_\Sigma(R))$ is balanced and for the remaining rings $L_\Sigma(G_\Sigma(R))$ is a null graph, and hence balanced. □

It is evident from the above result that the smallest order of a non-local ring whose line signed graph of signed unit graph is not balanced is 6, which is apparent from the graph shown in Figure 1. Besides this, Figure 1 also helps us to visualize the possibility that the signed unit graph may be balanced, although its line signed graph is not necessarily balanced.

Example 1. If $R \cong \mathbb{Z}_6$, then $G(R) \cong C_6$. In this case, both $G_\Sigma(R)$ and $L_\Sigma(G_\Sigma(R))$ are shown in Figure 1.

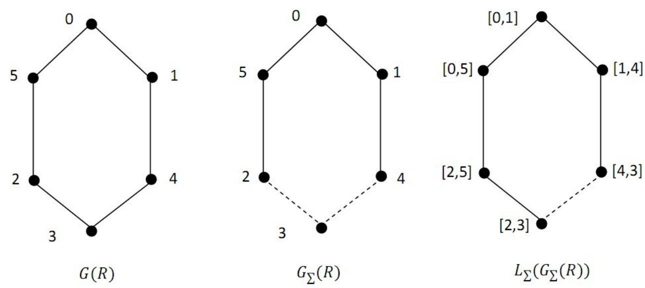

Figure 1. **The graph $G(\mathbb{Z}_6)$, $G_\Sigma(\mathbb{Z}_6)$ and $L_\Sigma(G_\Sigma(\mathbb{Z}_6))$**

One can see that $G_\Sigma(R)$ is balanced, while $L_\Sigma(G_\Sigma(R))$ is not balanced, due to presence of precisely one negative edge in the cycle.

Theorem 3. *Let R be a finite commutative ring with unity such that $|Z^0(R)| \leq 2$. Then $L_\Sigma(G_\Sigma(R))$ is balanced.*

Proof. Let R be a ring with non-zero identity such that $|Z^0(R)| \leq 2$. If $|Z^0(R)| = 1$, then R is either isomorphic to $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$. Since both the rings are local, so $L_\Sigma(G_\Sigma(R))$ is balanced due to Theorem 1.

If $|Z^0(R)| = 2$, then R is $\mathbb{Z}_9$ or $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since the first two rings are local, by Theorem 1, $L_\Sigma(G_\Sigma(R))$ is balanced. Next, if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then in light of Lemma 1, $G(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is isomorphic to 2-copies of K_2. Thus $L_\Sigma(G_\Sigma(R))$ has two isolated vertices, and hence balanced trivially. □

Theorem 4. *Let R be the direct product of local rings, among which at least one has three or more units. Then $L_\Sigma(G_\Sigma(R))$ is not balanced.*

Proof. Let R be a finite commutative ring with unity. In view of [2, page 95], R can be written as $R_1 \times R_2 \times R_3 \times \cdots \times R_t$ $(t > 1)$, where each R_i is a local ring. If at least for one i, $|U(R_i)| \geq 3$, then there exist a vertex v_1 in $G_\Sigma(R)$ of negative degree greater than equal to three and that would create an all negative triangle in $L_\Sigma(G_\Sigma(R))$ namely; $[v_1, v_2] - [v_1, v_3] - [v_1, v_4] - [v_1, v_2]$, where $v_1 = (1, 0, 1, 1, \ldots, 1)$, $v_2 = (0, u_1, 0, 0, \ldots, 0)$, $v_3 = (0, u_2, 0, 0, \ldots, 0)$ and $v_4 = (0, u_3, 0, 0, \ldots, 0)$, where $u_i \in U(R_i)$. This shows that $L_\Sigma(G_\Sigma(R))$ is not balanced. □

From the above discussion, it may be noted that the necessary condition on a finite ring R for which $L_\Sigma(G_\Sigma(R))$ of given $G_\Sigma(R)$ is balanced is that each ring in the direct product has at most two units.

Proposition 1. Let R be a finite commutative ring with unity such that $|U(R)| \leq 2$. Then R is isomorphic to $\mathbb{Z}_2^{t-1} \times S$ $(t > 0)$, where $S \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{Z}_4$ or $\mathbb{Z}_6$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$.

Proof. Let R be a finite commutative ring with unity such that $|U(R)| \leq 2$. Since every finite commutative ring with unity can be written as $R \cong R_1 \times R_2 \times \cdots \times R_t$, where each R_i's a local ring. Without loss of generality, let us assume that $|R_1| \leq |R_2| \leq |R_3| \leq \cdots \leq |R_t|$. We know that the number of units of $R \cong R_1 \times R_2 \times \cdots \times R_t$ is given by $U(R) = U(R_1) \times U(R_2) \times \cdots \times U(R_t)$ and $|U(R)| = |U(R_1)| \times |U(R_2)| \times \cdots \times |U(R_t)|$. Now, if $|U(R)| = 1$, then $|U(R_i)| = 1 \ \forall\ i, (1 \leq i \leq t)$. Therefore, for each i, $R_i \cong \mathbb{Z}_2$, and hence $R \cong \mathbb{Z}_2^t$ $(t > 0)$.
Let $|U(R)| = 2$, then exactly one of R_i must have two units and remaining R_i must have one unit. Without loss of generality, suppose that $|U(R_i)| = 1 \ \forall\ i$ $(1 \leq i \leq t-1)$ and $|U(R_t)| = 2$. Now it is known that the commutative rings with unity having exactly two units is isomorphic to one of the following listed rings $\mathbb{Z}_3$ or $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, or $\mathbb{Z}_6$. Except $\mathbb{Z}_6$ all the listed rings are local, so R_t must be isomorphic to either $\mathbb{Z}_3$ or $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$ and $R_i \cong \mathbb{Z}_2 \ \forall\ i$ $(1 \leq i \leq t-1)$. Therefore $R \cong \mathbb{Z}_2^{t-1} \times S$ $(t > 0)$, where $S \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$. □

