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Research Article

Further results on the j-independence number of graphs

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Abstract: In a graph G of minimum degree δ and maximum degree Δ, a subset S of vertices of G is j-independent, for some positive integer j, if every vertex in S has at most $j-1$ neighbors in S. The j-independence number $\beta_j(G)$ is the maximum cardinality of a j-independent set of G. We first establish an inequality between $\beta_j(G)$ and $\beta_\Delta(G)$ for $1 \leq j \leq \delta-1$. Then we characterize all graphs G with $\beta_j(G) = \beta_\Delta(G)$ for $j \in \{1, \ldots, \Delta-1\}$, where the particular cases $j = 1, 2, \delta-1$ and δ are well distinguished.

Keywords: j-independence number, j-independent sets

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1. Introduction

We consider simple graphs $G = (V, E) = (V(G), E(G))$ of order $|V(G)| = n(G)$ and size $|E(G)| = m(G)$. Two vertices u and v are *neighbors* in G if they are adjacent; that is, if $uv \in E$. For any vertex $u \in V$, let $N_G(u)$ be the set of neighbors of u and let $N_G[u] = N_G(u) \cup \{u\}$. The *degree* of a vertex u is $d_G(u) = |N_G(y)|$. The *minimum* and *maximum degree* of a graph G are denoted $\delta(G)$ and $\Delta(G)$, respectively. If $S \subset V$, then $N_G(S) = \cup_{v \in S} N_G(v)$, and we denote by $G[S]$ the subgraph induced by S in G. Moreover, we write $N_S(x) = N_G(x) \cap S$, $d_S(x) = |N_S(x)|$. Clearly, $d_G(x) = d_V(x)$ for every $x \in V(G)$. When no confusion arises, we simply write V, E, δ and Δ.

The *path* (*cycle*, *complete graph*, *star*, respectively) of order n is denoted by P_n (C_n, K_n, $K_{1,n-1}$, respectively). The star $K_{1,3}$, is called a *claw*. If a graph G does not

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contain an induced subgraph that is isomorphic to some graph F, then we say that G is *F-free*. We say that G is *regular* if every vertex has the same degree and *semiregular* if $\Delta(G) - \delta(G) = 1$. If every vertex of G has degree d, we say G is d-regular.

For an integer $j \geq 1$ and a graph $G = (V, E)$, a subset S of V is *j-independent* if $\Delta(G[S]) < j$ and *j-dominating* if every vertex in $V - S$ has at least j neighbors in S. The *j-independence number* $\beta_j(G)$ is the maximum cardinality of a j-independent set of G. A maximum j-independent set of G is also called a *$\beta_j(G)$-set*. We denote by $\gamma_j(G)$ and $\Gamma_j(G)$ the minimum and maximum orders of a minimal j-dominating set with respect to inclusion and call $\gamma_j(G)$ the *j-domination number*. The concepts of j-independence and j-domination were introduced by Fink and Jacobson [3]. For more details on the j-independence and j-domination, we refer the reader to the survey by Chellali et al. [1].

Clearly for a graph G of order n and maximum degree Δ, $\gamma_1(G)$ is the domination number $\gamma(G)$, $\beta_1(G)$ is the independence number $\beta(G)$ and $\beta_\Delta(G) < \beta_{\Delta+1}(G) = n$. Moreover the sequence $(\beta_j(G))_{1 \leq j \leq n}$ is non-decreasing but few things are known on the rate of growth of this sequence [2].

In this paper, we establish an inequality between $\beta_j(G)$ and $\beta_\Delta(G)$ for $1 \leq j \leq \delta - 1$. Then we characterize graphs G of maximum degree Δ such that $\beta_j(G) = \beta_\Delta(G)$ for $1 \leq j \leq \Delta - 1$ and study more particularly the cases $j = 1, 2, \delta$ and $\delta - 1$. We will use the following results.

