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Research Article

Chromatic transversal Roman domination in graphs

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Abstract: For a graph G with chromatic number k, a dominating set S of G is called a chromatic-transversal dominating set (ctd-set) if S intersects every color class of any k-coloring of G. The minimum cardinality of a ctd-set of G is called the *chromatic transversal domination number* of G and is denoted by $\gamma_{ct}(G)$. A *Roman dominating function* (RDF) in a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $w(f) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function of a graph G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$. The concept of *chromatic transversal domination* is extended to Roman domination as follows: For a graph G with chromatic number k, a *Roman dominating function* f is called a *chromatic-transversal Roman dominating function* (CTRDF) if the set of all vertices v with $f(v) > 0$ intersects every color class of any k-coloring of G. The minimum weight of a chromatic-transversal Roman dominating function of a graph G is called the *chromatic-transversal Roman domination number* of G and is denoted by $\gamma_{ctR}(G)$. In this paper a study of this parameter is initiated.

Keywords: Domination, Coloring, Chromatic transversal Roman domination

AMS Subject classification: 05C69

1. Introduction

By a graph $G = (V, E)$ we mean a finite, connected, undirected and simple graph. The order of G is denoted by n. For graph theoretic terminology we in general follow [3].

One of the fastest growing areas within graph theory is the study of domination and related problems. A comprehensive treatment of fundamentals of domination is given in the book of Haynes et al. [12]. Surveys of several advanced topics in domination can be seen in the book edited by Haynes et al. [11]. Another area of research which has received much attention within graph theory is graph colorings which deals with the fundamental problem of partitioning a set of objects into classes according to certain conditions. Benedict Michael et al. [20] combined these two concepts to obtain a new variant of domination called the *chromatic transversal domination*. One more variant which combines domination and graph colorings known as dominator coloring is also well studied in literature [1, 7, 10, 18, 19].

A set $S \subseteq V$ is called a dominating set of G if every vertex in $V - S$ is adjacent to a vertex in S. The minimum cardinality of a dominating set in G is called the *domination number* of G and is dentoed by $\gamma(G)$. The *chromatic number* of a graph G is the minimum number of colors needed to color the vertices of G so that no two adjacent vertices share the same color and is denoted by $\chi(G)$. As defined by Benedict Michael et al. [20], for a graph G with chromatic number k, a dominating set S of G is called a chromatic-transversal dominating set (ctd-set) if S intersects every color class of any k-coloring of G. The minimum cardinality of a ctd-set of G is called the *chromatic transversal domination number* of G and is denoted by $\gamma_{ct}(G)$. E.J. Cockayne et al. [8] intoduced the concept of Roman domination. A *Roman dominating function* (RDF) in a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $w(f) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function of a graph G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$. An RDF of weight $\gamma_R(G)$ is called a γ_R-function of G or $\gamma_R(G)$-function. If V_0, V_1, V_2 are the sets of vertices assigned the values 0, 1 and 2 respectively under f, then there is a 1-1 correspondence between the function $f : V(G) \to \{0, 1, 2\}$ and the sets V_0, V_1, V_2 of $V(G)$. Thus f can be written as $f = (V_0, V_1, V_2)$. For a detailed study in Roman domination, one can refer to [2, 4–6, 8, 9, 13–17, 21–27]. The concept of *chromatic-transversal domination* is extended to Roman domination as follows: For a graph G with chromatic number k, a *Roman dominating function* f is called a *chromatic-transversal Roman dominating function* (CTRDF) if the set of all vertices v with $f(v) > 0$ intersects every color class of any k-coloring of G. The minimum weight of a chromatic-transversal Roman dominating function of a graph G is called the *chromatic-transversal Roman domination number* of G and is denoted by $\gamma_{ctR}(G)$. A CTRDF of weight $\gamma_{ctR}(G)$ is called a γ_{ctR}-function of G or a $\gamma_{ctR}(G)$-function. In this paper a study of this parameter is initiated.

2. Notation

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by n. A subgraph of G is a graph having all its vertices and edges in G. For any set $S \subseteq V$, the induced subgraph $G[S]$ is the maximal subgraph of G with respect to S. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *diameter* of a graph G is the maximum distance between the pair of vertices in G. The *degree* of a vertex v in a graph G is the number of edges that are incident to the vertex v and is denoted by $deg(v)$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G)$ and $\Delta(G)$. A vertex of degree zero is called an *isolated* vertex, while a vertex of degree one is called a *leaf* vertex or a *pendant* vertex of G. An edge incident to a leaf is called a *pendant edge*. A *strong support* is a vertex that is adjacent to at least two leaf vertices. A set S of vertices is called *independent* if no two vertices in S are adjacent. A simple graph in which every pair of distinct vertices are adjacent is called a *complete graph*. A *clique* of a simple graph G is a subset S of V such that $G[S]$ is complete. The *clique number* of a graph G, denoted by $\omega(G)$ is the number of vertices in a maximum clique of G. For $n \geq 4$, the *wheel* W_n is defined to be the graph obtained by connecting a single vertex to all the vertices of C_{n-1}, where C_{n-1} is a cycle on $n-1$ vertices and is called the *rim* of the wheel. For two positive integers r, s, the *complete bipartite* graph $K_{r,s}$ is the graph with partition $V(G) = X \cup Y$ such that $|X| = r$, $|Y| = s$, X and Y are independent and every two vertices belonging to different partite sets are adjacent to each other. A complete bipartite graph of the form $K_{1,n}$ is called a star graph. A connected graph without any cycle is called a tree and if G has exactly one cycle, then G is called a *unicyclic graph*. The *corona* of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2.