Theorem 5. *Let R be a finite commutative ring with unity, and let $L_\Sigma(G_\Sigma(R))$ be line signed graph of a signed unit graph $G_\Sigma(R)$. Then $L_\Sigma(G_\Sigma(R))$ is balanced if and only if either R is a local ring or R is isomorphic to $\mathbb{Z}_2^{t-1} \times S$, $(t \geq 1)$, where S is one of the following rings $\mathbb{Z}_2$ or $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$.*

Proof. Let R be a finite commutative ring with non-zero identity and let $L_\Sigma(G_\Sigma(R))$ be the line signed graph of a signed unit graph $G_\Sigma(R)$. From ([2], p. 95), R must be of the form $R \cong R_1 \times R_2 \times R_3 \times \cdots \times R_t$, $(t \geq 1)$, where each $R_i (1 \leq i \leq t)$ is a local ring.

Necessity: Let us assume that $L_\Sigma(G_\Sigma(R))$ is balanced and we wish to prove that R is isomorphic to $\mathbb{Z}_2^{t-1} \times S$, where S is one of the rings $\mathbb{Z}_2$ or $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$. We shall prove the result by contrapositive. To do this, suppose R is neither local nor $R \cong \mathbb{Z}_2^t \times S$, where S is one of the following rings $\mathbb{Z}_2$ or $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$. Then one should necessarily have $t > 1$ and R must have $|U(R)| \geq 2$. First, if $|U(R)| = 2$, then by Proposition 1 along with assumption R must be isomorphic to $\mathbb{Z}_2^{t-1} \times \mathbb{Z}_3$ $(t > 1)$. But for this R, $L_\Sigma(G_\Sigma(R))$ is a disjoint union of C_6 in which one cycle has exactly one negative edge, which indicates that $L_\Sigma(G_\Sigma(R))$ is not balanced.

Next, if $|U(R)| > 2$, then for choice of unit of R_i, we have the following two possibilities: (i) when at least one R_i have $|U(R_i)| \geq 3$; (ii) when at least two R_i have $|U(R_i)| = 2$.

For possibility (i) when at least one R_i have $|U(R_i)| \geq 3$, $L_\Sigma(G_\Sigma(R))$ is not balanced, due to Theorem 4. (ii) If at least two R_i's have $|U(R_i)| = 2$, then R have at least four units, i.e., $|U(R)| \geq 4$. Also we know that $G(R)$ is $|U(R)|$-regular. Therefore, there exist a vertex v in $G_\Sigma(R)$ having $d^-(v) = 2$ and $d^+(v) \geq 2$ as R_i's has two units. Clearly, corresponding to these edges incident on v in $G_\Sigma(R)$ there exist a complete graph of order k $(k \geq 4)$ in $L_\Sigma(G_\Sigma(R))$ with exactly one negative edge, and this indicates that $L_\Sigma(G_\Sigma(R))$ is not balanced. From the above analysis it is found that in each of the above possibilities $L_\Sigma(G_\Sigma(R))$ is not balanced. Thus by contrapositive the result follows.

Sufficiency: First let R be a local ring then due to Theorem 1, $L_\Sigma(G_\Sigma(R))$ is balanced. Next let $R \cong \mathbb{Z}_2^{t-1} \times S$,$(t \geq 1)$ where S is one of the following rings $\mathbb{Z}_2$ or $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$. For $t = 1$, R is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$ and all these rings are local, and hence by Theorem 1, $L_\Sigma(G_\Sigma(R))$ is balanced.

On the other hand if $t > 1$, then for $S \cong \mathbb{Z}_2$, we have $G(\mathbb{Z}_2^{t-1} \times \mathbb{Z}_2) \cong \cup_{i=1}^{2^{(t-1)}} K_2$, due to Lemma 1 and hence $L_\Sigma(G_\Sigma(R))$ is a null graph. Next, for $S \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, we have $G(\mathbb{Z}_2^{t-1} \times S) \cong$ $2^{(t-1)}$-copies of C_4, in which $G_\Sigma(R)$ consists of exactly one homogenous all positive C_4 and remaining components homogenous all-negative C_4. Note that in this case $L_\Sigma(G_\Sigma(R)) \cong G_\Sigma(R)$, this yields, $L_\Sigma(G_\Sigma(R))$ is balanced. □

Remark 1. It is worthwhile to note here that if $L_\Sigma(G_\Sigma(R))$ is balanced, then $|U(R)| \leq 2$, but converse is not true. As for instance if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, then $|U(R)| = 2$ and there is a vertex $(1, 0)$ in $G_\Sigma(R)$ having negative degree 3, and which create an all-negative triangle in $L_\Sigma(G_\Sigma(R))$. This shows that $L_\Sigma(G_\Sigma(R))$ is not balanced.