Theorem 1 (Fink, Jacobson [3]). *If G is a graph with $\Delta \geq k \geq 2$, then*

$$\gamma_k(G) \geq \gamma(G) + k - 2. \tag{1}$$

Theorem 2 (Jacobson, Peters, Rall [6]). *Let G be a graph of order n and minimum degree δ and let $k \leq \delta$ be a positive integer. Then*

$$\gamma_k(G) + \beta_{\delta-k+1}(G) \leq n.$$

Theorem 3 (Chellali et al. [1]). *Let G be a graph of order n and maximum degree Δ and let $k \leq \Delta$ be a positive integer. Then*

$$\gamma_k(G) + \beta_{\Delta-k+1}(G) \geq n.$$

If moreover G is d-regular, then $\gamma_k(G) + \beta_{d-k+1}(G) = n$.

2. An inequality between $\beta_j(G)$ and $\beta_\Delta(G)$

In this section, we strengthen the inequality $\beta_j(G) \leq \beta_\Delta(G)$ for $j \leq \delta - 1$.

Theorem 4. *Let j, δ, Δ be three positive integers with $j < \delta \leq \Delta$ and let G be a graph with minimum degree δ and maximum degree Δ. Then*

$$\beta_j(G) \leq \beta_\Delta(G) - \delta + j + 1. \tag{2}$$

Moreover, $\beta_j(G) = \beta_\Delta(G) - \delta + j + 1$ *if and only if the three following equalities are satisfied:*

$$\begin{cases} \beta_j & = n - \gamma_{\delta-j+1} & (a) \\ \gamma_{\delta-j+1} & = \gamma + \delta - j - 1 & (b) \\ \beta_\Delta & = n - \gamma. & (c) \end{cases}$$

Proof. By successively applying Theorems 2, 1 and 3 with $j = \delta - k + 1$, we get the following three inequalities.

$$\beta_j \quad \leq \quad n - \gamma_{\delta-j+1} \quad \leq \quad n - \gamma - \delta + j + 1 \quad \leq \quad \beta_\Delta - \delta + j + 1.$$

The equality occurs in (2) if and only if each of the previous inequalities is an equality. □

Corollary 1. *If* G *is a graph with minimum degree* $\delta \geq 3$ *and maximum degree* Δ*, then* $\beta_j(G) < \beta_\Delta(G)$ *for all* $j \leq \delta - 2$.

The next proposition gives a property of graphs attaining the bound in Theorem 4 for some value of j.

Proposition 1. *Let* $G = (V, E)$ *be a graph with maximum degree* Δ*, minimum degree* $\delta \geq 2$ *and* j *a positive integer with* $j \leq \delta - 1$*. If* $\beta_j(G) = \beta_\Delta(G) - \delta + j + 1$*, then every* $\beta_j(G)$*-set is a* $(j+1)$*-dominating set of* G *and thus* $\beta_j(G) \geq \gamma_{j+1}(G) \geq j + 1$.

Proof. Let S be a $\beta_j(G)$-set. If there is a vertex $y \in V - S$ such that $|N_S(y)| \leq j$, then $|N_{V-S}(y)| \geq \delta - j$. Let A be a subset of $N_{V-S}(y)$ with $|A| = \delta - j$. Then $S \cup A$ is a Δ-independent set of G with $|S| + \delta - j$ vertices, contradicting the hypothesis $\beta_\Delta(G) = \beta_j(G) + \delta - j - 1$. Hence S is a $(j+1)$-dominating set of G. □

Example 1. For $n \geq 4$ even, let H_n be the $(n-2)$-regular graph of order n. Equivalently, H_n is a complete graph K_n minus a perfect matching. Clearly, $\delta = \Delta = n - 2$. Moreover, we can check that $\gamma_j(H_n) = \beta_j(H_n) = j$ if j is even and $\gamma_j(H_n) = \beta_j(H_n) = j + 1$ if j is odd. Hence $\beta_j(H_n) = \beta_\Delta(H_n) - \delta + j$ if j is even, $\beta_j(H_n) = \beta_\Delta(H_n) - \delta + j + 1$ if j is odd and in the second case only, the three equalities (a), (b), (c) of Theorem 4 are satisfied. In addition, for j odd, every $\beta_j(H_n)$-set is isomorphic to H_{j+1} and is $(j+1)$-dominating.

