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***Research Article***

# On chromatic number and clique number in $k$-step Hamiltonian graphs

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**Abstract:** A graph $G$ of order $n$ is called $k-$step Hamiltonian for $k \geq 1$ if we can label the vertices of $G$ as $v_1, v_2, \ldots, v_n$ such that $d(v_n, v_1) = d(v_i, v_{i+1}) = k$ for $i = 1, 2, \ldots, n-1$. The (vertex) chromatic number of a graph $G$ is the minimum number of colors needed to color the vertices of $G$ so that no pair of adjacent vertices receive the same color. The clique number of $G$ is the maximum cardinality of a set of pairwise adjacent vertices in $G$. In this paper, we study the chromatic number and the clique number in $k-$step Hamiltonian graphs for $k \geq 2$. We present upper bounds for the chromatic number in $k-$step Hamiltonian graphs and give characterizations of graphs achieving the equality of the bounds. We also present an upper bound for the clique number in $k-$step Hamiltonian graphs and characterize graphs achieving equality of the bound.

**Keywords:** Hamiltonian graph, $k-$step Hamiltonian graph, chromatic number, clique number

**AMS Subject classification:** 05C69, 05C78

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## 1. Introduction

Throughout this paper, $G = (V(G), E(G))$ is a simple graph with $V(G)$ as its vertex set and $E(G)$ as its edge set. The open *neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$(or just $N(v)$) is the set $\{u : uv \in E(G)\}$. The *degree* of a vertex $v$, $\deg_G(v)$(or just $\deg(v)$) is the number of neighbors of $v$ in $G$, that is, $\deg(v) = |N_G(v)|$. We refer to $\delta(G)$ and $\Delta(G)$ as the minimum and maximum degree among all vertices of $G$, respectively. Also, a vertex $v \in V(G)$ is called a *pendant vertex* if $\deg(v) = 1$. Let $K_n$, $C_n$ and $P_n$ be a complete graph, a path and a cycle with $n$ vertices, respectively. The *distance* between two vertices $u$ and $v$ in $G$, $d(u, v)$, is the minimum length among all paths between $u$ and $v$ and the maximum distance $d(u, v)$ among two vertices $u, v$ of $G$ is the *diameter* of $G$ and is denoted by $\mathrm{diam}(G)$. For a set $A \subseteq V(G)$, $G[A]$ is the subgraph of $G$ induced by $A$. A *circulant graph* $C_m(a_1, a_2, \ldots, a_k)$ with $0 < a_1 < a_2 < \ldots < a_k < \frac{m+1}{2}$ is a graph of order $m$ with vertices $\{v_1, v_2, \ldots, v_m\}$ such that $v_i$ is adjacent to $v_{i+a_j}$ for all $a_j \in \{a_1, a_2, \ldots, a_k\}$, where the summation $i + a_j$ is taken modulo $m$. For a graph $G$, the *corona graph* of $G$, $\mathrm{cor}(G)$, is the graph obtained by adding a pendant vertex to every vertex of $G$. For other notation and terminology not defined here, we refer to [15].

A proper *vertex coloring* of a graph $G$ is an assignment of colors to the vertices of $G$ such that every pair of adjacent vertices receives different colors. The *chromatic number* of a graph $G$, denoted by $\chi(G)$, is the minimum number of colors required in a proper vertex coloring of $G$. If $G$ has a proper vertex coloring of $k$ colors, then $\chi(G) \leq k$. The study of chromatic number of graph is an active area of research, see for example [6, 11–14]. A *clique* in a graph $G$ is a set $S$ of pairwise adjacent vertices and the number of vertices in the maximum clique is referred to as the *clique number* of $G$, denoted by $\omega(G)$.

A graph $G$ is said to be *Hamiltonian* if $G$ has a spanning cycle referred as a *Hamiltonian cycle*. Although the Hamiltonicity problem is a widely studied subject in graph theory, no exact characterization for the existence of the Hamiltonian cycle has been found. A good survey on the developments of Hamiltonicity problem can be found in [4]. The concept of Hamiltonicity has been extended by Lau et al. [9] to $k-$step Hamiltonicity as follows: For a graph $G$ of order $n$, if we can arrange the vertices as $v_1, v_2, \ldots, v_n$ such that $d(v_n, v_1) = d(v_i, v_{i+1}) = k$ for $i = 1, 2, \ldots, n-1$ and $k \geq 1$, then we call $G$ a $k-$step Hamiltonian (or just $k-$SH) graph with $v_1, v_2, \ldots, v_n, v_1$ as the $k-$step Hamiltonian (or just $k-$SH) walk of $G$. The $k-$step Hamiltonicity of some family of graphs including trees, tripartite graphs, cycles, grid graphs, torus graphs, cubic graphs and subdivision of cycles, have been studied, see [1, 2, 5, 7–10].

In this paper, we continue the study of $k-$SH graphs by proving bounds for the chromatic number and the clique number in $k-$SH graphs, where $k \geq 2$. In Section 2 we give a proof for the fact that a $k-$SH graph has at least $2k + 1$ vertices. In

Section 3, we present upper bounds for the chromatic number in $k-$SH graphs and give characterizations of graphs achieving the equality of the bounds. In Section 4, we present an upper bound for the clique number in $k-$SH graphs and characterize graphs achieving equality of the bound. We make use of the following known results.

**Theorem 1 (Brooks' Theorem).** *For every connected graph $G$ other than an odd cycle or a complete graph, $\chi(G) \leq \Delta(G)$.*

**Theorem 2 (Chartrand et al. [3]).** *If $G$ is a connected graph of order $n$ and diameter $d$, then $\chi(G) \leq n - d + 1$.*

## 2. Preliminary

The following theorem has played an important role in several works on the subject of $k-$step Hamiltonian graphs, while the proof given in [8] does not have any argument for the bound $n \geq 2k + 1$.