In the following theorem, we characterize the rings R, for which both $G_\Sigma(R)$ and $L_\Sigma(G_\Sigma(R))$ are balanced:

Theorem 6. *Let R be a finite commutative ring with unity and $G_\Sigma(R)$ be its singed unit graph. Let $L_\Sigma(G_\Sigma(R))$ be a line signed unit graph of $G_\Sigma(R)$. Then $G_\Sigma(R)$ and $L_\Sigma(G_\Sigma(R))$ are both balanced if and only if R is either a local ring or R is isomorphic to $\mathbb{Z}_2^t \times S$, where $S \cong \mathbb{Z}_2$ or $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$.*

Proof. Invoking Theorem 5 and [8, Theorem 2.3], the proof follows. □

3. Rings for which $L_\Sigma(G_\Sigma(R))$ is Consistent

This section begins with the result depicting the $\mathcal{C}$-consistency of $L_\Sigma(G_\Sigma(R))$ for local ring. Next, to realize the impact of non-local rings on the $\mathcal{C}$-consistency of $L_\Sigma(G_\Sigma(R))$ we shall be doing almost everything for non-local rings. Further, we establish several sufficient conditions which work as a strong foundation for the main result.

Theorem 7. *Let R be a local ring and let $L_\Sigma(G_\Sigma(R))$ be a line signed graph of signed unit graph $G_\Sigma(R)$. Then $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent.*

Proof. Let R be a local ring. Then in light of Lemma 3, $G_\Sigma(R)$ is an all-positive, and hence corresponding $L_\Sigma(G_\Sigma(R))$ is also all-positive. Therefore, $L_\Sigma(G_\Sigma(R))$ is trivially $\mathcal{C}$-consistent. □

Theorem 8. *Let $R \cong \mathbb{Z}_2^t \times S$, $t \geq 1$, where S is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$. Then $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent.*

Proof. Let $R \cong \mathbb{Z}_2^t \times S$, $t \geq 1$. If S is taken to be $\mathbb{Z}_2$, then in light of Lemma 1, $G(\mathbb{Z}_2^t \times \mathbb{Z}_2) \cong \cup_{i=1}^{2^t} K_2$. Clearly, by Definition 2 along with the definition of line signed graph, we acquire the desired conclusion, i.e., $L_\Sigma(G_\Sigma(R))$ is trivially $\mathcal{C}$-consistent.
Next, if S is taken to be $\mathbb{Z}_3$, then in the similar vein, $G(\mathbb{Z}_2^t \times \mathbb{Z}_3)$ is isomorphic to 2^{t-1}-copies of C_6, in which $G_\Sigma(R)$ consists of two negative edges in one cycle and remaining other all negative components, which would create exactly one negative edge in one cycle and remaining other all negative components, respectively in $L_\Sigma(G_\Sigma(R))$. Therefore under the C-marking, two vertices in one component receive '$-$'ve sign and all six vertices in the other components receive '$+$'ve sign in $L_\Sigma(G_\Sigma(R))$, and hence it is $\mathcal{C}$-consistent.
Finally, if S is either $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, then $G(\mathbb{Z}_2^t \times S) \cong\ 2^t-$copies of C_4, in which $G_\Sigma(R)$ consists of precisely one homogenous all positive C_4 and remaining other homogenous all-negative C_4. Note that $L_\Sigma(G_\Sigma(R)) \cong G_\Sigma(R)$, this indicates that under the C-marking all the vertices in $L_\Sigma(G_\Sigma(R))$ receive '$+$'ive sign. It follows that $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent. □

Remark 2. Note that if a non-local ring R has $\mathbb{Z}_2$ as a quotient, then $L_\Sigma(G_\Sigma(R))$ need not be $\mathcal{C}$-consistent, as for instance, if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_7$, then R has $\mathbb{Z}_2$ as a quotient, but $L_\Sigma(G_\Sigma(R))$ is not $\mathcal{C}$-consistent.

To examine the impact of reduced rings on the $\mathcal{C}$-consistency of $L_\Sigma(G_\Sigma(R))$, we have established the following result:

Theorem 9. *For $t > 1$, let $R \cong \mathbb{Z}_2^{t-1} \times \mathbb{Z}_3$, or $\prod_{i=1}^{t} \mathbb{F}_i$, where either each $\mathbb{F}_i$'s is a field of characteristic 2 or each $\mathbb{F}_i$'s is a field of characteristic p_i $(p_i > 2)$, respectively. Then $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent.*

Proof. If $R \cong \mathbb{Z}_2^{t-1} \times \mathbb{Z}_3$ or $\mathbb{Z}_2^{t-1} \times \mathbb{Z}_2$, then the proof follows from Theorem 8. Next, if $R \cong \prod_{i=1}^{t} \mathbb{F}_i$, where either each $\mathbb{F}_i$'s is a field of characteristic 2, or each $\mathbb{F}_i$'s is a field of characteristic p_i $(p_i > 2)$, respectively, then to examine $\mathcal{C}$-consistency in $L_\Sigma(G_\Sigma(R))$ it is enough to calculate the negative degree of each vertex in $G_\Sigma(R)$. To do this, let us assume that $v = (v_1, v_2, \ldots, v_t)$ be an arbitrary vertex of $G_\Sigma(R)$. If $v \in U(R)$ or $v = \bar{0}$, then $d^-(v) = 0$. This yields $d^-(v) > 0$ only when $v_i \neq 0$ at least for one i $(1 \leq i \leq t)$ in v. Also one can verify that $d^-(v) = |U(R)| - L$, where $L = \prod_j |U(F_j)| \cdot \prod_i (|U(F_i)| - 1)$ in $G_\Sigma(R)$.
On the other hand, when each $\mathbb{F}_i$'s is a field of characteristic 2, then the value of 'L' turns out to be even. Therefore $d^-(v)$ is odd. Since v is arbitrary, $d^-(u)$ is also odd for $u \in V(G_\Sigma(R))$. Hence the negative degree of a vertex [u, v] in $L_\Sigma(G_\Sigma(R))$ is given by