It is worth noting that any additional condition on G allowing to strengthen Theorems 1, 2 or 3 allows to lower the bound in Theorem 4. For instance, Hansberg [4] proved that if a graph G with maximum degree $\Delta \leq n-2$ has less than $(\gamma(G)-1)(k-1)$ induced cycles C_4 for an integer k with $\Delta \geq k \geq 2$, then $\gamma_k(G) \geq \gamma(G) + k - 1$. This gives the following corollary by letting again $j = \delta - k + 1$.

Corollary 2. *Let* G *be a graph with maximum degree* $\Delta \leq n - 2$ *and minimum degree* δ*, and let* $j \leq \delta - 1$ *be a positive integer. If* G *has less than* $(\gamma(G) - 1)(\delta - j)$ *induced cycles* C_4*, then* $\beta_j(G) \leq \beta_\Delta(G) - \delta + j$.

3. Some families of graphs

In this section, we define some families of extremal graphs for equalities between the independence parameters which will be discussed in Section 4.

3.1. Families $\mathcal{G}(j,\Delta)$ and $\mathcal{F}(\delta,\Delta)$

Definition 1. Let j and Δ be integers with $1 \le j \le \Delta-1$. A connected graph $G=(V,E)$ of maximum degree Δ belongs to the family $\mathcal{G}(j,\Delta)$ if V admits a partition (X,Y), called a good partition, such that $d_X(y)=\Delta$ for all $y \in Y$ and $d_X(x) \le j-1$, $d_Y(x) \le 1$ for all $x \in X$.

We note that the set Y is independent and all its vertices have degree Δ while the set X is j-independent and all its vertices have degree at most $j < \Delta$. The good partition (X,Y) is unique since Y is the set of all the vertices of degree Δ.

Proposition 2. *Let j and Δ be positive integers with $j \le \Delta - 1$ and let G be a graph of family $\mathcal{G}(j,\Delta)$. Then the part X of the unique good partition (X,Y) of G is a maximum Δ-independent set and $\beta_j(G)=\beta_\Delta(G)$.*

Proof. Set X is Δ-independent comes from the fact it is j-independent. Now assume T is another Δ-independent set of G and let $y \in T \cap Y$. Then $d_{T\cap Y}(y) \le \Delta - 1$ and since $d_X(y)=\Delta$, $d_{X-T}(y) \ge 1$. Hence, since each vertex of X has at most one neighbor in Y, $|X-T| \ge |Y \cap T|$ and thus $|T| = |T\cap X| + |T \cap Y| \le |X|$. Whence X is a maximum Δ-independent set and consequently also a maximum j-independent set. Therefore $\beta_j(G)=\beta_\Delta(G)$. □

From Definition 1, the minimum degree of a graph G of $\mathcal{G}(j,\Delta)$ is at most j. Definition 2 describes the subfamily $\mathcal{F}(\delta,\Delta)$ of graphs of $\mathcal{G}(\delta,\Delta)$ of minimum degree δ.

Definition 2. Let δ and Δ be integers with $1 \le \delta \le \Delta - 1$. A connected graph $G=(V,E)$ of maximum degree Δ and minimum degree δ belongs to family $\mathcal{F}(\delta,\Delta)$ if V admits a partition (X,Y) such that $d_X(y)=\Delta$ for all $y \in Y$, $d_Y(x)=1$ for all $x \in X$ and the induced subgraph $G[X]$ is $(\delta-1)$-regular.

3.2. Families $\mathcal{L}(\Delta)$, $\mathcal{H}(\Delta-1,\Delta)$, $\mathcal{H}(\Delta,\Delta)$

Definition 3. Let $G=(V,E)$ be a connected graph of maximum degree $\Delta \ge 2$, and let (X,Y) be a partition of V such that Y is an independent set, X is a $(\Delta-1)$-independent set and $d_X(y)=\Delta$ for all $y \in Y$. The partition (X,Y) has Property $\mathcal{P}$ if for every $A \subseteq X$ and $B \subseteq Y$ such that $|A| < |B|$ and $B \subseteq N_Y(A)$, there exists a vertex v of $X - A$ such that $d_{(X-A)\cup B}(v) = \Delta$.