**Theorem 3 (Lau et al. [8]).** *The cycle $C_n$ for $n \geq 3$ is $k-$SH for $k \geq 2$ if and only if $n \geq 2k + 1$ and $gcd(n, k) = 1$.*

We provide in the following a proof for the above bound.

**Theorem 4.** *If $G$ is a $k-$SH graph of order $n$ for $k \geq 1$, then $n \geq 2k + 1$.*

*Proof.* The result is obvious if $k = 1$. Thus assume that $k \geq 2$. Let $G$ be a $k$-SH graph of order $n$, and let $W : v_1, v_2, \ldots, v_n, v_1$ be a $k-$SH walk in $G$. Thus $d(v_n, v_1) = d(v_i, v_{i+1}) = k$ for each $i = 1, 2, \ldots, n - 1$. Since $d(v_1, v_2) = k$, let $x_0, x_1, \ldots, x_k$ be a shortest path between $v_1$ and $v_2$, where $x_0 = v_1$ and $x_k = v_2$. Clearly each of $x_i$, $(i = 1, 2, \ldots, k - 1)$ lies on $W$. We follow the walk $W$ starting from $v_1$. We relabel the vertices $x_1, \ldots, x_{k-1}$ according to their place in $W$. Let the relabeled vertices be $x_{j_1}, x_{j_2}, \ldots, x_{j_{k-1}}$, where $x_{j_r}$ is before $x_{j_s}$ if $r < s$. For each $r \in \{1, 2, \ldots, k - 1\}$, $x_{j_r}$ has two consecutive vertices on $W$ namely $x'_{j_r}$ and $x''_{j_r}$, and without loss of generality, assume that $x'_{j_r}$ is on the left side of $x_{j_r}$ and $x''_{j_r}$ is on the right side of $x_{j_r}$ in $W$. Clearly $\{x'_{j_1}, x''_{j_1}\} \cap \{v_1, v_2\} = \emptyset$. For each $r = 2, \ldots, k - 1$, $\{x'_{j_r}, x''_{j_r}\} \not\subseteq \{v_1, v_2, x_{j_s}, x'_{j_s}, x''_{j_s}\}$ for $s < r$. So for each $r = 2, \ldots, k - 1$, $\{x'_{j_r}, x''_{j_r}\} - \{v_1, v_2, x_{j_s}, x'_{j_s}, x''_{j_s} : s < r\} \neq \emptyset$. So $n \geq k + 1 + 2 + k - 2 = 2k + 1$. □

For the sharpness of the bound in Theorem 4, consider the graph $G = C_{2k+1}$ for $k \geq 1$. By Theorem 3, $G$ is $k-$SH.

## 3. Chromatic number

We begin with the following bound.

**Theorem 5.** *If $G$ is a $k-$SH graph of order $n$ for $k \geq 2$, then $\chi(G) \leq \lceil \frac{n}{2} \rceil$. If equality holds, then $k = 2$.*

*Proof.* Let $G$ be a $k-$SH graph of order $n$ for $k \geq 2$. Without loss of generality, assume that $v_1, v_2, \ldots, v_n, v_1$ is a $k-$SH walk of $G$. Clearly, $d(v_i, v_{i+1}) = d(v_n, v_1) = k$ for $i = 1, 2, \ldots, n-1$. We define a vertex coloring $c$ of $G$ as follows: If $n$ is even, then for $i = 0, 1, \ldots, \frac{n}{2} - 1$, we let $c(v_{2i+1}) = c(v_{2i+2}) = i + 1$. If $n$ is odd, then for $i = 0, 1, \ldots, \frac{n-1}{2} - 1$, we let $c(v_{2i+1}) = c(v_{2i+2}) = i + 1$ and $c(v_n) = \frac{n-1}{2} + 1$. Clearly no pair of adjacent vertices receive the same color. The number of colors used in this proper vertex coloring $c$ is $\lceil \frac{n}{2} \rceil$. Therefore, $\chi(G) \leq \lceil \frac{n}{2} \rceil$, as required.

Assume now that $\chi(G) = \lceil \frac{n}{2} \rceil$. Since $G$ is $k-$SH for $k \geq 2$, clearly $G$ is connected and $G$ is not a complete graph. If $G$ is an odd cycle, then $\chi(G) = 3$, that is, $\frac{n+1}{2} = 3$. This means $G = C_5$. By Theorem 3, $C_5$ is $2-$SH. Thus assume that $G$ is not an odd cycle. Since $G$ is not a complete graph or an odd cycle, by Brooks' Theorem, $\chi(G) \leq \Delta(G)$. Therefore we have $\lceil \frac{n}{2} \rceil \leq \Delta(G)$. Let $v$ be a vertex of maximum degree, that is $\deg(v) = \Delta(G)$ and let $A = V(G) - N[v]$.

Assume that $n$ is even. Then, $n \leq 2\Delta(G)$ and $|A| = n - \Delta(G) - 1 \leq \Delta(G) - 1$. Since $G$ is $k-$SH, there exist two vertices $y, z \in A$ such that $y, v, z$ are consecutive vertices in a $k-$SH walk of $G$. Let $W$ be such $k-$SH walk and $W' = W - \{y, v, z\}$. Then it remains $\Delta(G)$ vertices from $N(v)$ in $W'$ and some vertices of $A$. Thus, clearly there exist two consecutive vertices $\alpha, \beta$ in $W$ with $\alpha, \beta \in N(v)$. Therefore, $k = 2$. The case $n$ odd is similarly verified. □

We next show that for each $n \geq 5$, there exists a graph achieving equality of the bound in Theorem 5.

