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Research Article

Graphoidally independent infinite cactus

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Abstract: A graphoidal cover of a graph G (not necessarily finite) is a collection ψ of paths (not necessarily finite, not necessarily open) satisfying the following axioms: (GC-1) Every vertex of G is an internal vertex of at most one path in ψ, and (GC-2) every edge of G is in exactly one path in ψ. The pair (G, ψ) is called a graphoidally covered graph and the paths in ψ are called the ψ-edges of G. In a graphoidally covered graph (G, ψ), two distinct vertices u and v are ψ-adjacent if they are the ends of an open ψ-edge. A graphoidally covered graph (G, ψ) in which no two distinct vertices are ψ-adjacent is called ψ-independent and the graphoidal cover ψ is called a totally disconnecting graphoidal cover of G. Further, a graph possessing a totally disconnecting graphoidal cover is called a graphoidally independent graph. The aim of this paper is to establish complete characterization of graphoidally independent infinite cactus.

Keywords: Graphoidal cover of a graph, graphoidally covered graphs, graphoidally independent graphs, cactus

AMS Subject classification: 05C69

1. Introduction

Throughout this paper, we shall follow the notation and terminology of West [17] for graphs, except that a graph could be infinite in which case the reader is referred to Ore [14].

The concept of graphoidal covers for finite graphs was first introduced by Acharya and Sampathkumar [4] in 1987. There are several variations of the concept of graphoidal

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covers such as acyclic graphoidal cover, geodesic graphoidal cover, induced graphoidal cover and simple graphoidal cover. For results on these topics one may refer to [7–9, 11]. Many interesting notions based on graphoidal covers of a graph like graphoidal covering number [4, 10, 15], graphoidal labeling [16], graphoidal length [6, 12] etc., have been introduced and are being studied extensively. A comprehensive review of graphoidal covers is given in [5]. In [1], Acharya and Gupta extended the notion of graphoidal covers to infinite graphs and extended the notion of domination to graphoidally covered graphs [1–3].

Given a graph $G = (V, E)$, by a *graphoidal cover of* G we mean a collection ψ of non-trivial paths in G, which are not necessarily open and not necessarily finite, satisfying the following axioms:

(GC-1) Every vertex of G is an internal vertex of at most one path in ψ, and

(GC-2) Every edge of G belongs to exactly one path in ψ.

For a given graphoidal cover ψ of a graph G, the paths in ψ are called ψ-*edges* of G, and the ordered pair (G, ψ) is called a *graphoidally covered graph*. In this definition, when G is infinite, a ψ-edge could possibly be infinite; in particular, it may be a one-way infinite path having one end-vertex or a two-way infinite path having no end-vertex. Further, a finite open ψ-edge has two distinct end-vertices while a closed ψ-edge is a cycle and has only one coincident end-vertex, which is specified by ψ.

The set of all graphoidal covers of G is denoted by $\mathcal{G}_G$. Clearly, $\psi =: E(G)$, the edge set of G, is a graphoidal cover of G and is called the trivial graphoidal cover of G; a graphoidal cover that is not trivial is referred as a non-trivial graphoidal cover.

Definition 1. [1] Two distinct vertices u and v of G are said to be ψ-*adjacent* if there exists a finite open path $P \in \psi$ with u and v as its end vertices.

Definition 2. [1] A graphoidal cover ψ of a graph G is called a *totally disconnecting graphoidal cover* [1] of G if no two distinct vertices in G are ψ-adjacent.

Clearly, for a graphoidally covered graph (G, ψ), the graphoidal cover ψ is a totally disconnecting graphoidal cover of G if ψ contains no open path of finite length i.e., every ψ-edge is either a one-way infinite path, a two-way infinite path or a cycle. A cycle or an infinite path are trivial examples of graphs G which do admit a totally disconnecting graphoidal cover.

However, not every graph G (finite or infinite) possesses a totally disconnecting graphoidal cover $\psi \in \mathcal{G}_G$. The graphs shown in Figure 1 are examples of graphs (finite and infinite) which cannot have a totally disconnecting graphoidal cover, since every graphoidal cover ψ of each of these graphs must necessarily contain an open finite ψ-edge.