3. Some Standard Graphs

In this section γ_{ctR} values for paths, cycles and complete bipartite graphs are determined. To begin with we state the following theorem proved in [8].

Theorem 1. *[8] For the classes of paths P_n and cycles C_n, $\gamma_r(P_n) = \gamma_r(C_n) = \lceil \frac{2n}{3} \rceil$.*

Theorem 2. *For paths P_n,*

$$\gamma_{ctR}(P_n) = \begin{cases} n & \text{if } n \leq 4 \\ \lceil \frac{2n}{3} \rceil & \text{if } n \geq 5. \end{cases}$$

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. It is clear that $\chi(P_n) = 2$ and $\gamma_{ctR}(P_n) \geq \gamma_R(P_n)$. When $n = 2$, choose a γ_R-function of P_2 which assigns 1 to both the vertices of P_2. Then clearly $\gamma_{ctR}(P_2) = 2$. When $n = 3$, there is a unique γ_R-function of P_3 which assigns 2 to the central vertex and 0 to the end vertices. Thus $\gamma_{ctR}(P_3) = 3$. When $n = 4$, any γ_R-function of P_4 will assign either 2 to v_2, 1 to v_4 and 0 elsewhere or 2 to v_3, 1 to v_1 and 0 elsewhere. In both the cases either $\{v_1, v_3\}$ or $\{v_2, v_4\}$ form a color class of any χ-coloring of P_4. Hence $\gamma_{ctR}(P_4) = 4$. For $n \geq 5$, let f be a

γ_R-function of P_n defined as

$$f(v_i) = \begin{cases} 2, & i = 3j-1, 1 \le j \le \lfloor \frac{n+1}{3} \rfloor \\ 1, & i = n \text{ and } n \equiv 1 \pmod 3 \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $\{v_2, v_5\}$ intersects both the color classes of any χ-coloring of P_n. Hence $\gamma_{ctR}(P_n) \le \lceil \frac{2n}{3} \rceil$. Thus $\gamma_{ctR}(P_n) = \lceil \frac{2n}{3} \rceil$. □

Corollary 1. *For paths P_n, $\gamma_{ctR}(P_n) = \gamma_R(P_n)$ if and only if $n \neq 3, 4$.*

A similar proof can be given for cycles C_n. Hence the following theorem is stated without proof.

Theorem 3. *For cycles C_n,*

$$\gamma_{ctR}(C_n) = \begin{cases} n & \text{if } n = 4 \text{ and } n \text{ is odd} \\ \lceil \frac{2n}{3} \rceil & \text{otherwise.} \end{cases}$$

Corollary 2. *For cycles C_n, $\gamma_{ctR}(C_n) = \gamma_R(C_n)$ if and only if $n \neq 3, 4, 5$.*

Theorem 4. *For wheels $G = W_n$,*

$$\gamma_{ctR}(W_n) = \begin{cases} n & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. When n is even, $\chi(G) = 4$. Hence for every $v \in V(G)$. $\{v\}$ is a color class of a χ-partition of G. Thus $\gamma_{ctR}(G) = n$. When n is odd, $\chi(G) = 3$. Let $f : V(G) \to \{0, 1, 2\}$ be a function defined by $f(w) = 2$, $f(x) = f(y) = 1$, $f(z) = 0$ for every $z \in V(G) \backslash \{x, y, w\}$, where w is the central vertex and x, y are two adjacent vertices on the rim of the wheel. Clearly $\{w, x, y\}$ intersects every color class of any χ-coloring of G. Hence $\gamma_{ctR}(G) \le 4$. Further since $\chi(G) = 3$, $|V_2 \cup V_1| \ge 3$. But $|V_1| = 3$ is not possible. Thus $|V_2| = 1$ and $|V_1| = 2$ which implies that $\gamma_{ctR}(G) \ge 4$. Hence $\gamma_{ctR}(G) = 4$. (Refer Figure 1). □

4. Bipartite Graphs

In the following theorem we prove that for any bipartite graph G, $\gamma_{ctR}(G)$ lies between $\gamma_R(G)$ and $\gamma_R(G) + 1$.