**Proposition 1.** *For each $n \geq 5$, there exists a $2-$SH graph $G$ of order $n$ with $\chi(G) = \lceil \frac{n}{2} \rceil$.*

*Proof.* If $G$ is a graph of order $n \leq 4$, then by Theorem 4, $G$ is not 2-SH. Thus, we consider $n \geq 5$. Let $G = G_n$ be a graph obtained from the complete graph $K_n$, $(n \geq 5)$ with $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ by removing the edges of the Hamiltonian cycle $v_1, v_2, \ldots, v_n, v_1$. The graph $G$ is connected since $K_n$, $n \geq 5$ has $\lfloor \frac{n-1}{2} \rfloor$ edge-disjoint Hamiltonian cycles. Note that for each $i = 1, 2, \ldots, n$, $v_i$ is adjacent to every $v_j$ for $j \notin \{i+1, i-1\}$ with the summations $i+1$ and $i-1$ are taken in modulo $n$ and so $d(v_i, v_{i+1}) = 2$. Therefore, $v_1, v_2, \ldots, v_n, v_1$ is a $2-$SH walk of $G$ and thus $G$ is $2-$SH. By Theorem 5, $\chi(G) \leq \lceil \frac{n}{2} \rceil$.

We next prove that $\chi(G) \geq \lceil \frac{n}{2} \rceil$. For even $n$, clearly $\{v_{2i} : 1 \leq i \leq \frac{n}{2}\}$ is a clique. Therefore, $\chi(G) \geq \omega(G) \geq \frac{n}{2}$. Now, we prove by induction on $n$ that for each odd $n \geq 5$, $\chi(G) \geq \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$. For the base step assume that $n = 5$. Then $G = G_5 = C_5$. Clearly, $\chi(G) = 3 \geq \frac{n+1}{2}$. Assume the result holds for all odd $n'$ with $5 \leq n' < n$.

Now, consider the graph $G = G_n$ for odd $n$. Let $c$ be a proper vertex coloring of $G$. Since $G$ is not a complete graph, there exist $i, j$ such that $c(v_i) = c(v_j)$. Clearly, $j = i - 1$ or $j = i + 1$. Without loss of generality, we assume that $j = i + 1$. Now, remove $v_i$ and $v_{i+1}$ and also the edge $v_{i-1}v_{i+2}$ to obtain $G_{n-2}$. Clearly, the restriction of $c$ on $G_{n-2}$ is a proper vertex coloring of $G_{n-2}$. By the induction hypothesis, we have $|\{c(v) : v \in V(G_{n-2})\}| \geq \lceil \frac{n-2}{2} \rceil = \frac{n-1}{2}$. Therefore, for the graph $G = G_n$, we have $|\{c(v) : v \in V(G)\}| \geq 1 + \lceil \frac{n-2}{2} \rceil = \frac{n+1}{2}$, as desired. □

As another example of families of graphs achieving the equality in the bound in Theorem 5, consider the complete graph $K_{\frac{n}{2}}$ for even $n \geq 6$ with vertices $v_1, v_2, \ldots, v_{\frac{n}{2}}$. Let $G = \text{cor}(K_{\frac{n}{2}})$ with $V(G) = V\left(K_{\frac{n}{2}}\right) \cup \{u_1, u_2, \ldots, u_{\frac{n}{2}}\}$ such that $u_i$ is adjacent to $v_i$ in $G$ for $i = 1, 2, \ldots, \frac{n}{2}$. Clearly, $d(v_i, u_j) = 2$ for $i \neq j$. Also, it is clear that $\chi(G) = \chi\left(K_{\frac{n}{2}}\right) = \frac{n}{2}$ because we can color each vertex $u_i$ for $i = 1, 2, \ldots, \frac{n}{2}$ with one of the color in the set $\{1, 2, \ldots, \frac{n}{2}\}$ that is different from the color of $v_i$. The $2-$SH walk is then given by the sequence of vertices $v_1, u_2, v_3, u_4, \ldots, u_{\frac{n}{2}-1}, v_{\frac{n}{2}}, u_1, v_2, \ldots, v_{\frac{n}{2}-1}, u_{\frac{n}{2}}, v_1$ when $\frac{n}{2}$ is odd and $u_1, v_2, u_3, v_4, \ldots, u_{\frac{n}{2}-1}, v_{\frac{n}{2}}, u_2, v_1, u_{\frac{n}{2}}, v_{\frac{n}{2}-1}, \ldots, u_4, v_3, u_1$ when $\frac{n}{2}$ is even.

Now, we propose the following problem.

**Problem 1.** Characterize all $2-$SH graphs $G$ of order $n$ with $\chi(G) = \lceil \frac{n}{2} \rceil$.

**Theorem 6.** *There is no forbidden induced subgraph characterization for $2-$SH graphs of order $n$ with chromatic number $\lceil \frac{n}{2} \rceil$.*

*Proof.* Let $G$ be a graph of order $a$. The result is obvious if $a \leq 2$. Thus, assume that $a \geq 3$. Let $b = 2\lceil \frac{a}{2} \rceil$. Then we form the graph $\text{cor}(K_b)$. Identify each vertex of $G$ with a pendant vertex of $\text{cor}(K_b)$ to obtain a graph $H$ of order $2b$. Since the adding edges are between pendant vertices of $\text{cor}(K_b)$, clearly $\chi(H) = \chi(\text{cor}(K_b)) = b$. As before, one can easily see that the graph $\text{cor}(K_b)$ is $2-$SH and no two pendant vertices of $\text{cor}(K_b)$ are consecutive in the $2-$SH walk. Therefore, a $2-$SH walk in $\text{cor}(K_b)$ is also a $2-$SH walk in $H$ and thus $H$ is $2-$SH. Thus $G$ is an induced subgraph of $H$, where $H$ is a $2-$SH graph with $\chi(H) = \left\lceil \frac{|V(H)|}{2} \right\rceil$. □