Definition 4. Let $\Delta \geq 2$ be an integer. A connected graph $G = (V, E)$ of maximum degree Δ belongs to family $\mathcal{L}(\Delta)$ if V admits a partition (X, Y), called a good partition, such that $d_X(y) = \Delta$ for all $y \in Y$, $d_X(x) \leq \Delta - 2$ for all $x \in X$ and the partition (X, Y) satisfies Property $\mathcal{P}$.

Proposition 3. *Let $G = (V, E)$ be a connected graph of maximum degree $\Delta \geq 2$. If G belongs to $\mathcal{L}(\Delta)$, then the set X of every good partition (X, Y) of V is a maximum Δ-independent set and $\beta_{\Delta-1}(G) = \beta_\Delta(G)$.*

Proof. Let $G \in \mathcal{L}(\Delta)$ and let (X, Y) be a good partition. Clearly, the $(\Delta - 1)$-independent set X is also Δ-independent. Now, assume G admits another Δ-independent set T and let $T \cap X = T_X$, $T \cap Y = T_Y$. Since every vertex y of T_Y has Δ neighbors in X but less than Δ neighbors in T_X, y has at least one neighbor in $X - T_X$. Hence $T_Y \subseteq N_Y(X - T_X)$. If $|X - T_X| < |T_Y|$, then by Property $\mathcal{P}$, there exists a vertex v in T_X such that $d_{T_X \cup T_Y}(v) = \Delta$, in contradiction to the definition of T. Whence $|T_Y| \leq |X - T_X|$ and thus $|T| \leq |X|$. Therefore X is a maximum Δ-independent set of G and consequently a maximum $(\Delta - 1)$-independent set, and $\beta_{\Delta-1}(G) = \beta_\Delta(G)$. $\square$

The following proposition shows that for $j = \Delta - 1$, Proposition 3 is stronger than Proposition 2.

Proposition 4. *For every integer $\Delta \geq 2$, $\mathcal{G}(\Delta - 1, \Delta) \subset \mathcal{L}(\Delta)$ and the inclusion is strict.*

Proof. Let G be a graph of $\mathcal{G}(\Delta - 1, \Delta)$ and let (X, Y) be its unique good partition. Since $d_Y(x) \leq 1$ for all $x \in X$, $|B| \leq |A|$ for every pair of subsets $A \subseteq X$ and $B \subseteq Y$ such that $B \subseteq N_Y(A)$. Hence the partition (X, Y) satisfies Property $\mathcal{P}$ and $G \in \mathcal{L}(\Delta)$. The third example L_n given at the end of this subsection shows that the inclusion is strict. $\square$

Property $\mathcal{P}$ is not easy to check. The following property $\mathcal{Q}$, simpler but weaker than $\mathcal{P}$, is sometimes useful.

Definition 5. Let G be a connected graph of maximum degree $\Delta \geq 2$, the vertex set V of which admits a partition (X, Y) such that Y is an independent set, X is a $(\Delta - 1)$-independent set and $d_X(y) = \Delta$ for all $y \in Y$. The partition (X, Y) has Property $\mathcal{Q}$ if for every pair y_1, y_2 of vertices of Y such that $N_X(y_1) \cap N_X(y_2) \neq \emptyset$, there exist at least two non-adjacent vertices x_1, x_2 in $N_X(y_1) \cap N_X(y_2)$ such that $d_X(x_1) = d_X(x_2) = \Delta - 2$.