Thus, a given graph G (finite or infinite) may or may not possess a totally disconnecting graphoidal cover. This gives rise to the following definition:

Definition 3. [1] A graph G is said to be a **graphoidally independent graph** if G admits a totally disconnecting graphoidal cover ψ, and in this case, the corresponding graphoidally covered graph (G, ψ) (or the graph G) is said to be ψ**-independent**.

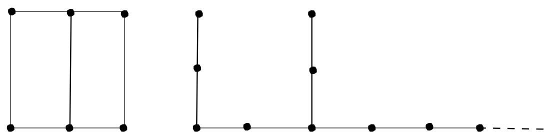

Figure 1. Graphs which are not graphoidally independent.

A complete characterization of graphoidally independent finite graphs is given in [2] and the problem remains open for infinite graphs. However, graphoidally independent infinite trees, infinite unicyclic graphs and infinite 2-edge connected graphs are characterised in [13]. In this paper, we explore graphoidally independent infinite cactus. Here we list some terminology, definitions and theorems from the papers [1, 2, 13] which we will be using in our further study of graphoidally independent graphs.

Definition 4. [1] Given any subgraph H of G, $P \in \psi$ is called H-forming if $E(H) \cap E(P) \neq \phi$.

Definition 5. [1] A *free path* in a graph is a maximal path each edge of which is a bridge.

Theorem 1. *[13] If an infinite graph is graphoidally independent then the following hold:*

(a) The graph has at most one pendant vertex.

(b) The graph has no free path of finite length.

Theorem 2. *[13] An infinite tree T is graphoidally independent if and only if T has at most one pendant vertex.*

Theorem 3. *[13] An infinite unicyclic graph G is graphoidally independent if and only if*

(a) G has at most one pendant vertex.

(b) G has no free path of finite length.

Definition 6. [1] Given a graph $G = (V, E)$, by a **one-way hyperchain** in G we mean a sequence (finite or infinite) $(x_1, P_1, x_2, P_2, \ldots, x_k, P_k, x_{k+1}, \ldots), k \geq 2$, where $x_1, x_2, \ldots$ are distinct vertices in G and each P_i is either a cycle or an infinite path in G such that the following conditions are satisfied:

(HC1) for each i, $x_i \in V(P_i)$ and x_i is the end-vertex of P_i, if P_i is a one-way infinite path.

(HC2) $x_{i+1} \in V(P_i) - \{x_i\}$ for each i.

(HC3) $V(P_i) \cap V(P_{i+1}) = \{x_{i+1}\}$ for every i.

(HC4) $|i-j| \geq 2$ implies $V(P_i) \cap V(P_j) = \phi$.

The one-way hyperchain in the above definition could be finite or infinite. The finite hyperchain $(x_1, P_1, x_2, \ldots, x_k, P_k)$ is said to have length k.

Next, we define the term 'cycle-chain' which we will be using to prove the main result of the paper.

Definition 7. In a graph G, a finite hyperchain $(x_1, P_1, x_2, P_2, \ldots, x_k, P_k), k \geq 2$ of length k in which each P_i is a cycle is called a **cycle-chain** of length k in G.

It is established in [13] that the necessary conditions found in Theorem 1 for a graph to be graphoidally independent are sufficient as well for an infinite tree and for an infinite unicyclic graph to be graphoidally independent. A natural question arises: "Are these two necessary conditions sufficient, in general, for an arbitrary infinite graph to be graphoidally independent?" The answer to the above question is in negative. The following example of an infinite graph (Figure 2) which satisfies the two necessary conditions and is yet not graphoidally independent substantiates this fact. For the graph in Figure 2, every graphoidal cover ψ certainly contains a finite open ψ-edge and hence the graph can not be graphoidally independent.

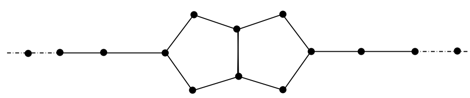

Figure 2. An infinite graph which is not graphoidally independent.