$$d^-[u, v] = d^-(u) + d^-(v) - 2$$

Thus the negative degree of each vertex in $L_\Sigma(G_\Sigma(R))$ is even, and hence under a canonical marking, each vertex of $L_\Sigma(G_\Sigma(R))$ receive a positive sign.
Next, if each $\mathbb{F}_i$'s is a field of characteristic p_i $(p_i > 2)$, respectively, then the value of 'L' turns out to be even. Therefore, $d^-(v)$ is even. Since v is arbitrary, $d^-(u)$ is also even for $u \in V(G_\Sigma(R))$. Thus by following the similar procedure as done above it is found that the negative degree of a vertex $[u, v]$ in $L_\Sigma(G_\Sigma(R))$ is even, and hence under a canonical marking each vertex of $L_\Sigma(G_\Sigma(R))$ receive a positive sign. Thus the result follows. □

In the following result we establish the sufficient condition to determine the rings R for which $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent by imposing the condition on the set of nonzero zero-divisors of R.

Theorem 10. *Let R be a ring with unity such that $|Z^0(R)| \leq 4$. Then $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent.*

Proof. Let R be a ring with $1 \neq 0$ such that $|Z^0(R)| \leq 4$. To show the desired result, we shall tackle each case separately.
If $|Z^0(R)| = 1$, then the possible rings are $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$. Since both the rings are local, $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent due to Theorem 7. Next, if $|Z^0(R)| = 2$, then upto isomorphism such rings are $\mathbb{Z}_9$ or $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Now it is easy to see that the first two listed rings are local, so by Theorem 7, $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent and for the remaining one $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent due to Theorem 8. If $|Z^0(R)| = 3$, then

in this case there are eight commutative rings upto isomorphism, namely, $\mathbb{Z}_6$ or $\mathbb{Z}_8$ or $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$ or $\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2-2\rangle}$ or $\frac{\mathbb{Z}_2[x,y]}{\langle x,y\rangle^2}$ or $\frac{\mathbb{Z}_4[x]}{\langle 2,x\rangle^2}$ or $\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$ or $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$. Note that except $\mathbb{Z}_6$ all other rings are local, thus $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent for these rings. For $\mathbb{Z}_6$ we have $L_\Sigma(G_\Sigma(\mathbb{Z}_6))$ is isomorphic to C_6 with one negative edge and under a $\mathcal{C}$-marking exactly two vertices will be assigned with '$(-)$'ive sign. Hence $L_\Sigma(G_\Sigma(\mathbb{Z}_6))$ is $\mathcal{C}$-consistent. Finally if $|Z^0(R)| = 4$, then the possible non-isomorphic commutative rings are $\mathbb{Z}_2 \times \mathbb{F}_4$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_{25}$ or $\frac{\mathbb{Z}_5[x]}{\langle x^2\rangle}$. For the first two rings $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent due to Theorem 9 and the rest two listed rings are local, therefore $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent for each of the listed rings. □

The following remark may help to visualize the $\mathcal{C}$-consistency of $L_\Sigma(G_\Sigma(R))$ when a ring has 'k' nonzero zero-divisors up to 14 of a particular kind.

Remark 3. If R is finite commutative ring with $1 \neq 0$ such that $|Z^0(R)| = k$, where $k \in \{2, 4, 6, 8, 10, 12, 14\}$. Then $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent.

Proof. Let R be finite commutative ring with $1 \neq 0$ such that $|Z^0(R)| = k$, where $k \in \{2, 4, 6, 8, 10, 12, 14\}$. In order to show the result we shall make use of Theorem 7 and Theorem 9. For each k we shall tackle the case separately. The cases for $k = 2$ and $k = 4$ have been covered in Theorem 10. Now for $k = 6$, there are exactly five nonzero commutative rings, each having exactly 6 nonzero zero-divisors, viz., $\mathbb{Z}_{49}$, $\frac{\mathbb{Z}_7[x]}{\langle x^2\rangle}$, $\mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{F}_4 \times \mathbb{F}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The first two rings are local, so in view of Theorem 7, $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent. Now for the remaining rings $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent, due to Theorem 9.
Next, for $k = 8$, there are exactly ten nonzero commutative rings, each having exactly eight nonzero zero divisors, viz., $\mathbb{Z}_{27}$, $\frac{\mathbb{Z}_3[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_3[x,y]}{\langle x,y\rangle^2}$, $\frac{\mathbb{Z}_9[x]}{\langle 3x,x^2\rangle}$, $\frac{\mathbb{Z}_9[x]}{\langle 3x,x^2-3\epsilon\rangle}$, where $\epsilon \in \sum_2$, $\frac{\mathbb{F}_9[x]}{\langle x^2\rangle}$, $GR(81, 9)$, $\mathbb{Z}_5 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{F}_8$ and $\mathbb{Z}_3 \times \mathbb{Z}_7$. Among the listed ten rings, first seven ring are local and rest of them are of the form as given in Theorem 9. Therefore, invoking Theorem 7 and Theorem 9, $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent.
Next for $k = 10$, there are exactly five nonzero commutative rings, each having exactly 10 nonzero zero-divisors, viz., $\mathbb{Z}_{121}$, $\frac{\mathbb{Z}_{11}[x]}{\langle x^2\rangle}$, $\mathbb{Z}_5 \times \mathbb{Z}_7$, $\mathbb{Z}_3 \times \mathbb{F}_9$, and $\mathbb{F}_4 \times \mathbb{F}_8$. All the mentioned rings are either local or of the form as appeared in Theorem 9. So by the same arguments as given above, $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent.
Next for $k = 12$, there are exactly six nonzero commutative rings, each having exactly 12 nonzero zero-divisors, viz., $\mathbb{Z}_{169}$, $\frac{\mathbb{Z}_{13}[x]}{\langle x^2\rangle}$, $\mathbb{Z}_7 \times \mathbb{Z}_7$, $\mathbb{Z}_3 \times \mathbb{Z}_{11}$, $\mathbb{Z}_5 \times \mathbb{F}_9$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$. All the listed rings are either local or appeared in Theorem 9. So by giving the similar arguments as given above, $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent.
Next for $k = 14$, there are 7 rings with 14 nonzero zero-divisors, viz., $\mathbb{Z}_{39}$, $\mathbb{Z}_{55}$, $\mathbb{F}_8 \times \mathbb{F}_8$, $\mathbb{Z}_7 \times \mathbb{F}_9$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_9$, and $\mathbb{Z}_3 \times \frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$. Here for the first five rings $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent, due to Theorem 9 and for the remaining two rings $\mathbb{Z}_3 \times \mathbb{Z}_9$, and $\mathbb{Z}_3 \times \frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$, their signed unit graphs are isomorphic and hence their line signed graphs are also isomorphic. Now in order to check the $\mathcal{C}$-consistency, we shall calculate the negative degree of each vertex in $L_\Sigma(G_\Sigma(\mathbb{Z}_3 \times \mathbb{Z}_9))$. Let $[v_1, v_2] \in V(L_\Sigma(G_\Sigma(\mathbb{Z}_3 \times \mathbb{Z}_9)))$, where $v_1, v_2 \in V(G_\Sigma(\mathbb{Z}_3 \times \mathbb{Z}_9))$. In view of the Definition 2, it