**Proposition 2.** *The difference $\left\lceil \frac{|V(G)|}{2} \right\rceil - \chi(G)$ can be arbitrarily large in a 2-SH graph $G$.*

*Proof.* Let $n \geq 7$ be an odd integer, and $r = \frac{n+1}{2} - 3$. Consider the graph $G = C_{2(r+3)-1}$. By Theorem 3, $G$ is $2-$SH. Then, $\left\lceil \frac{|V(G)|}{2} \right\rceil - \chi(G) = \left\lceil \frac{2(r+3)-1}{2} \right\rceil - 3 = r = \frac{n-5}{2}$. □

**Theorem 7.** *For each $k \geq 3$, there exists a $k-$SH graph $G$ of order $n$ with $\chi(G) = \lceil \frac{n}{k} \rceil$.*

*Proof.* Let $k \geq 3$ and consider the graph $G = C_{2k+1}$. By Theorem 3, $G$ is $k-$SH. Since $G$ is an odd cycle of order $2k+1$, then $\chi(G) = 3 = \lceil \frac{2k+1}{k} \rceil$. □

We propose the following conjecture.

**Conjecture 1.** If $G$ is a $k-$SH graph of order $n$ for $k \geq 1$, then $\chi(G) \leq \lceil \frac{n}{k} \rceil$.

We next present another upper bound for the chromatic number in a $k-$SH graph.

**Theorem 8.** *If $G$ is a $k-SH$ graph of order $n$ for $k \geq 2$, then $\chi(G) \leq n-k$, with equality if and only if $k = 2$ and $G = C_5$.*

*Proof.* Let $G$ be a $k-$SH graph of order $n$ for $k \geq 2$. Clearly, $G$ is connected. Since $G$ is $k-$SH, we have $\text{diam}(G) \geq k$ and thus, by Theorem 2, we have $\chi(G) \leq n-k+1$. If $\chi(G) = n-k+1$, then we obtain the contradiction $2k+1 \leq n \leq 2k-1$ from Theorems 4 and 5. Thus $\chi(G) \leq n-k$.

We next prove the equality part. Assume that $\chi(G) = n-k$. Since $G$ is $k-$SH, it follows by Theorem 4 that $n \geq 2k+1$. If $k \geq 3$, then by Theorem 5, $\chi(G) < \lceil \frac{n}{2} \rceil$ and so $n < 2k+1$, a contradiction. Therefore, $k = 2$ and thus $n \geq 5$. Now, we have $\chi(G) = n-2 \leq \lceil \frac{n}{2} \rceil$. If $n$ is even, then $n \leq 4$, a contradiction. If $n$ is odd, then $n \leq 5$ and thus $n = 5$. Therefore, $G$ is a $2-$SH graph of order 5. We can easily check that $G = C_5$.

The converse is clear. □

Let $C_m(1,2)$ be the circulant graph of order $m$. Abd Aziz et al. [1] obtained the following sufficient condition for the graph $C_m(1,2)$ to be $k-$SH.

**Theorem 9 (Abd Aziz et al. [1]).** *If $gcd(m, 2j-1) = 1$ for $m \geq 6$ and $2 \leq j \leq \lceil \frac{m-1}{4} \rceil$, then $C_m(1,2)$ is $j-SH$.*

They then gave a construction namely $B-$construction that produces a $(k+1)-$SH graph from any given $k-$SH graph $G$. The construction is as follows:

***B*-Construction.** Let $G$ be a $k-$SH graph of order $n$ for $k \geq 1$ with a given $k-$SH walk $v_1, v_2, \ldots, v_n, v_1$. Consider the graph $\text{cor}(G)$ with the new $n$ vertices $u_1, u_2, \ldots, u_n$ such that $u_i$ is adjacent to $v_i$ for $i = 1, 2, \ldots, n$. Then, the $B$-construction produces a graph $B(G)$ from $G$ as follows:

(i) For odd $n$, $B(G) = \text{cor}(G)$.

(ii) For even $n$, $B(G)$ is obtained from $\text{cor}(G)$ by the following scheme:

**Step 1.** For an integer $m$, $m \geq 6$ and $k \leq \lceil \frac{m-1}{4} \rceil - 1$ with $\gcd(m, 2k+1) = 1$, the circulant graph $C_m(1,2)$ is $(k+1)-$SH by Theorem 9. Let $C_m^1(1,2)$ and $C_m^2(1,2)$ be two copies of $C_m(1,2)$. Without loss of generality, assume that

$u_{1,1}, u_{1,2}, \ldots, u_{1,m}, u_{1,1}$ (respectively $u_{2,1}, u_{2,2}, \ldots, u_{2,m}, u_{2,1}$) is a $(k+1)-$SH walk of $C_m^1(1,2)$ (respectively $C_m^2(1,2)$). Note that $d(u_{i,j}, u_{i,j+1}) = k+1$ for $i = 1, 2$ and for $j = 1, 2, \ldots, m$, where the summation $j+1$ is taken in modulo $m$.

**Step 2**. Identify the vertices $u_{1,1}, u_{1,m}, u_{2,1}$ and $u_{2,m}$ to the vertices $u_n, u_1, v_n$ and $v_1$, respectively.

The $i$-th iterated construction $B$ of $G$, $B^i(G)$ for any $i \geq 1$ is defined recursively by $B^1(G) = B(G)$, $B^2(G) = B(B(G))$ and $B^i(G) = B(B^{i-1}(G))$ for $i \geq 2$.