Thus, the necessary conditions in Theorem 1 are not sufficient, in general, for an infinite graph. A natural curiosity arises that are the conditions sufficient for a graph having more than one cycle to be graphoidally independent and we proceed to examine that. Two infinite cacti G and H shown in Figure 3(i) and Figure 3(ii) trivially satisfy both necessary conditions of Theorem 1 and G is graphoidally independent while H is not graphoidally independent as every graphoidal cover ψ of H contains an open ψ-edge of finite length.
From the above discussion we conclude that an infinite cactus satisfying the necessary conditions of Theorem 1 may or may not be graphoidally independent. The problem before us is that "What additional conditions are required to ascertain that an infinite cactus is graphoidally independent?" We started with our investigation in this direction and completely characterized graphoidally independent infinite cactus.

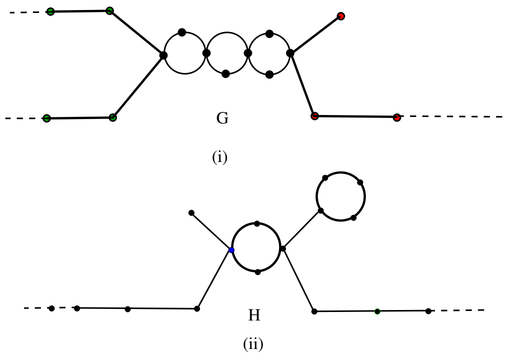

Figure 3. Examples of infinite cacti (i) graphoidally independent (ii) not graphoidally independent

2. Main Theorem

Before coming to our main theorem, we study some properties of a totally disconnecting graphoidal cover of a graphoidally independent graph. We know that a graphoidally independent graph G has a graphoidal cover ψ such that every ψ-edge is either a one-way infinite path, a two-way infinite path or a cycle. The following theorem further presents some necessary conditions on a totally disconnecting graphoidal cover of a graph containing a two-way infinite path.

Theorem 4. *If a connected graph G has a totally disconnecting graphoidal cover ψ, then the following two conditions hold:*

(i) ψ can have at most one two-way infinite path.

(ii) In case (G, ψ) has a two-way infinite ψ-edge, then G cannot have any pendant vertex.

Proof. (i) Suppose ψ contains two two-way infinite paths, namely P_1 and P_2. Since every vertex of a two-way infinite path in ψ is an internal vertex, P_1 and P_2 are vertex disjoint. Let u and v be two arbitrary vertices in $V(P_1)$ and $V(P_2)$ respectively. Since G is connected, there exists a u-v path in G. Let Q be a shortest u-v path in G such that Q has no edge common with P_1 and P_2. Since G is ψ-independent, $Q \notin \psi$. Let $H = \{Q_1, Q_2, \ldots, Q_s\}$ be the set of Q-forming ψ-edges. Let $Q'_i = Q_i \cap Q$. Let the ordering of $Q'_i s$ is such that the last vertex of Q'_i is the first vertex of Q'_{i+1}. Therefore $|V(Q'_i) \cap V(Q'_{i+1})| = 1$. Let $V(Q'_i) \cap V(Q'_{i+1}) = \{q_i\}$. Then q_i is end-vertex of at least one of Q_i and Q_{i+1}. Since u is internal vertex of P_1, it is end-vertex of Q_1 and therefore, q_1 is an internal vertex of Q_1 and an end-vertex of Q_2. Also since no ψ-edge has two end-vertices, q_2 is an internal vertex of Q_2. Continuing like this, we see that q_{s-1} is an end-vertex of Q_s. Since v is an internal vertex of P_2, v is an end-vertex of Q_s. Thus, Q_s is a ψ-edge with two end-vertices, namely q_{s-1} and

v, which is a contradiction. Hence ψ can not contain more than one two-way infinite path.

(ii) Let ψ contain a two-way infinite path, namely P and let $v \in V(G)$ be a pendant vertex. Since P has no end-vertex, $v \notin V(P)$. Let Q be a shortest path from v to P and let $V(P) \cap V(Q) = \{w\}$. Then v and w are end-vertices of some Q-forming ψ-edges. Using similar argument as above, we get a Q-forming ψ-edge of finite length, a contradiction. Hence the proof.