Theorem 5. *For bipartite graphs G,*

$$\gamma_R(G) \le \gamma_{ctR}(G) \le \gamma_R(G) + 1.$$

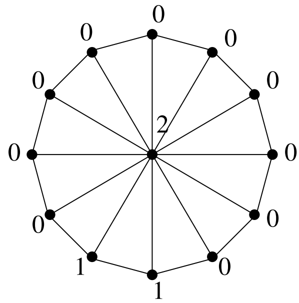

Figure 1. **The wheel W_{13} with $\gamma_{ctR}(W_{13}) = 4$**

Proof. Let (X, Y) be the bipartition of $V(G)$. Clearly $\chi(G) = 2$. If for every γ_R-function $f = (V_0, V_1, V_2)$ of G, the distance between any 2 vertices of $V_1 \cup V_2$ is even, then $V_1 \cup V_2$ is either X or Y. Thus, either X or Y is a color class of a χ-partition which does not intersect $V_1 \cup V_2$ in which case $\gamma_{ctR}(G) > \gamma_R(G)$. Now define $g : V(G) \rightarrow \{0, 1, 2\}$ by $g(x) = 1$ for some $x \in V_0$ and $g(x) = f(x)$ otherwise. Then g is a γ_{ctR}-function of G. Thus $\gamma_{ctR}(G) = \gamma_R(G) + 1$.
If for some γ_R-function of G say $f = (V_0, V_1, V_2)$, there is a pair of vertices $x, y \in V_1 \cup V_2$ such that $d(x, y)$ is odd, then $V_1 \cup V_2$ intersects both the color classes X and Y. Hence $\gamma_{ctR}(G) = \gamma_R(G)$. Thus $\gamma_R(G) \leq \gamma_{ctR}(G) \leq \gamma_R(G) + 1$. □

Corollary 3. *For a bipartite graph G, $\gamma_{ctR}(G) = \gamma_R(G)$ if and only if there exists a γ_R-function $f = (V_0, V_1, V_2)$ of G such that there are at least 2 vertices u, v in $V_1 \cup V_2$ with $d(u, v)$ as an odd number.*

Theorem 6. *For complete bipartite graphs $G = K_{r,s}$, $r \leq s$, $s \geq 2$*

$$\gamma_{ctR}(G) = \begin{cases} 3 & \text{if } r = 1 \\ 4 & \text{otherwise.} \end{cases}$$

Proof. Let (X, Y) be the bipartition of G with $|X| = r$, $|Y| = s$. If $r = 1$, then $G = K_{1,s}$ and clearly $\gamma_{ctR}(G) = 3$. If $r = 2$, $\gamma_R(G) = 3$ and $V_2 \cup V_1 = X$, where $f = (V_0, V_1, V_2)$ is a γ_R-function of G. Such an assignment is unique. But $V_2 \cup V_1$ does not intersect Y which forms a color class of any χ-coloring of G. Thus $\gamma_{ctR}(G) \geq 4$. Now by assigining 2 to a vertex in X and a vertex in Y, it is evident that $\gamma_{ctR}(G) \leq 4$. Thus $\gamma_{ctR}(G) = 4$. When $r \geq 3$, it is clear that $\gamma_{ctR}(G) = \gamma_R(G) = 4$. □

Corollary 4. *For complete bipartite graphs $G = K_{r,s}$, $r \leq s$, $s \geq 2$, $\gamma_{ctR}(G) = \gamma_R(G)$ if and only if $r \neq 1, 2$.*

5. Split Graphs

A graph G is said to be a *split graph* if $V(G)$ can be partitioned into two sets X and Y such that X induces a complete graph and Y is independent. In this section we determine $\gamma_{ctR}(G)$, where G is a split graph. For this purpose we consider $k \le |X|$ vertices in X as follows: Let $G = G_1$ and $v_1 \in X$ such that $deg_{G_1}(v_1) = \Delta(G_1)$. Remove all the neighbors of v_1 in Y. Let G_2 be the resulting graph and $v_2 \in X$ such that $deg_{G_2}(v_2) = \Delta(G_2)$. Remove the neighbors of v_2 in Y. Repeat the process until all the vertices in Y are removed. Let $v_1, v_2, \ldots, v_k$ be the vertices in X whose neighbors in Y were removed successively. Then k is called the *split number* of G.
In all the results that follow in this section, a split graph G means a graph G with partition (X, Y) where X induces a complete graph and Y is independent.

Theorem 7. *For a split graph G, $\gamma_{ctR}(G) = |X| + k$, where k is the split number of G.*

Proof. Since every vertex in Y is not adjacent to at least one vertex in X; $\chi(G) = |X|$ and $\gamma_{ctR}(G) > |X|$. Let k be the split number of G and let $v_1, v_2, \ldots, v_k$ be the corresponding vertices in X as described above. Now any γ_{ctR}-function of G will assign a total weight of 2 to each $N[v_i]$, $1 \le i \le k$ and 1 to the vertices in $X - \{v_1, v_2, \ldots, v_k\}$. Hence $\gamma_{ctR}(G) \ge |X| - k + 2k \ge |X| + k$.
Now define $f : V(G) \to \{0, 1, 2\}$ by

$$f(v) = \begin{cases} 2 & \text{if } v = v_i, 1 \le i \le k \\ 1 & \text{if } v \in X \setminus \{v_1, v_2, \ldots, v_k\} \\ 0 & \text{if } v \in Y. \end{cases}$$