**Theorem 10 (Abd Aziz et al. [1]).** *If $G$ is $k-$SH for $k \geq 1$, then $B(G)$ is $(k+1)-$SH.*

From the $B$-construction above, we can obtain the following two results.

**Lemma 1.** *If $G$ is a $k-$SH graph of order $n$ with $\chi(G) \geq 5$ and $H$ is a graph obtained from $G$ by $B$-construction, then $\chi(H) = \chi(G)$.*

*Proof.* Let $G$ be a $k$-SH graph of order $n$ with $\chi(G) \geq 5$ and $H$ be a graph obtained from $G$ by applying the $B$-construction. If $n$ is odd, then by the above construction, $H = B(G) = \text{cor}(G)$. Clearly, $\chi(H) = \chi(G)$ because in $H$, we can color every vertex $u_i$ for $i = 1, 2, \ldots, n$ with one of the color in the set $\{1, 2, \ldots, \chi(G)\}$ that is not the color of $v_i$.

If $n$ is even, then $H = B(G)$ is obtained from $\text{cor}(G)$ as described above. Since the circulant graph $C_m(1,2)$ contains a triangle, $\chi(C_m(1,2)) \geq 3$ and it is not difficult to see that for $m \geq 6$, $\chi(C_m(1,2)) = 3$ when $m \equiv 0 \pmod 3$ and $\chi(C_m(1,2)) = 4$ when $m \not\equiv 0 \pmod 3$. Note that in $B(G)$, we have $v_n = u_{2,1}$, $u_n = u_{1,1}$, $v_1 = u_{2,m}$ and $u_1 = u_{1,m}$. As before, we can color $\text{cor}(G)$ with $\chi(G)$ colors. Let $c$ be this coloring. Now, we can color $C_m^1(1,2)$ and $C_m^2(1,2)$ in such a way that the vertices $u_{1,1}$ and $u_{1,m}$ receive $c(u_n)$ and $c(u_1)$, respectively, and the vertices $u_{2,1}$ and $u_{2,m}$ receive $c(v_n)$ and $c(v_1)$, respectively. Therefore, we have $\chi(H) = \chi(G)$. □

**Theorem 11.** *For each $l \geq 5$, there exists a chain of graphs $H_2 \subseteq H_3 \subseteq H_4 \subseteq \ldots$ such that $\chi(H_i) = l$ for each $i \geq 2$, $H_i$ is $i-$SH, and $\dfrac{n(H_{i+1})}{n(H_i)} > 2$.*

*Proof.* Let $l \geq 5$ and $H_2 = G_{2l}$ be the graph defined in the proof of Proposition 1. Then, we know that $H_2$ is $2-$SH with $\chi(H_2) = l$. Now, for each $i > 2$, let $H_i = B^{i-2}(H_2)$. By Theorem 10 and Lemma 1, for each $i > 2$, $H_i$ is an $i-$SH graph with $\chi(H_i) = l$. Clearly, from the construction of $H_i$ for $i \geq 2$, we have $\dfrac{n(H_{i+1})}{n(H_i)} > 2$. □

**Corollary 1.** *For each $k \geq 3$, there exists a $k-$SH graph $G$ such that $\chi(G) < \dfrac{n(G)}{2^{k-1}}$.*

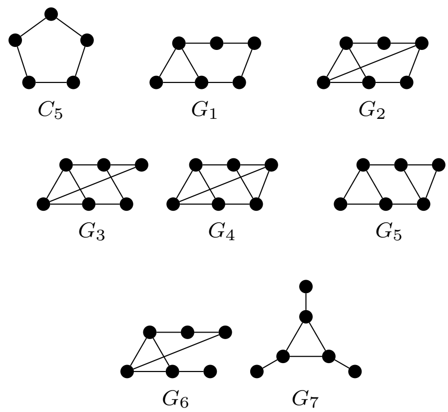


**Figure 1.** **The graphs in $\mathcal{F}$.**

## 4. Clique number

In this section, we present an upper bound for the clique number in a $k-$SH graph and characterize the graphs achieving the equality of the bound. Let $\mathcal{F}$ be the family of graphs shown in Figure 1.

**Theorem 12.** *If $G$ is a $k-$SH graph of order $n$ for $k \geq 2$, then $\omega(G) \leq n-k-1$, with equality if and only if $k = 2$ and $G \in \mathcal{F}$.*

*Proof.* Let $G$ be a $k-$SH graph of order $n$ for $k \geq 2$ and $S$ be a maximum clique in $G$. Let $v_1$ and $v_2$ be two consecutive vertices on a $k-$SH walk of $G$ and assume that $P : x_0, x_1, \ldots, x_{k-1}, x_k$ is a shortest path in $G$ from $v_1$ to $v_2$, where $v_1 = x_0$ and $v_2 = x_k$. Clearly, $|S \cap \{x_0, \ldots, x_k\}| \leq 2$, otherwise we will have a shorter $v_1, v_2$-path. Therefore, $\omega(G) \leq n-k+1$. Suppose that $\omega(G) = n-k+1$. Then, $|S \cap \{x_0, \ldots, x_k\}| = 2$. Let $x_i, x_{i+1} \in S \cap \{x_0, \ldots, x_k\}$. Without loss of generality, we can assume that $x_i \neq v_1$. Then, there is no vertex at distance $k$ from $x_i$, a contradiction. Thus, $\omega(G) \leq n-k$.