is apparent that if either $v_1 \in U(\mathbb{Z}_3 \times \mathbb{Z}_9)$ or $v_1 \in M(\mathbb{Z}_3 \times \mathbb{Z}_9)$, then $d^-(v_1) = 0$. It may easily be seen that the number of negative edges incident on v_1 in $G_\Sigma(R)$ is 6, when neither v_1 nor v_2 belongs to any of $U(\mathbb{Z}_3 \times \mathbb{Z}_9)$ or $M(\mathbb{Z}_3 \times \mathbb{Z}_9)$. Thus $d^-(v_1) = 6$ and $d^-(v_2) = 6$ and hence the negative degree of vertex $[v_1, v_2]$ in $L_\Sigma(G_\Sigma(\mathbb{Z}_3 \times \mathbb{Z}_9))$ is given by $6 + 6 - 2 = 10$, which is even. Therefore, under the $\mathcal{C}$-marking each vertex of $L_\Sigma(G_\Sigma(R))$ receive the positive sign. Hence for each ring R having $|Z^0(R)| \leq k$, where $k \in \{2, 4, 6, 8, 10, 12, 14\}$, $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent. □

By imposing the condition on cardinality of rings we have the following result:

Theorem 11. *Let R be finite commutative ring with $1 \neq 0$ such that $|R| < 10$. Then $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent.*

Proof. Let R be a finite commutative ring with $1 \neq 0$ such that $|R| < 10$. If $|R| = 2$ or 3 or 5 or 7, then in each case R is precisely local. Therefore, by Theorem 7, $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent for each R.
Next, if $|R| = 4$, then all non-isomorphic rings of order 4 are, viz., $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/\langle x^2 \rangle$ or $\mathbb{F}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Among the listed rings, the first three rings are local, so $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent, due to Theorem 7. For the remaining ring $\mathbb{Z}_2 \times \mathbb{Z}_2$, $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent, due to absence of a cycle.
Now if $|R| = 6$, then the only ring is $\mathbb{Z}_6$ for which $G(R)$ is isomorphic to C_6 and $L_\Sigma(G_\Sigma(\mathbb{Z}_6))$ is a cycle C_6 consisting of one negative edges. Now, under the canonical marking only two vertices in $L_\Sigma(G_\Sigma(\mathbb{Z}_6))$ receive the negative sign. Therefore, $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent.
Next, let $|R| = 8$. Then there are 'ten' non-isomorphic rings that are listed as follows $\mathbb{Z}_8$ or $\mathbb{F}_8$ or $\mathbb{Z}_2[x]/\langle x^3 \rangle$ or $\frac{\mathbb{Z}_2[x,y]}{\langle x,y \rangle^2}$ or $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2 \rangle}$ or $\frac{\mathbb{Z}_4[x]}{\langle 2,x \rangle^2}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{F}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$. Among the listed rings the first six rings are local, so by Theorem 7 that $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent. For the rings $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{F}_4$, $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent due to Theorem 9. Now for the remaining rings we found that $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent due to Theorem 8.
Finally, let $|R| = 9$. Then R is isomorphic to one of the rings viz., $\mathbb{Z}_9$ or $\mathbb{Z}_3[x]/\langle x^2 \rangle$ or $\mathbb{F}_9$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$. Note that the first three listed rings are local, and the remaining one is reduced, therefore $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent due to Theorem 7 and Theorem 9 respectively. □

Remark 4. It is apparent from Theorem 11 that the smallest order of a non-local ring R for which $L_\Sigma(G_\Sigma(R))$ is not $\mathcal{C}$-consistent is 10. Interestingly, however, one can easily verify that the only commutative ring with $1 \neq 0$ of order 10 is $\mathbb{Z}_{10}$, for which $L_\Sigma(G_\Sigma(R))$ is not $\mathcal{C}$-consistent.