□

Now we proceed with the main theme of the paper i.e., characterizing graphoidally independent infinite cactus.

Theorem 5. *An infinite cactus G is graphoidally independent if and only if G satisfies the following:*

1. *G has no free path of finite length, and*
2. *If P and Q are two one-way infinite edge-disjoint free paths and $\mathfrak{C} = (x_1, C_1, x_2, C_2, \ldots, x_k, C_k)$ $k \geq 1$ is a cycle-chain such that for $1 \leq i \leq j \leq k$,*

$$V(P) \cap V(C_i) \neq \phi$$

$$V(Q) \cap V(C_j) \neq \phi,$$

then the end-vertex of at least one of P and Q in G belongs to $V(\mathfrak{C})$.

Proof. Let G be a graphoidally independent graph and $\psi \in \mathcal{G}_G$ be such that G is ψ-independent. In view of Theorem 1, condition (1) is necessary. Now, assume that there exist two edge-disjoint one-way infinite free paths P and Q and a cycle-chain $\mathfrak{C} = (C_1, x_2, C_2, x_3, \ldots, x_k, C_k)$, $k \geq 1$ satisfying $V(P) \cap V(C_i) \neq \phi$ and $V(Q) \cap V(C_j) \neq \phi$ for some $1 \leq i \leq j \leq k$. If

$$V(P) \cap V(C_i) = \{u_i\},$$

$$V(Q) \cap V(C_j) = \{u_j\},$$

then assume that u_i and u_j are not respective end-vertices of P and Q in G. Let $p \in V(P)$ and $q \in V(Q)$ be end-vertices of P and Q in G respectively. Since G is a graphoidally independent graph, all edges of P and Q are covered by some ψ-edges. Let $P_1, P_2, P_3, \ldots$ and $Q_1, Q_2, Q_3, \ldots$ be P-forming and Q-forming ψ-edges respectively. Then u_i is internal vertex of some P-forming ψ-edge P_r and u_j is internal vertex of some Q-forming ψ-edge Q_s. This implies C_i is a ψ-cycle with u_i as its end-vertex. This further implies that for $i \leq l \leq j$, x_l is the end-vertex of C_l. In particular, x_j is the end-vertex of ψ-cycle C_j. Therefore, u_j is an internal vertex of C_j. But u_j is an internal vertex of Q_s as well, which is a contradiction to the definition of graphoidal cover of G. This proves the necessity of condition (2).

Conversely, let G be an infinite cactus satisfying conditions (1) and (2). We will define a graphoidal cover ψ such that G is ψ-independent. Conditions (1) and (2) together imply that G has at most one pendant vertex. This gives rise to two cases.

Case I. G has a pendant vertex, say w_0.
Let P_0 be a free path in G emanating from w_0. Then P_0 is a one-way infinite path with w_0 as its end-vertex. Let $\psi_0 = \{P_0\}$. Consider the subgraph $G_1 = G - E(P_0)$. If G_1 does not have any vertex of degree at least one, then G is ψ-independent, where $\psi = \psi_0$ and we are done. If not, let

$$V_1 := \{v \in V(P_0) : \deg_{G_1}(v) > 0\}.$$

For each vertex $u \in V_1$, let $\mathcal{C}_1(u)$ denote the set of all cycles in G_1 containing u and let $\mathcal{P}_1(u)$ denote a maximal set of edge-disjoint one-way infinite paths in G_1, each emanating from u such that every edge of each of the paths in $\mathcal{P}_1(u)$ is a bridge. Let

$$S_1 := \bigcup_{u \in V_1} \{\{\bigcup_{C \in \mathcal{C}_1(u)} C\} \cup \{\bigcup_{P \in \mathcal{P}_1(u)} P\}\}.$$

Let

$$\psi_1(u) := \{\bigcup_{C \in \mathcal{C}_1(u)} C\} \cup \{\bigcup_{P \in \mathcal{P}_1(u)} P\},$$

with u as the end-vertex of each cycle and path in $\psi_1(u)$. Further, let $\psi_1 := \bigcup_{u \in V_1} \psi_1(u)$.
Now consider the subgraph $G_2 = G_1 - E(S_1)$. Let $V_2 := \{v \in V(S_1) : \deg_{G_2}(v) > 0\}$. We can iterate the process and obtain a sequence of sets of one-way infinite paths and cycles $S_0, S_1, S_2, S_3, \ldots$ in such a way that

1. $S_0 = \{P_0\}$
2. Each edge in S_i belongs to a cycle or a one-way infinite path emanating from a vertex in $V(S_{i-1})$.
3. $\bigcup_i E(S_i) = E(G)$.
4. $E(S_i) \cap E(S_j) = \phi$, for $i \neq j$.