Then clearly f is a CTRDF of G as X intersects every color class of any χ-coloring of G. Hence $\gamma_{ctR}(G) \le |X| + k$. Thus, $\gamma_{ctR}(G) = |X| + k$. □

Corollary 5. *For a split graph G, $\gamma_{ctR}(G) = \gamma_R(G)$ if and only if every vertex in X is a strong support.*

Corollary 6. *For a split graph G, $\gamma_{ctR}(G) = n$ if and only if every vertex in X is of degree at most $|X|$.*

6. Realization

Theorem 8. *Given two positive integers a, b with $2 \le a \le b$, there exists a graph G such that $\gamma_{ctR}(G) = b$ and $\gamma_R(G) = a$.*

Proof. If $a = b = 2$, then for the graph K_2, $\gamma_{ctR}(K_2) = \gamma_R(K_2) = 2$. Hence, we assume that $3 \le a \le b$. Consider the graph $H \circ 2K_1$ where H is a tree and

take a copy of K_{b-a+2}. If $a < b$, a is even, then join a vertex of K_{b-a+2} to a vertex of H in $H \circ 2K_1$, where $|V(H)| = \frac{a-2}{2}$. For the resulting graph G, clearly $\gamma_{ctR}(G) = b - a + 2 + 2\left(\frac{a-2}{2}\right) = b$ and $\gamma_R(G) = 2 + 2\left(\frac{a-2}{2}\right) = a$.
If $a \leq b$, a is odd, then join a vertex of K_{b-a+2} to a vertex of H in $H \circ 2K_1$, where $|V(H)| = \frac{a-3}{2}$ and in turn join a K_2 to one of the vertices of H. For the resulting graph G, $\gamma_{ctR}(G) = b - a + 2 + 2\left(\frac{a-3}{2}\right) + 1 = b$ and $\gamma_R(G) = 2 + 2\left(\frac{a-3}{2}\right) + 1 = a$.
If $a = b$ and a is even, then consider G to be the graph $H \circ 2K_1$ where $|V(H)| = \frac{a}{2}$. Then $\gamma_{ctR}(G) = 2 \times \frac{a}{2} = a$ and $\gamma_R(G) = a$. Hence, the theorem holds. □

7. Bounds

For K_2, $\gamma_{ctR}(K_2) = 2$ and $\gamma_{ctR}(K_1) = 1$. Thus one can easily observe that for $n \geq 3$, $3 \leq \gamma_{ctR}(G) \leq n$.

Theorem 9. *For any graph G, $\gamma_{ctR}(G) = 3$ if and only if G is either a K_3 or a star.*

Proof. Suppose $\gamma_{ctR}(G) = 3$. Then there exists a γ_{ctR}-function $f = (V_0, V_1, V_2)$ of G such that either $|V_1| = 3$, $|V_2| = 0$ or, $|V_1| = 1$ and $|V_2| = 1$. In the first case, clearly $G = K_3$. In the latter case, $\chi(G) \leq 2$. Since G is connected, G is bipartite. Thus the vertex in V_2 say w is adjacent to every vertex in $V(G)$. Hence G is a star. □

Next we prove that, for any tree T, $\gamma_{ctR}(T)$ is bounded above by $\frac{4n}{5}$ and characterize those trees which attain this bound. For this purpose we state the following theorems proved in [2].

Theorem 10. *[2] If T is an n-vertex tree with $n \geq 3$, then $\gamma_R(T) \leq \frac{4n}{5}$.*

Theorem 11. *[2] If T is an n-vertex tree, then $\gamma_R(T) = \frac{4n}{5}$ if and only if $V(T)$ can be partitioned into sets inducing P_5 such that the subgraph induced by the central vertices of these paths are connected.*

Theorem 12. *For any tree T with $n \geq 5$, $\gamma_{ctR}(T) \leq \frac{4n}{5}$ and equality holds if and only if either $T = T_1$ (as given in Figure 2) or $V(T)$ can be partitioned into sets inducing P_5 such that the subgraph induced by the central vertices of these paths are connected.*

Proof. Since T is a tree, $\gamma_R(T) \leq \gamma_{ctR}(T) \leq \gamma_R(T) + 1$. If $\gamma_R(T) < \frac{4n}{5}$, then $\gamma_{ctR}(T) < \frac{4n}{5} + 1$. Thus $\gamma_{ctR}(T) \leq \frac{4n}{5}$. If $\gamma_R(T) = \frac{4n}{5}$, then by Theorem 11, T is as described in the statement of the theorem. If $T = P_5$, then $\gamma_{ctR}(T) = 4$. Otherwise, define $f : \{0, 1, 2\} \to \gamma(T)$ by

$$f(v) = \begin{cases} 0, & \text{if } v \text{ is a support vertex} \\ 1, & \text{if } v \text{ is a leaf} \\ 2, & \text{otherwise.} \end{cases}$$