Proposition 5. *Let G be a connected graph of maximum degree $\Delta \geq 2$, the vertex set V of which admits a partition (X, Y) such that Y is an independent set, X is a $(\Delta - 1)$-independent set and $d_X(y) = \Delta$ for all $y \in Y$. If the partition (X, Y) satisfies Property $\mathcal{P}$, then it satisfies Property $\mathcal{Q}$.*

Proof. Let y_1, y_2 be two vertices of Y such that $N(y_1) \cap N(y_2)$ contains a vertex x of X and let $A = \{x\}$, $B = \{y_1, y_2\}$. By $\mathcal{P}$, there exists a vertex x_1 in $X - \{x\}$ of degree Δ in $(X - \{x\}) \cup \{y_1, y_2\}$. Since $d_X(x_1) \le \Delta - 2$, necessarily $d_X(x_1) = \Delta - 2$ and x_1 is not adjacent to x but is adjacent to both y_1, y_2. The same argument applied to the sets $A_1 = \{x_1\}$ and B shows that there exists a vertex x_2 in X, possibly equal to x, of degree $\Delta - 2$ in X, not adjacent to x_1 but adjacent to both y_1, y_2. Hence Property $\mathcal{Q}$ is satisfied. □

If a graph G has a partition satisfying the definition of $\mathcal{Q}$ and if some vertex x of X has more than one neighbor in Y, then x belongs to an induced 4-cycle C_4 of G. This gives the following corollary of Proposition 5.

Corollary 3. *Every C_4-free graph of $\mathcal{L}(\Delta)$ is in $\mathcal{G}(\Delta - 1, \Delta)$.*

In Section 4, we are specially interested in graphs of $\mathcal{L}(\Delta)$ with minimum degree $\Delta - 1$ or Δ. So we give the following definitions.

Definition 6. $\mathcal{H}(\Delta - 1, \Delta)$ is the subset of graphs of $\mathcal{L}(\Delta)$ of minimum degree $\Delta - 1$, i.e., the semiregular graphs of minimum degree $\Delta - 1$ of $\mathcal{L}(\Delta)$, and $\mathcal{H}(\Delta, \Delta)$ is the subset of graphs of $\mathcal{L}(\Delta)$ of minimum degree Δ, i.e., the regular graphs of $\mathcal{L}(\Delta)$.

Note that for every good partition (X, Y) of the vertex set of a graph G of $\mathcal{L}(\Delta)$ and for every $x \in X$, $d_Y(x) \ge 1$ if $G \in \mathcal{H}(\Delta - 1, \Delta)$ and $d_Y(x) \ge 2$ if $G \in \mathcal{H}(\Delta, \Delta)$.

In what precedes, the integers j, δ, Δ are often given as functions of the order n of G. We give below some examples of families.

1. $\mathcal{F}(1, \Delta) = \{K_{1,\Delta}\}$ and $\mathcal{F}(1, n-1)$ is the set of all stars $K_{1,n-1}$.

2. For $n \ge 4$ even, consider again the $(n-2)$-regular graph H_n consisting of a clique of order n minus a perfect matching. The partition (X, Y) where Y consists of two non-adjacent vertices satisfies Property $\mathcal{P}$ and shows that $H_n \in \mathcal{H}(n-2, n-2)$.

3. For $n \ge 6$ even, let L_n be a connected graph obtained from two disjoint cliques of order $n/2$ by adding two non-adjacent edges joining two different vertices of the first clique to two different vertices of the second one. For this graph, $\Delta = n/2$, $\delta = n/2 - 1$ and the partition (X, Y), where Y consists of two non-adjacent vertices of degree $n/2$, satisfies Property $\mathcal{P}$. Hence $L_n \in \mathcal{H}(n/2 - 1, n/2) \subseteq \mathcal{L}(n/2)$. But L_n does not belong to $\mathcal{F}(n/2 - 1, n/2)$, nor to $\mathcal{G}(n/2 - 1, n/2)$, since two vertices of X have two neighbors in Y.

4. Equalities between independence parameters

In this section, we give characterizations or properties of graphs G such that $\beta_i(G) = \beta_j(G)$ for some values of i and j. We are particularly interested in the case $j = \Delta(G)$, where by Corollary 1, $\beta_i(G) = \beta_\Delta$ can only occur if $i \ge \delta(G) - 1$. If G has several components G_k, then $\Delta(G) = \max_k \Delta(G_k)$, $\delta(G) = \min_k \delta(G_k)$ and the problem is interesting only if all components have the same minimum and maximum degrees. In this case, the properties of graphs G such that $\beta_i(G) = \beta_j(G)$ hold for each component. Therefore it is sufficient in what follows to consider connected graphs.