Suppose that $\omega(G) = n-k$. Clearly $1 \leq |S \cap \{x_0, \ldots, x_k\}| \leq 2$. Suppose that $|S \cap \{x_0, \ldots, x_k\}| = 1$. Let $x_i \in S \cap \{x_0, \ldots, x_k\}$. If $x_i = v_1$, then $v_2$ is the only vertex at distance $k$ from $x_i$ in $G$, a contradiction. Thus $x_i \neq v_1$. Similarly, $x_i \neq v_2$. But then there is no vertex at distance $k$ from $x_i$ in $G$, a contradiction. Next suppose that $|S \cap \{x_0, \ldots, x_k\}| = 2$. Let $x_i, x_{i+1} \in S \cap \{x_0, \ldots, x_k\}$. Since $|S| = n-k$, clearly there exists a vertex $y \notin S \cup V(P)$. Without loss of generality, assume that $x_i \neq v_1$.

Then there exists at most one vertex at distance $k$ from $x_i$ (possibly $d(x_i, y) = k$), a contradiction.

We conclude that, $\omega(G) \leq n - k - 1$, as desired. We next prove the equality part. Assume that $\omega(G) = n-k-1$. Let $v_1, v_2, x_0, x_1, \ldots, x_k$ and $S$ be as described above.

**Claim 1.** $k = 2$.
*Proof of Claim 1.* Suppose $k \geq 3$. Clearly $|S| \geq 2$. According to $|S \cap \{x_0, \ldots, x_k\}| \leq 2$, we have three possibilities.

Suppose that $|S \cap \{x_0, \ldots, x_k\}| = 0$. If there exists a vertex $x_i$ for $i = 1, 2, \ldots, k-1$ such that $x_i$ is adjacent to some vertex $y \in S$, then there is no vertex at distance $k$ from $x_i$ in $G$, a contradiction. Therefore, every vertex $x_i$ for $i = 1, 2, \ldots, k-1$ has no neighbor in $S$. But then, $v_2$ is the only vertex at distance $k$ from $v_1$ in $G$, a contradiction.

Next suppose that $|S \cap \{x_0, \ldots, x_k\}| = 1$. Let $x_i \in S \cap \{x_0, \ldots, x_k\}$. Since $|S| = n-k-1$, there exists a vertex $y \notin S \cup V(P)$. Suppose $x_i = v_1$. If $d(v_1, y) \neq k$, then $v_2$ is the only vertex at distance two from $v_1$ in $G$, a contradiction. Therefore, $d(v_1, y) = k$ and thus $y$ is adjacent to $x_{k-1}$. Clearly, $y$ is not adjacent to $x_0, x_1, \ldots, x_{k-2}$. But now, $x_1$ is at distance at most $k-1$ to other vertices of $G$, a contradiction. Therefore, $x_i \neq v_1$ and similarly $x_i \neq v_2$. But then, $x_i$ is at distance $k$ to at most one vertex of $G$ (possibly $d(x_i, y) = k$), a contradiction.

Next, suppose that $|S \cap \{x_0, \ldots, x_k\}| = 2$. Let $x_i, x_{i+1} \in S \cap \{x_0, \ldots, x_k\}$. Since $|S| = n-k-1$, there exist two vertices $y, z \notin S \cup V(P)$. Assume that $x_i = v_1$. Since $G$ is $k-$SH, there is another vertex at distance $k$ from $v_1$ different from $v_2$. Clearly, that vertex is either $y$ or $z$. Without loss of generality, assume that $d(v_1, y) = k$. Since $k \geq 3$, $y$ has no neighbor in $S$. Also, it is clear that $y$ is not adjacent to $x_1, x_2, \ldots, x_{k-2}$. Thus, $y$ is adjacent to $x_{k-1}$. But now, $x_{i+1}$ is at distance $k$ to at most one vertex of $G$ (possibly $d(x_{i+1}, z) = k$), a contradiction. Therefore, $x_i \neq v_1$. Similarly, $x_i \neq x_{k-1}$. Now, assume that $x_i = x_1$. Clearly $d(x_i, y) = d(x_i, z) = k$ since $G$ is $k-$SH. Again, since $k \geq 3$, $y$ and $z$ have no neighbor in $S$. Also, $y$ and $z$ are not adjacent to $v_1, x_1, \ldots, x_{k-1}$. Thus, both $y$ and $z$ are adjacent to $v_2$. But now, $v_2$ is the only vertex at distance $k$ from $v_1$, a contradiction. Therefore, $x_i \neq x_1$. Similarly, $x_i \neq x_{k-2}$ ($k \geq 4$). Next, consider $x_i \neq x_1$ or $x_i \neq x_{k-2}$ ($k \geq 5$). Again $d(x_i, y) = d(x_i, z) = k$. Clearly, $y$ and $z$ are adjacent to some vertex in $\{v_1, x_1, \ldots, x_{k-1}, v_2\} - \{x_i\}$. But, every $x_i, y-$path and every $x_i, z-$path created by joining $y$ and $z$ to any of those vertices has length at most $k-1$, a contradiction. So the proof of Claim 1 is complete. $\diamondsuit$

We now prove that $G \in \mathcal{F} = \{C_5, G_1, G_2, G_3, G_4, G_5, G_6, G_7\}$. Note that $|S| = \omega(G) = n-3$. Let $v_1$ and $v_2$ be two consecutive vertices on a 2−SH walk of $G$ and assume that $P : x_0, x_1, x_2$ is a shortest path in $G$ from $v_1$ to $v_2$, where $v_1 = x_0$ and $v_2 = x_2$. Since $k = 2$, clearly $|S| \geq 2$. According to $|S \cap \{x_0, x_1, x_2\}| \leq 2$, we have three cases.