We now turn our attention towards proving the sufficient conditions in Theorem 12 and Theorem 13, which are essential to achieve the main result(Theorem 14).

Theorem 12. *Let R be a finite commutative ring with $1 \neq 0$. Suppose that $R \cong \prod_{i=1}^{t} R_i$ ($t > 1$), where each R_i's is a local ring with maximal ideal M_i, respectively. If $|M_i|$ is even for some i ($1 \leq i \leq t$), then $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent.*

Proof. Let R be a finite commutative ring such that $R \cong \prod_{i=1}^{t} R_i$ ($t > 1$), where each R_i's is a local ring with maximal ideal M_i, respectively. To examine the $\mathcal{C}$-consistency of $L_\Sigma(G_\Sigma(R))$, first, we shall compute the negative degree of each vertex in $G_\Sigma(R)$). Now let us assume that $|M_i|$ is even for some i and let $u = (u_1, u_2, \ldots, u_t)$ and $v = (v_1, v_2, \ldots, v_t)$ be two arbitrary elements of R, i.e., $u, v \in V(G_\Sigma(R))$. It is easy to check that the presence of negative edges on u(or v) in $G_\Sigma(R)$ is seen only when some $u_i \in M_i$ and $u_j \in U(R_j)$. Now we shall tackle each of these cases:

Case 1 Let $u_1 \in M_1$. Then the edge (u, v) is negative if and only if $u + v \in U(R)$. Since $u_1 \in M_1$, it gives $v_1 \in U(R_1)$ and $v_i \in M_i$ for some i ($2 \leq i \leq t$). Thus in $G_\Sigma(R)$, the number of negative edges incident on u is a multiple of $|U(R_1)|$, which is even.

Case 2 Let $u_1 \in U(R_1)$. Now suppose that the edge (u, v) is negative, then there are two possibilities for v_i ($2 \leq i \leq t$): (i) $v_i \in M_i$ and (ii) $v_i \in U(R_i)$.

(i) If $v_i \in M_i$ for some i, then the number of such v's are multiple of $|U(R_1)|$.

(ii) If $v_i \in U(R_i)$ for all i, then the number of such v's are multiple of $|M_1|$.

Thus in both the possibilities, the negative degree of u is even. From the cases as mentioned earlier, it is evident that $d^-(u)$ is even in $G_\Sigma(R)$ and u is arbitrary in each case. Therefore the negative degree of a vertex $[u, v] \in V(L_\Sigma(G_\Sigma(R)))$, associated with $u, v \in V(G_\Sigma(R))$ is given by

$$d^-[u, v] = d^-(u) + d^-(v) - 2$$

which is even. This shows that the negative degree of each vertex in $L_\Sigma(G_\Sigma(R))$ is even and hence under $\mathcal{C}$-marking each vertex will receive '+'ive sign. Thus the result follows. □

Theorem 13. *Let R be a finite commutative ring with $1 \neq 0$. Suppose that $R \cong \prod_{i=1}^{t} R_i$ ($t > 1$), where each R_i's is a local ring with maximal ideal M_i, respectively. If $|U(R_i)|$ is even for all i ($1 \leq i \leq t$), then $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent.*

Proof. Let R be a finite commutative ring with $1 \neq 0$ of the form $\prod_{i=1}^{t} R_i$ ($t > 1$), where each R_i's is a local ring with maximal ideal M_i, respectively. Let $u = (u_1, u_2, \ldots, u_t)$ and $v = (v_1, v_2, \ldots, v_t)$ be two arbitrary elements of $V(G_\Sigma(R))$ and if they are adjacent, then corresponding to these vertices, the vertex $[u, v] \in V(L_\Sigma(G_\Sigma(R)))$. In view of the Definition 2, it is apparent that if either $u_i \in U(R_i)$ or $u_i \in M_i$ for all i, then $d^-(u) = 0$. Thus, a negative edge on u occurs only when some $u_i \in U(R_i)$ and some $u_j \in M_j$. It may easily be seen that the number of negative edges incident on u in $G_\Sigma(R)$ is given by $|U(R)| - (\prod_i(|U(R_i| - |M_i|) \cdot \prod_j(|U(R_j)|))$ which is even. Now we proceed similar to that as we did in the previous theorem to compute the negative degree of a vertex $[u,v] \in V(L_\Sigma(G_\Sigma(R)))$ and found that $d^-[u,v]$ is even. Since $[u,v]$ is arbitrary, it follows that through the $\mathcal{C}$-marking each vertex of $L_\Sigma(G_\Sigma(R))$ receive '+'ive sign. Hence $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent. □

Remark 5. It is bring here to note that the sufficient conditions furnished in Theorem 12 and Theorem 13 are not mutually exclusive, as for instance; if $R \cong \mathbb{Z}_4 \times \mathbb{Z}_8$, then R satisfies the conditions of each of the above mentioned theorem.

The following result characterizes rings for which $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent.