It is clear that f is $\gamma_{ctR}(T)$-function with weight $\frac{4n}{5}$. Thus, $\gamma_{ctR}(T) = \frac{4n}{5}$. Thus, in all the cases $\gamma_{ctR}(T) \leq \frac{4n}{5}$.
Now suppose that $\gamma_{ctR}(T) = \frac{4n}{5}$. If $\gamma_{ctR}(T) = \gamma_R(T) = \frac{4n}{5}$, then by Theorem 11, T is of the required type as mentioned in the statement. If $\gamma_{ctR}(T) = \gamma_R(T) + 1$, then $\gamma_R(T) = \frac{4n}{5} - 1$. Hence, $V(T)$ will be partitioned into sets $W_1, W_2, \ldots, W_{n/5}$ such that $|W_i| = 5$, $1 \leq i \leq n/5$ and any γ_R-function of T will assign a total weight of 4 to each of the sets W_i except one say W_1 and W_1 will be assigned a total weight of 3. Clearly each W_i, $2 \leq i \leq n/5$, will induce a P_5. Let v_1, v_2, v_3, v_4, v_5 be vertices in W_2 which form a P_5 in that order. Let f be a γ_R-function of T which assigns 2 to v_2, zero to v_1, v_3, 1 to v_4, v_5, a total weight of 4 to the vertices in W_i, $3 \leq i \leq \frac{n}{5}$ and a total weight of 3 to the vertices in W_1. Clearly, v_4 and v_5 belong to different color classes of any χ-coloring of T. Hence, $f(v_4) = f(v_5) = 1$ implies that $\gamma_{ctr}(T) = \gamma_R(T)$, which is not the case. Hence, $\bigcup_{i=2}^{n/5} W_i = \emptyset$ and $V(T) = W_1$ and $|W_1| = 5$. Hence T is either P_5 or $K_{1,4}$ or T_1 as given in Figure 2. Further $\gamma_{ctR}(T) = 4$. If $T = K_{1,4}$, then $\gamma_{ctR}(T) = 3$ which is not the case. Hence, T is either P_5 or T_1 as given in Figure 2 (Refer Figure 3).
Converse part is straightforward. □

Theorem 13. *For any graph G, $\gamma_{ctR}(G) \geq \omega(G)$ and equality holds if and only if $G = K_n$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{ctR}-function of G and H be a maximum complete subgraph in G. Then, $|V(H)| = \omega(G)$. Further, $\chi(G) \geq \omega(G)$ and $|V_2 \cup V_1| \geq \chi(G)$ which implies that $|V_2 \cup V_1| \geq \omega(G)$. That is, $\gamma_{ctR}(G) \geq \omega(G)$.
Suppose that $\gamma_{ctR}(G) = \omega(G)$. Then $|V_2 \cup V_1| \geq \omega(G)$ implies that $|V_2 \cup V_1| \geq \gamma_{ctR}(G)$. That is $|V_2 \cup V_1| \geq 2|V_2| + |V_1|$. But $|V_2 \cup V_1| \leq 2|V_2| + |V_1|$. Hence $|V_2 \cup V_1| = 2|V_2| + |V_1|$. Thus $|V_2| = 0$ and $|V_1| = n = \gamma_{ctR}(G) = \omega(G)$. Hence, G is a complete graph.
Conversely if $G = K_n$, then clearly $\gamma_{ctR}(G) = \omega(G)$. □

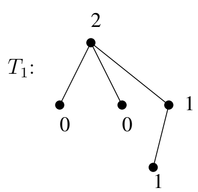

Figure 2. The tree T_1 with $\gamma_{ctR}(T_1) = 4$

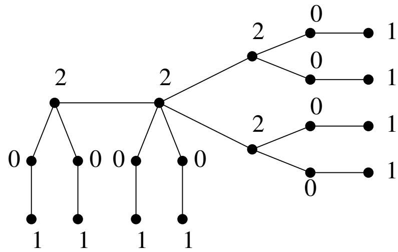

Figure 3. **A tree T with $\gamma_{ctR}(T) = \frac{4n}{5}$**

8. Graphs with $\gamma_{ctR}(G) = n$

In this section, graphs with $\gamma_{ctR}(G) = n$ are investigated.

Theorem 14. *If G is a bipartite graph with $\gamma_{ctR}(G) = n$, then $diam(G) \leq 3$.*

Proof. Since G is a bipartite graph, $\chi(G) = 2$. Suppose that $diam(G) \geq 4$. Let $Q = (v_1, v_2, v_3, \ldots, v_{diam(G)+1})$ be a diametral path in G. Define $f : V(G) \to \{0, 1, 2\}$ by $f(v_2) = 2$, $f(v_1) = f(v_3) = 0$, $f(v) = 1$ for every $v \in V(G)\backslash\{v_1, v_2, v_3\}$. Since v_4, v_5 are in different color classes, f is a CTRDF with $f(V) < n$, a contradiction. Thus $diam(G) \leq 3$. □

Theorem 15. *Let G be a bipartite graph. Then $\gamma_{ctR}(G) = n$, if and only if $G = P_2, P_3, P_4$ or C_4.*