4.1. Graphs G with $\beta_{j-1}(G) = \beta_j(G)$ for some $j \in \{2, \ldots, \Delta\}$

It is known [2] that the j-independence number $\beta_j(G)$ of a graph G may be larger than its maximum minimal j-domination number $\Gamma_j(G)$. The next result shows that this is no more true when $\beta_{j-1}(G) = \beta_j(G)$.

Proposition 6. *Let G be a connected graph with maximum degree Δ and j an integer with $2 \le j \le \Delta$. If $\beta_{j-1}(G) = \beta_j(G)$ and S is a maximum $(j-1)$-independent set, then S is a minimal j-dominating set of G and thus $\beta_j(G) \le \Gamma_j(G)$.*

Proof. Let S be a $\beta_{j-1}(G)$-set. If some vertex y of $V - S$ has less than j neighbors in S, then $S \cup \{y\}$ is a j-independent set larger than S in contradiction to $|S| = \beta_{j-1}(G) = \beta_j(G)$. Hence S is a j-dominating set. This j-dominating set is minimal since $d_S(x) < j$ for all $x \in S$. Therefore $\beta_j(G) \le \Gamma_j(G)$. □

For the particular case $j = \Delta$, it was shown in [1] that $\beta_\Delta(G) \ge \Gamma_\Delta(G)$, which gives the following corollary.

Corollary 4. *Let G be a connected graph with maximum degree $\Delta \ge 2$ such that $\beta_{\Delta-1}(G) = \beta_\Delta(G)$. Then every maximum $(\Delta-1)$-independent set is a minimal Δ-dominating set and $\beta_\Delta(G) = \Gamma_\Delta(G)$.*

4.2. Graphs G with $\beta_j(G) = \beta_\Delta(G)$ for some $j \in \{1, \ldots, \Delta - 1\}$

We give a necessary and sufficient condition for a connected graph G to satisfy $\beta_j(G) = \beta_\Delta(G)$.

Theorem 5. *Let j and Δ be positive integers with $j \le \Delta - 1$ and let G be a connected graph of maximum degree Δ. Then $\beta_j(G) = \beta_\Delta(G)$ if and only if $G \in \mathcal{G}(j, \Delta)$ when $j \le \Delta - 2$, $G \in \mathcal{L}(\Delta)$ when $j = \Delta - 1$.*

Proof. The part "if" is a consequence of Propositions 2 and 3. To prove the part "only if", we consider a maximum j-independent set X of G and $Y = V - X$. Since $\beta_j(G) = \beta_\Delta(G)$ and $(\beta_k(G))_k$ is a non-decreasing sequence, X is a maximum k-independent set of G for $j+1 \le k \le \Delta$ and by Proposition 6, X is a Δ-dominating set of G. Thus $d_X(y) = \Delta$ for all $y \in Y$ and Y is independent.
In the case $j \le \Delta - 2$, suppose that some vertex x of X has at least two neighbors y_1 and y_2 in Y and consider the set $S = (X - \{x\}) \cup \{y_1, y_2\}$. For $i = 1, 2$, $d_S(y_i) = \Delta - 1$, and for all $v \in S \cap X$, $d_S(v) \le j + 1 \le \Delta - 1$. Hence S is a Δ-independent set larger

than X, in contradiction to $\beta_\Delta(G) = |X|$. Therefore $d_Y(x) \le 1$ for all x in X and thus $G \in \mathcal{G}(j, \Delta)$.
Now let $j = \Delta - 1$ and suppose that there exist two subsets $A \subseteq X$ and $B \subseteq Y$ such that $|A| < |B|$, $B \subseteq N_Y(A)$ and $d_{(X-A)\cup B}(x) < \Delta$ for every vertex x of $X - A$. Since $B \subseteq N_Y(A)$, every vertex of B has a neighbor in A and thus at most $\Delta - 1$ neighbors in the set $S = (X - A) \cup B$. Since moreover $d_S(x) < \Delta$ for every vertex x of $X - A$, S is a Δ-independent set larger than X, a contradiction. Therefore the partition (X, Y) satisfies Property $\mathcal{P}$ and $G \in \mathcal{L}(\Delta)$. □

When G is C_4-free, the following corollary follows from Corollary 3.