**Case 1.** $|S \cap \{x_0, x_1, x_2\}| = 0$.

Since $G$ is $2-$SH, there exist two vertices $y_1, y_2$ at distance two from $x_1$ and clearly, $y_1, y_2 \in S$. Therefore, $x_1$ is not adjacent to $y_1$ and also not adjacent to $y_2$. Assume that $|S| \geq 3$ and consider $y_3 \in S$. Since $v_1, v_2$ are consecutive vertices in the $2-$SH walk, we have at least four other vertices $x_1, y_1, y_2, y_3$ in the $2-$SH walk of $G$. Clearly, two vertices in $S$ will be consecutive in the $2-$SH walk, a contradiction. Thus, we assume that $|S| = 2$. Since $G$ is $2-$SH, there exists a vertex at distance two from $v_1$ different from $v_2$ and clearly, that vertex is from $S$. Without loss of generality, let $d(v_1, y_2) = 2$. Thus $v_1$ is adjacent to $y_1$. Since $d(x_1, y_2) = 2$ and $y_2$ is not adjacent to $v_1$, clearly $y_2$ is adjacent to $v_2$. If $y_1$ is adjacent to $v_2$, then $x_1$ is the only vertex at distance two from $y_1$, a contradiction. Therefore, $y_1$ is not adjacent to $v_2$. Thus, we have $G = C_5$.

**Case 2.** $|S \cap \{x_0, x_1, x_2\}| = 1$.
Let $x_i \in S \cap \{x_0, x_1, x_2\}$. Since $|S| = n - 3$, there exists a vertex $y \notin S \cup V(P)$. If $x_1 \in S$, then there exists at most one vertex at distance two from $x_1$ in $G$ (possibly $d(x_1, y) = 2$), a contradiction. Therefore, $x_1 \notin S$. Without loss of generality, assume that $x_0 \in S$. Clearly, $d(x_0, y) = 2$ since $G$ is $2-$SH. Now, follow the $2-$SH walk starting from $x_2, x_0, y$. Assume that $|S| \geq 4$ and consider $y_1, y_2, y_3 \in S \setminus \{x_0\}$. The next vertex after $y$ in the $2-$SH walk is either $x_1$ or some vertex in $S \setminus \{x_0\}$. Suppose the next vertex after $y$ is $x_1$. Then the next vertex after $x_1$ should be in $S \setminus \{x_0\}$, say $y_1$. But then the next vertex after $y_1$ in the $2-$SH walk does not exist, a contradiction. Therefore, the next vertex after $y$ in the $2-$SH walk is from $S \setminus \{x_0\}$, say $y_1$. Then the next vertex after $y_1$ should be $x_1$ and the next vertex after $x_1$ is from $S \setminus \{x_0, y_1\}$, say $y_2$. But again there is no next vertex after $y_2$ in the $2-$SH walk, a contradiction. Therefore $2 \leq |S| \leq 3$.

Assume that $|S| = 2$. Let $S = \{x_0, y_1\}$. Since $G$ is $2-$SH, the two vertices at distance two from $x_1$ are $y_1$ and $y$. Therefore, $x_1$ is not adjacent to $y_1$ and also not adjacent to $y$. Since $d(x_0, y) = 2$, $y$ is adjacent to both $y_1$ and $x_2$. And since $\omega(G) = 2$, $x_2$ is not adjacent to $y_1$. Thus we have $G = C_5$.

Next assume that $|S| = 3$. Let $S = \{x_0, y_1, y_2\}$.

Assume that $y$ is adjacent to $x_1$. Then, the next vertex after $y$ in the $2-$SH walk is either $y_1$ or $y_2$. Without loss of generality, assume that the next vertex after $y$ is $y_1$. The next vertex after $y_1$ should be $x_1$. Thus $y_1$ is not adjacent to $x_1$. Then, the next vertex after $x_1$ should be $y_2$ and thus $x_1$ is not adjacent to $y_2$. Now, the vertices $x_2, x_0, y, y_1, x_1, y_2$ are consecutive in the $2-$SH walk. Assume that $y$ is adjacent to $x_2$. Suppose $y$ is not adjacent to $y_2$. Since $d(y, y_1) = 2$, $y_1$ is adjacent to $x_2$. If $y_2$ is adjacent to $x_2$, then $x_0$ is the only vertex at distance two from $x_2$ in $G$, a contradiction. Therefore $y_2$ is not adjacent to $x_2$ and thus we have $G = G_5$. Suppose next $y$ is adjacent to $y_2$. If $y_2$ is adjacent to $x_2$, then $x_1$ is the only vertex at distance two from $y_2$ in $G$, a contradiction. Therefore $y_2$ is not adjacent to $x_2$. If $y_1$ is adjacent to $x_2$, then we have $G = G_4$, otherwise we have $G = G_5$. Assume next $y$ is not adjacent to $x_2$. Since $d(y, y_1) = 2$, $y$ is adjacent to $y_2$. If $y_2$ is adjacent to $x_2$, then $x_1$ is the only vertex at distance two from $y_2$ in $G$, a contradiction. Therefore $y_2$ is not adjacent to $x_2$. If $y_1$ is not adjacent to $x_2$, again $x_1$ is the only vertex at distance two from $y_2$ in $G$, a contradiction. Therefore $y_1$ is adjacent to $x_2$. Thus we have $G = G_3$.