Theorem 14. *Let R be a finite commutative ring with $1 \neq 0$. Suppose that $R \cong \prod_{i=1}^{t} R_i$ $(t \geq 1)$, where each R_i is a local ring with maximal ideal M_i, respectively. Then $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent if and only if any one of the following conditions hold:*

(i) *R is local;*

(ii) *$|U(R_i)|$ is even $\forall$ i;*

(iii) *$R \cong \mathbb{Z}_2^{t-1} \times \mathbb{Z}_3$, or $\prod_{i=1}^{t} \mathbb{F}_i$, where each $\mathbb{F}_i$'s is field of characteristic 2;*

(iv) *$|M_i|$ is even for some i.*

Proof. **Necessity:** Let R be a finite commutative ring with non-zero identity and by ([2], p. 95) R be expressed as $R_1 \times R_2 \times \cdots \times R_t$ $(t \geq 1)$ where each R_i is a local ring with maximal ideal M_i, respectively. Suppose $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent and we claim that R satisfies one of the conditions $(i) - (iv)$.
We shall prove the result by contrapositive. For this, let us assume that R does not satisfy any of the above conditions, and then we are tempted to show that $L_\Sigma(G_\Sigma(R))$ is not $\mathcal{C}$-consistent. Under this assumption, t must be strictly greater than 1. Since each R_i is local, $|R_i| = p_i^{k_i}$, where each p_i's is prime and $(1 \leq i \leq t)$. To violate condition (ii), there must be some R_i's which are field of characteristic 2. Therefore, for given positive integers 't' and 'j', the precise form of R is $\mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_j \times R_{j+1} \times R_{j+2} \times \cdots \times R_t$ $(t \geq 1)$, where each $\mathbb{F}_i$'s $(1 \leq i \leq j)$ is field of characteristic 2. Note that each R_i's $(j+1 \leq i \leq t)$ is local and to disobey the condition (iii) and condition (iv), $j < t$ and $|M_i|$ must be odd for all i, respectively.
Now to demonstrate the study on $j < t$, we shall deal with the following two cases: (i) $j+1 = t$ and (ii) $j+1 < t$, in the realm of following two possibilities: $a)$ when at least one of the $\mathbb{F}_i$'s is isomorphic to $\mathbb{Z}_2$; $b)$ when none of the $\mathbb{F}_i$'s is isomorphic to $\mathbb{Z}_2$.
Case (i): Let $j+1 = t$;
Then $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_j \times R_{j+1}$, where each $\mathbb{F}_i$'s is field of characteristic 2 and R_{j+1} is local, but not field of characteristic 2.
$i(a)$; Let us assume that at least one of $\mathbb{F}_i$'s is isomorphic to $\mathbb{Z}_2$, but $R \ncong \mathbb{Z}_2^{t-1} \times \mathbb{Z}_3$. Consider three distinct elements $a_{j+1}, b_{j+1}, c_{j+1} \in U(R_{j+1})$. Then, one may have a 3 cycle in $L_\Sigma(G_\Sigma(R))$, viz., $[\overline{z}_1, \overline{z}_2] - [\overline{z}_1, \overline{z}_3] - [\overline{z}_1, \overline{z}_4] - [\overline{z}_1, \overline{z}_2]$,

where $\overline{z}_1 = (\underbrace{1,1,1,\ldots,1}_{j-times},0)$, $\overline{z}_2 = (\underbrace{0,0,\ldots,0}_{j-times},a_{j+1})$, $\overline{z}_3 = (\underbrace{0,0,\ldots,0}_{j-times},b_{j+1})$, and $\overline{z}_4 = (\underbrace{0,0,\ldots,0}_{j-times},c_{j+1})$, corresponding to a vertex of degree at least 3 in $G_\Sigma(R)$. Now we shall calculate negative degree of each vertex in $G_\Sigma(R)$. Making use of combinatorics, we found that $d^-(\overline{z}_1)$ is even and $d^-(\overline{z}_2)$, $d^-(\overline{z}_3)$, $d^-(\overline{z}_4)$ are all odd. In view of Lemma 2, we found that $d^-[\overline{z}_1,\overline{z}_2]$, $d^-[\overline{z}_1,\overline{z}_3]$, $d^-[\overline{z}_1,\overline{z}_4]$ are all odd, which ensures that through a canonical marking all 3 vertices receive '$-$' sign, and hence the above mentioned three cycle in $L_\Sigma(G_\Sigma(R))$ is not $\mathcal{C}$-consistent. Therefore, $L_\Sigma(G_\Sigma(R))$ is not $\mathcal{C}$-consistent.

$i(b)$ Next, assume that none of the $\mathbb{F}_i$'s is isomorphic to $\mathbb{Z}_2$. Consider $a_i,b_i,c_i \in U(\mathbb{F}_i)$ $(1 \le i \le j)$. Then, one may choose a 3 cycle in $L_\Sigma(G_\Sigma(R))$, corresponding to a vertex of degree at least 3 in $G_\Sigma(R)$, viz., $[\overline{z}_1,\overline{z}_2]-[\overline{z}_1,\overline{z}_3]-[\overline{z}_1,\overline{z}_4]-[\overline{z}_1,\overline{z}_2]$, where $\overline{z}_1 = (\underbrace{0,0,0,\ldots,0}_{j-times},1)$, $\overline{z}_2 = (\underbrace{a_1,a_2,\ldots,a_j}_{j-times},0)$, $\overline{z}_3 = (\underbrace{b_1,b_2,\ldots,b_j}_{j-times},0)$, and $\overline{z}_4 = (\underbrace{c_1,c_2,\ldots,c_j}_{j-times},0)$ and $\overline{z}_2$, $\overline{z}_3$, $\overline{z}_4$ are all distinct. Now we shall proceed to compute negative degree of each vertex for the verification of $\mathcal{C}$-consistency in $L_\Sigma(G_\Sigma(R))$. Making use of combinatorics, one can see that $d^-(\overline{z}_1)$ is odd and $d^-(\overline{z}_2)$, $d^-(\overline{z}_3)$, $d^-(\overline{z}_4)$ are all even. Proceeding in the similar vein, we found that $d^-[\overline{z}_1,\overline{z}_2]$, $d^-[\overline{z}_1,\overline{z}_3]$, $d^-[\overline{z}_1,\overline{z}_4]$ are all odd, which assure that through the canonical marking all 3 vertices receive '$(-)$*ive*' sign, and hence the above mentioned three cycle is not $\mathcal{C}$-consistent. This shows that $L_\Sigma(G_\Sigma(R))$ is not $\mathcal{C}$-consistent.