Let $\psi = \psi_0 \cup \psi_1 \cup \psi_2 \cup \cdots$. Then by the construction, ψ is a graphoidal cover of G and each ψ-edge is either a one-way infinite path or a cycle. Thus, no two distinct vertices in G are ψ-adjacent and therefore G is ψ-independent.

Case II. G has no pendant vertex.
We divide this case further into two subcases as follows.

Subcase I. For any cycle C in G and for any one-way infinite free path P in G satisfying $V(P) \cap V(C) \neq \phi$, $V(P) \cap V(C)$ is an end-vertex of P in G.
We begin with any arbitrary cycle C_0 in G. Let $u_0 \in V(C_0)$ and let $\psi_0 = \{C_0\}$ with

u_0 as the end-vertex of C_0. We can continue as in Case I by replacing P_0 by C_0 and obtain a graphoidal cover ψ of G such that G is ψ-independent.

Subcase II. G has a cycle C_0 and a one-way infinite free path Q_0 such that $V(Q_0) \cap V(C_0) \neq \phi$ and the unique vertex $V(Q_0) \cap V(C_0)$ is not the end-vertex of Q_0.
Because of condition (2), there can be at most one such one-way infinite free path Q_0 having a vertex in common with C_0 such that $V(Q_0) \cap V(C_0)$ is not the end-vertex of Q_0. Let z be the end-vertex of Q_0 in G. Let $\psi_0 = \{Q_0\}$ with z_0 as the end-vertex of Q_0. Consider the subgraph $G_1 = G - E(Q_0)$. Let $V_1 := \{v \in V(Q_0) : \deg_{G_1}(v) > 0\}$. If $V_1 = \phi$, then $\psi = \psi_0$ is the desired graphoidal cover of G. Now suppose $V_1 \neq \phi$. Since G does not contain any pendant vertex, $z_0 \in V_1$. For $u \in V_1$, each edge e in $E(G_1)$, incident with u, lies either on a cycle in G_1 or on a one-way infinite path emanating from u in G_1 such that each edge of the path is a bridge in G_1. For each $u \in V_1$, let $\mathcal{C}_1(u)$ denote the set of all cycles in G_1 containing u and $\mathcal{P}_1(u)$ denote a maximal set of edge-disjoint one-way infinite paths in G_1, each emanating from u such that every edge of each path in $\mathcal{P}_1(u)$ is a bridge. Since Q_0 is a free path, maximality of Q_0 implies that $\mathcal{P}_1(z_0)$ is empty. Let $S_1 := \bigcup_{u \in V_1 - \{z\}} \{\{\bigcup_{C \in \mathcal{C}_1(u)} C\} \cup \{\bigcup_{P \in \mathcal{P}_1(u)} P\}\}$ and $S_1' := \bigcup_{C \in \mathcal{C}_1(z_0)} C$. For $u \in V_1, (u \neq z_0)$ let,

$$\psi_1(u) := \{ \bigcup_{C \in \mathcal{C}_1(u)} C\} \cup \{ \bigcup_{P \in \mathcal{P}_1(u)} P\},$$

with u as the end-vertex of each cycle and each one-way infinite path in $\psi_1(u)$.
Consider the subgraph $G_2 := G_1 - (E(S_1) \cup E(S_1'))$. Let