Next, assume that $y$ is not adjacent to $x_1$. Then $y$ is adjacent to at least one of $y_1$ or $y_2$ since $d(x_0, y) = 2$. Without loss of generality, assume that $y$ is adjacent to $y_2$. If $x_1$ is adjacent to $y_2$, then $y_2$ is at distance two to at most one vertex of $G$, a contradiction. Therefore, $x_1$ is not adjacent to $y_2$. If $y_2$ is adjacent to $x_2$, then $x_1$ is the only vertex at distance two from $y_2$ in $G$, a contradiction. Thus, $y_2$ is not adjacent to $x_2$. Suppose $y$ is adjacent to $y_1$. Then clearly $y$ is adjacent to $x_2$, for otherwise there is no next vertex after $y$ in the $2-$SH walk. Then, $x_1$ is the next vertex after $y$ in the $2-$SH walk. Then, the next vertex after $x_1$ is either $y_1$ or $y_2$. But then the next vertex in the $2-$SH walk does not exist, a contradiction. Therefore, $y$ is not adjacent to $y_1$. Assume that $y$ is not adjacent to $x_2$. Then the next vertex after $y$ in the $2-$SH walk is $y_1$. Then the next vertex after $y_1$ is $x_1$. Thus $y_1$ is not adjacent to $x_1$. Then the next vertex after $x_1$ is $y_2$. If $y_1$ is not adjacent to $x_2$, then, $x_1$ is the only vertex at distance two from $y_2$ in $G$, a contradiction. Therefore, $y_1$ is adjacent to $x_2$. Thus we have $G = G_6$. Now, assume that $y$ is adjacent to $x_2$. Then, the next vertex after $y$ in the $2-$SH walk is either $x_1$ or $y_1$. Suppose the next vertex after $y$ is $x_1$. Then, the next vertex after $x_1$ is either $y_1$ or $y_2$. But in either case, there exists no next vertex in the $2-$SH walk, a contradiction. Therefore, the next vertex after $y$ is $y_1$. Then, the next vertex after $y_1$ must be $x_1$ and so $y_1$ is not adjacent to $x_1$. Then clearly the next vertex after $x_1$ is $y_2$. If $y_1$ is adjacent to $x_2$, then $G = G_2$, otherwise $G = G_1$.

**Case 3.** $|S \cap \{x_0, x_1, x_2\}| = 2$.
Without loss of generality, assume that $x_0, x_1 \in S$. Since $|S| = n - 3$, there exist two vertices $y_1, y_2 \notin S \cup V(P)$. Clearly, the vertices $y_1, x_1, y_2$ are consecutive in the $2-$SH walk. Also, $x_0$ is consecutive with one of $y_1$ or $y_2$ in the $2-$SH walk. Without loss of generality, assume that $x_0$ is consecutive with $y_1$. Now, follow the $2-$SH walk starting from $x_2, x_0, y_1, x_1, y_2$. Assume that $|S| \geq 4$ and consider $z_1, z_2 \in S \setminus \{x_0, x_1\}$. Then the next vertex after $y_2$ must be in $S \setminus \{x_0, x_1\}$. Without loss of generality, assume that the next vertex after $y_2$ in the $2-$SH walk is $z_1$. But then the next vertex after $z_1$ in the $2-$SH walk does not exist, a contradiction. Therefore $2 \leq |S| \leq 3$.

Assume that $|S| = 2$. Since $d(x_0, y_1) = d(x_1, y_1) = d(x_1, y_2) = 2$, it follows that $x_0$ is adjacent to $y_2$, $x_2$ is adjacent to $y_1$ and $y_1$ is adjacent to $y_2$. If $x_2$ is adjacent to $y_2$, then $x_1$ is the only vertex at distance two from $y_2$ in $G$, a contradiction. Therefore, $x_2$ is not adjacent to $y_2$. Thus we have $G = C_5$.

Next assume that $|S| = 3$. Let $S = \{x_0, x_1, z\}$. Then, clearly the next vertex after $y_2$ in the $2-$SH walk is $z$ and so $y_2$ is not adjacent to $z$. If $z$ is adjacent to $x_2$, then $y_2$ is the only vertex at distance two from $z$ in $G$, a contradiction. Therefore, $z$ is not adjacent to $x_2$.

Assume that $y_1$ is not adjacent to $y_2$. Since $d(x_0, y_1) = d(z, y_2) = 2$, clearly $y_1$ is adjacent to $z$ and $y_2$ is adjacent to $x_0$. Suppose now $y_2$ is adjacent to $x_2$. If $y_1$ is adjacent to $x_2$, then $G = G_3$, otherwise $G = G_6$. Suppose next $y_2$ is not adjacent to $x_2$. If $y_1$ is adjacent to $x_2$, then $G = G_6$, otherwise $G = G_7$.

Assume next $y_1$ is adjacent to $y_2$. Now, look at the adjacency between $x_0$ and $y_2$. First, assume that $x_0$ is adjacent to $y_2$. Consider the adjacency between $y_1$ and $z$. Suppose $y_1$ is not adjacent to $z$. Since $d(x_1, y_1) = 2$, $y_1$ is adjacent to $x_2$. If $x_2$ is adjacent to $y_2$, then $G = G_5$, otherwise $G = G_1$. Suppose next $y_1$ is adjacent to $z$. Now, assume that $y_1$ is not adjacent to $x_2$. If $x_2$ is adjacent to $y_2$, then $G = G_2$, otherwise $G = G_6$. Next, assume that $y_1$ is adjacent to $x_2$. If $x_2$ is adjacent to $y_2$, then $G = G_4$, otherwise $G = G_3$. Now, assume that $x_0$ is not adjacent to $y_2$. Since $d(x_0, y_1) = d(x_1, y_2) = 2$, it follows that $y_1$ is adjacent to $z$ and $y_2$ is adjacent to $x_2$. If $y_1$ is adjacent to $x_2$, then $G = G_5$, otherwise $G = G_1$.

For the converse, it is not difficult to show that any graph $G \in \mathcal{F}$ is $2-$SH with $\omega(G) = n - 3$. □

**Conflict of interest.** The authors declare that they have no conflict of interest.

**Data Availability.** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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