Case (ii): Let $j+1<t$;

Then, $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_j \times R_{j+1} \times R_{j+2} \times \cdots \times R_t$ $(t>2)$, where each $\mathbb{F}_i$'s is field of characteristic 2 and each R_k's $(j+1 \le k \le t)$ is local, but not field of characteristic 2.

$(ii)a)$ Let us assume that some of $\mathbb{F}_i$'s is isomorphic to $\mathbb{Z}_2$. Consider $a_k,b_k \in U(R_k)$,$\forall k$ $(j+1 \le k \le t)$. Then, one may have a 3 cycle in $L_\Sigma(G_\Sigma(R))$, viz., $[\overline{z}_1,\overline{z}_2]-[\overline{z}_1,\overline{z}_3]-[\overline{z}_1,\overline{z}_4]-[\overline{z}_1,\overline{z}_2]$, where $\overline{z}_1 = (\underbrace{1,1,1,\ldots,1}_{j-times},\underbrace{0,0,\ldots,0}_{(t-j)-times})$,

$\overline{z}_2 = (\underbrace{0,0,\ldots,0}_{j-times},\underbrace{a_{j+1},a_{j+2},\ldots,a_t}_{(t-j)-times})$, $\overline{z}_3 = (\underbrace{0,0,\ldots,0}_{j-times},\underbrace{b_{j+1},b_{j+2},\ldots,b_t}_{(t-j)-times})$, and

$\overline{z}_4 = (\underbrace{0,0,\ldots,0}_{j-times},\underbrace{a_{j+1},a_{j+2},\ldots,b_t}_{(t-j)-times})$, which occur corresponding to a vertex of degree at least 3 in $G_\Sigma(R)$. Note that the units of each R_k's in $\overline{z}_4$ is chosen in such a way that $\overline{z}_2$, $\overline{z}_3$, and $\overline{z}_4$ are all remain distinct. Now we shall calculate negative degree of each vertex for the verification of $\mathcal{C}$-consistency in $L_\Sigma(G_\Sigma(R))$. Now using the arguments, analogues to those used in the previous cases, we conclude that $d^-[\overline{z}_1,\overline{z}_2]$, $d^-[\overline{z}_1,\overline{z}_3]$, $d^-[\overline{z}_1,\overline{z}_4]$ are all odd, which ensures that through the canonical marking all 3 vertices receive '$(-)$'ive sign, and hence the above mentioned three cycle is not $\mathcal{C}$-consistent. Therefore, $L_\Sigma(G_\Sigma(R))$ is not $\mathcal{C}$-consistent.

$(ii)b)$; Assume that none of the $\mathbb{F}_i$'s is isomorphic to $\mathbb{Z}_2$. Then, $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times$

$\mathbb{F}_t \times R_{j+1} \times R_{j+2} \times \cdots \times R_t$ $(t \geq 1)$, where each $\mathbb{F}_i$'s is field of characteristic 2. Consider $a_i, b_i, c_i \in U(\mathbb{F}_i)$ $(1 \leq i \leq j)$. Then, one may choose a 3 cycle in $L_\Sigma(G_\Sigma(R))$, viz., $[\overline{z}_1, \overline{z}_2] - [\overline{z}_1, \overline{z}_3] - [\overline{z}_1, \overline{z}_4] - [\overline{z}_1, \overline{z}_2]$, corresponding to a vertex of degree at least 3 in $G_\Sigma(R)$, where $\overline{z}_1 = (\underbrace{0, 0, 0, \ldots, 0}_{j-times}, \underbrace{1, 1, \ldots, 1}_{(t-j)-times})$, $\overline{z}_2 = (\underbrace{a_1, a_2, \ldots, a_j}_{j-times}, \underbrace{0, 0, 0, \ldots, 0}_{(t-j)-times})$, $\overline{z}_3 = (\underbrace{b_1, b_2, \ldots, b_j}_{j-times}, \underbrace{0, 0, 0, \ldots, 0}_{(t-j)-times})$, and $\overline{z}_4 = (\underbrace{c_1, c_2, \ldots, c_j}_{j-times}, \underbrace{0, 0, 0, \ldots, 0}_{(t-j)-times})$ and $\overline{z}_2$, $\overline{z}_3$, $\overline{z}_4$ are all distinct. Making use of combinatorics, one can see that $d^-(\overline{z}_1)$ is odd and $d^-(\overline{z}_2)$, $d^-(\overline{z}_3)$, $d^-(\overline{z}_4)$ are all even, and therefore $d^-[\overline{z}_1, \overline{z}_2]$, $d^-[\overline{z}_1, \overline{z}_3]$, $d^-[\overline{z}_1, \overline{z}_4]$ are all odd, which ensures that under the canonical marking all 3 vertices will be marked with '$(-)ive$'sign. It indicates that the above mentioned three cycle is not $\mathcal{C}$-consistent. Therefore, $L_\Sigma(G_\Sigma(R))$ is not $\mathcal{C}$-consistent. Thus by contrapositive the necessity holds .

Sufficiency: For the sufficiency, let R satisfies one of the above conditions, then we wish to prove that $L_\Sigma(G_\Sigma(R))$ is $\mathcal{C}$-consistent. The proof is immediate from Theorems 7, 9, 12 and 13. □

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