$$V_2 := \{u \in V(S_1) \cup V(S_1') : \deg_{G_2}(u) > 0\}.$$

Due to condition (2), for $u \in V_2 \cap V(S_1)$, each edge e in $E(G_2)$ incident with u lies either on a cycle in G_2 or on a one-way infinite path in G_2 emanating from u such that each edge of the path is a bridge. Also due to condition (2), there can be at most one vertex $u^* \in V_2 \cap V(S_1')$ and at most one one-way infinite free path Q_1 in G_2 containing u^* such that u^* is not the end-vertex of Q_1 in G_2. Let z_1 be the end-vertex of Q_1 in G_2 and let C_1 be the cycle in $\mathcal{C}_1(z_0)$ in G_1 containing u^*. We define

$$\psi_1(z_0) := \bigcup_{C \in \mathcal{C}_1(z_0)} C,$$

with u^* as the end-vertex of C_1 and z_0 as the end-vertex of all other cycles in $\mathcal{C}_1(z_0)$.
Let $\psi_1 = \bigcup_{u \in V_1} \psi_1(u)$.
Now, for each $u \in V_2$, let $\mathcal{C}_2(u)$ denote the set of all cycles in G_2 containing u and let $\mathcal{P}_2(u)$ denote a maximal set of edge-disjoint one-way infinite paths in $G_2 - E(Q_1)$, each emanating from u such that every edge of each path in $\mathcal{P}_2(u)$ is a bridge. Let

$$S_2 := \{Q_1\} \bigcup_{u \in V_2} \{\{ \bigcup_{C \in \mathcal{C}_2(u)} C\} \cup \{ \bigcup_{P \in \mathcal{P}_2(u)} P\}\}.$$

Let $\psi_2(u) := \{\bigcup_{C\in\mathcal{C}_2(u)} C\} \cup \{\bigcup_{P\in\mathcal{P}_2(u)} P\}, u \neq u^*$, and

$$\psi_2(u^*) := \{Q_1\}\bigcup\{\bigcup_{P\in\mathcal{P}_2(u^*)} P\}\bigcup\{\bigcup_{C\in\mathcal{C}_2(u^*)} C\}.$$

For each $u \in V_2(u \neq u^*)$, u is the end-vertex of each path and cycle in $\psi_2(u)$ while in $\psi_2(u)$ the end-vertex of Q_1 is z_1 and the end-vertex of all other paths and cycles is u^*. Let $\psi_2 := \bigcup_{u\in V_2} \psi_2(u)$. Now, consider the subgraph $G_3 = G_2 - S_2$. We can iterate the process and obtain a sequence of sets of paths and cycles $S_0, S_1, \ldots$ such that

1. $S_0 = \{Q_0\}$
2. Each path and cycle in S_i has a vertex in common with some edge or cycle in S_{i-1}.
3. For each i, S_{2i} has at most one one-way infinite free path not having its end-vertex in S_{2i-1}.
4. $\bigcup_i E(S_i) = E(G)$.
5. $E(S_i) \cap E(S_j) = \phi$, for $i \neq j$.

Let $\psi = \bigcup_i \psi_i$. Then by construction, ψ is a graphoidal cover of G and each ψ-edge is either a one-way infinite path or a cycle. Thus, no two distinct vertices in G are ψ-adjacent and hence G is graphoidally independent. □

An example of a cactus which fails condition 2 of the above theorem is exhibited in Figure 3(ii) and therefore is not a graphoidally independent graph.

Corollary 1. *Every 2-edge connected infinite cactus is graphoidally independent.*

Proof. All blocks of a 2-edge connected cactus G are cycles. Therefore conditions (1) and (2) of the Theorem 5 hold trivially and hence G is a graphoidally independent graph. □

Corollary 2. *A one-way hyperchain, $\mathcal{H} = (x_1, P_1, x_2, P_2, ...)$, is graphoidally independent if and only if for each i, for which P_i is a one-way infinite path, x_i is the end-vertex of P_i in $\mathcal{H}$.*

Proof. Since every hyperchain is a cactus, the result follows from Theorem 5. □

3. Concluding Remarks

In this paper we focused on the problem of characterizing graphoidally independent infinite graphs i.e., the graphs which possess a graphoidal cover ψ such that no two distinct vertices in $V(G)$ are ψ-adjacent. We observed that the two necessary conditions: (i) G has at most one pendant vertex, and (ii) all free paths in G are of infinite length, are sufficient as well for infinite trees and infinite unicyclic graphs to be graphoidally independent. However, we noticed that the conditions are not sufficient in case of infinite cactus and we proceeded to find additional conditions on infinite cactus to be graphoidally independent and established the complete characterization of graphoidally independent infinite cactus. So, the problem remains open for arbitrary infinite graphs. This raises the following problem:

Problem 1. Characterize graphoidally independent infinite graphs in general.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] B.D. Acharya and P. Gupta, *Domination in graphoidal covers of a graph*, Discrete Math. **206** (1999), no. 1-3, 3–33.
https://doi.org/10.1016/S0012-365X(98)00389-6.

[2] ______, *Further results on domination in graphoidally covered graphs*, AKCE Int. J. Graphs. Combin. **4** (2007), no. 2, 127–138.

[3] B.D. Acharya, P. Gupta, and D. Jain, *On graphs whose graphoidal domination number is one*, AKCE Int. J. Graphs. Combin. **12** (2015), no. 2-3, 133–140.
https://doi.org/10.1016/j.akcej.2015.11.007.

[4] B.D. Acharya and E. Sampathkumar, *Graphoidal covers and graphoidal covering number of a graph*, Indian J. Pure Appl. Math. **18** (1987), no. 10, 882–890.

[5] S. Arumugam, B.D. Acharya, and E. Sampathkumar, *Graphoidal covers of a graph: a creative review*, Proceedings of the National workshop on Graph Theory and its Applications, Manonmaniam Sundaranar University, Tirunelveli, Eds. S. Arumugam, B.D. Acharya and E. Sampathkumar, Tata McGraw Hill, 1996, pp. 1–28.

[6] S. Arumugam, P. Gupta, and R. Singh, *Bounds on graphoidal length of a graph*, Electro. Notes Discrete Math. **53** (2016), 113–122.
https://doi.org/10.1016/j.endm.2016.05.010.

[7] S. Arumugam and C. Pakkiam, *Graphoidal bipartite graphs*, Graphs Combin. **10** (1994), no. 2, 305–310.
https://doi.org/10.1007/BF02986680.

[8] ———, *Graphs with unique minimum graphoidal cover*, Indian J. Pure Appl. Math. **25** (1994), no. 11, 1147–1147.

[9] S. Arumugam, I. Rajasingh, and P.R.L. Pushpam, *Graphs whose acyclic graphoidal covering number is one less than its maximum degree*, Discrete Math. **240** (2001), no. 1-3, 231–237.
https://doi.org/10.1016/S0012-365X(00)00350-2.

[10] ———, *A note on the graphoidal covering number of a graph*, J. Discrete Math. Sci. Cryptogr. **5** (2002), no. 2, 145–150.

[11] S. Arumugam and J.S. Suseela, *Acyclic graphoidal covers and path partitions in a graph*, Discrete Math. **190** (1998), no. 1-3, 67–77.
https://doi.org/10.1016/S0012-365X(98)00032-6.

[12] P. Gupta, M. Agarwal, and R. Singh, *On graphoidal length of a tree in terms of its diameter*, AKCE Int. J. Graphs Comb. **17** (2020), no. 3, 703–707.
https://doi.org/10.1016/j.akcej.2019.12.012.

[13] P. Gupta and D. Jain, *Graphoidally independent infinite graphs*, AKCE Int. J. Graphs Comb. **18** (2021), no. 2, 95–100.
https://doi.org/10.1080/09728600.2021.1953946.

[14] O. Ore, *Theory of Graphs*, Amer. Math. Soc. Colloq. Publ. ,38, Providence, 1962.

[15] C. Pakkiam and S. Arumugam, *On the graphoidal covering number of a graph*, Indian J. Pure Appl. Math. **20** (1989), no. 4, 330–333.

[16] I. Sahul Hamid and A. Anitha, *On label graphoidal covering number-I*, Trans. Comb. **1** (2012), no. 4, 25–33.
https://doi.org/10.22108/toc.2012.2271.

[17] D.B. West, *Introduction to Graph Theory*, Prentice hall Upper Saddle River, 2001.