Proof. Suppose that G is a tree. If $diam(G) = 3$ and $G \neq P_4$, then G is a bistar. Now by assigning 2 to the support vertices and zero to the leaf vertices, a CTRDF is obtained of weight lesser than n, a contradiction. Hence, $G = P_4$. If $diam(G) = 2$ and $G \neq P_3$, then G is a star. Clearly $\gamma_{ctR}(G) = 3 < n$, a contradiction. Hence, $G = P_3$. If $diam(G) = 1$, then $G = P_2$.
Suppose that G is not a tree. Then G has only even cycles. If G has a cycle $C_k = (v_1, v_2, \ldots, v_k)$, $k \geq 6$, then by assigning 2 to v_1, zero to v_2 and v_k and 1 elsewhere, a CTRDF is obtained of weight lesser than n, a contradiction. Hence any cycle in G is C_4.
Next we claim that $G = C_4$. Suppose there exists a vertex $w \in V(G)\backslash V(C_4)$ which is adjacent to a vertex in C_4. Without loss of generality let w be adjacent to v_1, then by assigning 2 to v_1, zero to v_2, w and 1 elsewhere, a CTRDF is obtained of weight lesser than n, a contradiction. Thus, $G = C_4$.
Converse is obvious. □

Theorem 16. *Let G be a unicyclic graph with cycle C_k. Then $\gamma_{ctR}(G) = n$, if and only if either $G = C_4$ or the following holds* *(i) k is odd.*

(ii) Every vertex not in C_k is at a distance at most 2 from C_k.

(iii) Every vertex not in C_k is of degree at most 2.

(iv) Every vertex in C_k is of degree at most 3.

Proof. If G is bipartite, then by Theorem 15, $G = C_4$. Suppose that G is not bipartite. Then G contains an odd cycle which proves (i). Further $\chi(G) = 3$ and all the three colors are used to color the vertices of the odd cycle in G by any χ-coloring of G. Suppose that there is a vertex not in C_k at a distance at least 3 from C_k. Then there exists at least 3 vertices say a_1, a_2, a_3 not in C_k and form a P_3 in that order. Now by assigning 2 to a_2, zero to a_1, a_3 and 1 elsewhere, a CTRDF is obtained of weight lesser than n, a contradiction. Hence, (ii) is proved.
To prove (iii), suppose that there is a vertex w not in C_k of degree more than 2. Let w_1, w_2 be 2 neighbors of w not in C_k. Then by assigning 2 to w, zero to w_1, w_2 and 1 elsewhere, a CTRDF is obtained of weight lesser than n, a contradiction. Hence, (iii) is proved. A similar contradiction can be arrived if there is a vertex in C_k of degree more than 3 which proves (iv).
Conversely suppose G is of the given type. If $G = C_4$, then $\gamma_{ctR}(G) = 4$. Suppose that G satisfies the given conditions. Since k is odd, $\chi(G) = 3$. Now no vertex in C_k can be assigned zero by any γ_{ctR} function of G. For, otherwise the vertex which is assigned zero can be colored with a unique color by some χ-coloring of G. The other 2 colors can be used to color the rest of the vertices. Further by conditions (ii), (iii) and (iv), one can infer that if some vertex not in C_k is assigned zero, then the corresponding neighbor which is assigned 2 is adjacent to exactly one vertex assigned zero. Thus, $\gamma_{ctR}(G) = n$. □

Theorem 17. *Let G be a non-bipartite graph with $\chi(G) = w(G)$. Then $\gamma_{ctR}(G) = n$ if and only if there exists a maximum clique H in G such that the following holds.*

(i) Each component of the subgraph induced by $V(G)\backslash V(H)$ is a K_2 or a K_1.

(ii) Every vertex in H has at most one neighbor not in H.

Proof. Let H be a maximum clique in G. As in the proof of Theorem 16, one can prove that every vertex not in H is at a distance at most 2 from H. Next we claim that if w is a vertex not in H at a distance 2 from H, then $deg(w) = 1$. Suppose to the contrary that $deg(w) > 1$. Then there exist two vertices $w_1, w_2 \in N(w)$ such that $w_1, w_2 \notin V(H)$. Now by assigning 2 to w, zero to w_1, w_2 and 1 elsewhere, a CTRDF is obtained of weight lesser than n, a contradiction.
Again as in the proof of Theorem 16, it can be proved that every vertex not in H is of degree at most 2. Thus each component of the subgraph induced by $V(G)\backslash V(H)$ is a K_2 or a K_1 which proves (i).

Suppose there is a vertex in H say w which has 2 neighbours w_1, w_2 not in H. Then by assigning 2 to w, zero to w_1, w_2 and 1 elsewhere, a CTRDF is obtained of weight lesser than n, a contradiction.
Converse is straightforward. □

Remark 1. Characterization of split graphs G, with $\gamma_{ctR}(G) = n$ can also be derived using Theorem 17.

In the following theorems, graphs with $\chi(G) = w(G) + 1$ and $\gamma_{ctR}(G) = n$ are characterized. For this purpose we define two families $\mathcal{G}_1, \mathcal{G}_2$ of graphs as follows.
A graph $G \in \mathcal{G}_1$ if G satisfies the following conditions.

(i) G is non bipartite

(ii) No two odd cycles in G are disjoint.

(iii) If B is the set of all vertices in G which lie in every odd cycle, then each component of the subgraph induced by $V(G)\backslash B$ is a K_2 or a K_1.

(iv) Every vertex in B has at most two neighbors not in B.

(v) If a vertex in B has two neighbors x, y not in B, then every odd cycle in G contains either x or y (Refer Figure 4).

For the graph G given in Figure 4, one can infer that G contains 4 odd cycles and $B = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$. The vertex w_1 has two neighbors x, y not in B and every odd cycle in G contains either x or y. Further G satisfies all the conditions of $\mathcal{G}_1$. Hence $G \in \mathcal{G}_1$.
A graph $G \in \mathcal{G}_2$ if $V(G)$ can be partitioned into two sets such that one set induces a complete subgraph H_1 of order $\omega(G)-2$ and the other set induces a subgraph $H_2 \in \mathcal{G}_1$ such that the following holds.

(i) If there is an odd cycle say C in H_2 such that every vertex in C is adjacent to every vertex in H_1, then every vertex in H_1 is adjacent to at most one vertex not in C (with respect to H_2). (Refer Figure 5).

(ii) If no such odd cycle exists, then every vertex in B (as mentioned in the definition of $\mathcal{G}_1$) is adjacent to every vertex in H_1 and in turn every vertex in H_1 is adjacent to at most two vertices not in B. (with respect to H_2). If a vertex in H_1 is adjacent to two vertices not in B, then both the vertices have a common neighbor in B.

For the graph G given in Figure 5, clearly $H_2 \in \mathcal{G}_1$ and there is an odd cycle C in H_2 in which every vertex of C is adjacent to every vertex of H_1 and no vertex in H_1 has a neighbor in $V(H_2)\backslash V(C)$. Hence, $G \in \mathcal{G}_2$.

Theorem 18. *Let G be a graph with $\chi(G) = w(G)+1$ and $w(G) = 2$. Then $\gamma_{ctR}(G) = n$ if and only in $G \in \mathcal{G}_1$.*

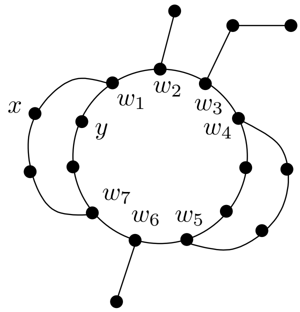

Figure 4. **A graph $G \in \mathcal{G}_1$ with $\gamma_{ctR}(G) = n$**

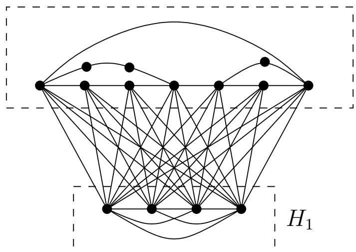

Figure 5. **A graph $G \in \mathcal{G}_2$ with $\gamma_{ctR}(G) = n$**

Proof. Let $\gamma_{ctR}(G) = n$. Since $\chi(G) = 3$, G is not bipartite. Suppose that G has two odd cycles which does not have a vertex in common. Let v_1, v_2, v_3 be three vertices in that order in one of the odd cycles. Then define $f : V(G) \to \{0, 1, 2\}$ by $f(v_2) = 2$, $f(v_1) = f(v_3) = 0$ and $f(v) = 1$ for every $v \in V(G)\backslash\{v_1, v_2, v_3\}$. Now it is clear that f is a CTRDF of G of weight lesser than n, a contradiction. Thus (ii) is proved.
To prove (iii), suppose to the contrary that some component of the subgraph induced by $V(G)\backslash B$ is neither a K_2 nor a K_1. Then there exists vertices v_1, v_2, v_3 which form a path in that order. As before we get a CTRDF of weight lesser than n, a contradiction. Thus (iii) is proved.
To prove (iv), suppose that there is a vertex w in B which has at least three neighbors not in B. Choose 2 vertices $x, y \notin B$ which are neighbors of w such that either x, y belong to the same odd cycle or x is in one odd cycle and y not in any odd cycle or both x, y does not belong to any odd cycle, or x, y belong to different odd cycles. In the first three cases by assigning 2 to w and zero to x, y and 1 elsewhere, will give a CTRDF of weight lesser than n, as in each case all the three colors will be used to the vertices assigned the value 1 by any χ-coloring of G. Hence, we get a contradiction. If x, y belong to different odd cycles, then choose a vertex $z \notin B$ which is adjacent to w and different from x and y. Now by assigning 2 to w, zero to y, z and 1 elsewhere, will give a CTRDF of weight lesser than n, as all three colors will be used to color the vertices in the odd cycle containing x by any χ-coloring of G. Thus, a contradiction

is obtained and (iv) is proved.
To prove (v), let w be a vertex in B which has two neighbors x, y not in B. We claim that every odd cycle in G contains either x or y. Suppose to the contrary that some odd cycle does not contain both x and y, then by assigning 2 to w, zero to x, y and 1 elsewhere, will give a CTRDF of weight lesser than n, a contradiction. Thus (iv) is proved and hence, $G \in \mathcal{G}_1$.
Conversely suppose G is a graph satisfying the given conditions. No vertex in B can be assigned zero by any γ_{ctr}-function of G, as the vertices in B lie in every odd cycle and $\{v\}$ is color class for every $v \in B$ in some χ-coloring of G. By conditions (iii), (iv) and (v), if any γ_{ctR}-function assigns zero to a vertex not in B, then the corresponding vertex which is assigned 2 is adjacent to exactly one vertex assigned zero. Thus $\gamma_{ctR}(G) = n$. □

Remark 2. For odd cycles C_n, $\gamma_{ctR}(C_n) = n$ can also be derived from Theorem 18.

Theorem 19. *Let G be a graph with $\chi(G) = w(G)+1$ and $w(G) \geq 3$. Then $\gamma_{ctR}(G) = n$ if and only in $G \in \mathcal{G}_2$.*

Proof. Let H_1 be a complete subgraph of order $\omega(G) - 2$. Let H_2 be the subgraph induced by $V(G)\backslash V(H)$. First we claim that $H_2 \in \mathcal{G}_1$. Let $w(G) = r$. Since $w(H_2) \geq 3$, H_2 is not bipartite, which proves (i) of the definition of $\mathcal{G}_1$. To prove (ii) of $\mathcal{G}_1$, suppose to the contrary that there are two odd cycles in H_2 which are disjoint. Since $\chi(G) = w(G) + 1$, the $(r + 1)^{th}$ color say c is used to color some vertex in H_2. In any χ-coloring of G, we have the following possibilities. The color c will be used in

(a) None of the two cycles

(b) Both the cycles

(c) Exactly one cycle.

Let v_1, v_2, v_3 be a path in that order in one of the cycles (in case (c), choose them to be in the cycle which does not use the color c). Now by assigning 2 to v_2, zero to v_1, v_3 and 1 elsewhere, a CTRDF is obtained of weight lesser than n. Thus $\gamma_{ctR}(G) < n$, a contradiction. Hence, (ii) of $\mathcal{G}_1$ is proved. Now to prove every component of the subgraph induced $V(H_2)\backslash B$ is a K_2 or K_1, suppose that there are vertices v_1, v_2, v_3 which form a path in that order exist in $V(H_2)\backslash B$. Then as discussed earlier, a CTRDF is obtained of weight lesser than n, as some vertex in B will be assigned the color c by every χ-coloring of G. Thus, (iii) of $\mathcal{G}_1$ is proved. As in the proof of Theorem 18, conditions (iv) and (v) can be proved. Thus $H_2 \in \mathcal{G}_1$.
Now to prove condition (i) of $\mathcal{G}_2$, suppose there is an odd cycle C in H_2 such that every vertex in C is adjacent to every vertex in H_1. Then we claim that every vertex in H_1 is adjacent to at most one vertex not in C (with respect to H_2). For otherwise, if there are 2 vertices x, y not in C adjacent to a vertex w in H_1. Then by assigning 2 to w, zero to x, y and 1 elsewhere, a CTRDF is obtained of weight lesser than n,

as all the three colors, other than the $r-2$ colors used in H_1 are used to color the vertices of C. Thus we get a contradiction. Hence, condition (i) of $\mathcal{G}_2$ is proved.
To prove condition (ii) of $\mathcal{G}_2$, suppose that no such odd cycle (as mentioned above) exists. We claim that every vertex in B is adjacent to every vertex of H_1. Suppose to the contrary that some vertex w in B is not adjacent to a vertex in H_1. Then clearly the $(r-2)$ colors used to color the vertices of H_1 and 2 colors used to color the vertices of H_2 are sufficient for the entire graph G which implies that $\chi(G) = w(G)$ which is not the case. Hence our claim holds. Next we claim that every vertex in H_1 is adjacent to at most 2 vertices not in B (with respect to H_2). This fact can be proved in a way similar to the proof of condition (iv) of Theorem 18. Finally we claim that if a vertex in H_1 is adjacent to two vertices not in B, then both the vertices have a common neighbor in B. Suppose to the contrary that a vertex w in H_1 is adjacent to two vertices x, y not in B and both x, y does not have a common neighbor in B, then by assigning 2 to w, zero to x, y and 1 elsewhere, a CTRDF is obtained of weight lesser than n, a contradiction. Summing the above arguments, condition (ii) of $\mathcal{G}_2$ holds and thus, $G \in \mathcal{G}_2$.
As in the proof of Theorem 18, the converse part is proved. □

Remark 3. For wheels W_n with even order, $\gamma_{ctR}(G) = n$, can also be derived from Theorem 19.

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