Corollary 5. *Let j and Δ be positive integers with $j \le \Delta - 1$ and let G be a C_4-free connected graph of maximum degree Δ. Then $\beta_j(G) = \beta_\Delta(G)$ if and only if $G \in \mathcal{G}(j, \Delta)$.*

The application of Theorem 5 to the particular cases $j = 1$ and $j = 2$ gives the following statements.

Theorem 6. *Let G be a connected graph with maximum degree Δ. Then $\beta(G) = \beta_\Delta(G)$ if and only if G is the cycle C_4 or a star.*

Proof. If $\Delta = 1$, then G is the path P_2. If $\Delta \ge 2$ then, by Theorem 5, $\beta(G) = \beta_\Delta(G)$ if and only if $G \in \mathcal{L}(2)$ or $G \in \mathcal{G}(1, \Delta)$ with $\Delta \ge 3$. Graphs of $\mathcal{L}(2)$ are the path P_3 and the cycle C_4 while graphs of $\mathcal{G}(1, \Delta)$ for $\Delta \ge 3$ are the stars $K_{1,n-1}$ with $n \ge 4$. □

Theorem 7. *Let G be a connected graph with maximum degree $\Delta \ge 2$. Then $\beta_2(G) = \beta_\Delta(G)$ if and only if G is a path P_n or a cycle C_n if $\Delta = 2$, $G \in \mathcal{L}(3)$ if $\Delta = 3$ and $G \in \mathcal{G}(2, \Delta)$ if $\Delta \ge 4$.*

Example 2. Let T be a tree consisting of $p \ge 1$ disjoint stars $K_{1,k}$ with $k \ge 3$ and $p - 1$ further edges attaching leaves of stars such that T results in a chain of stars. Then $n = p\,(k + 1)$, $\Delta = k$ and $\beta_2(T) = \beta_\Delta(T) = kp$. We see that $T \in \mathcal{G}(2, \Delta)$ for $\Delta = k \ge 3$. When $\Delta = 3$, by Proposition 4, $\mathcal{G}(2, 3) \subseteq \mathcal{L}(3)$.

Example 3. Let G be the connected graph obtained from $p \ge 3$ disjoint paths $P_2 = x_i y_i$ for $i \in \{1, 2, \ldots, p\}$ by joining vertices x_i to a new vertex x and by joining vertices y_i to another new vertex y. Then $n = 2p + 2$, $\Delta = p$, and $\beta_2(G) = \beta_\Delta(G) = 2p$. We see that $G \in \mathcal{G}(2, \Delta)$ for $p \ge 3$. When $\Delta = 3$, $\mathcal{G}(2, 3) \subseteq \mathcal{L}(3)$ as in Example 2.

By Corollary 1, $\beta_j(G) = \beta_\Delta(G)$ implies $j \ge \delta - 1$. In the following subsection, we consider the particular cases $j = \delta$ and $j = \delta - 1$.

4.3. Graphs G with $\beta_\delta(G) = \beta_\Delta(G)$ or $\beta_{\delta-1}(G) = \beta_\Delta(G)$

Replacing j by δ in Theorem 5 and Corollary 5 gives the following characterization.

Theorem 8. *A connected graph G of maximum degree Δ and minimum degree $\delta \leq \Delta - 1$ satisfies $\beta_\delta(G) = \beta_\Delta(G)$ if and only if it belongs to $\mathcal{F}(\delta, \Delta)$ when $\delta \leq \Delta - 2$ and to $\mathcal{H}(\Delta-1, \Delta)$ when $\delta = \Delta - 1$. If moreover G is C_4-free, then $\beta_\delta(G) = \beta_\Delta(G)$ if and only if it belongs to $\mathcal{F}(\delta, \Delta)$ for every value of δ.*

Replacing j by $\delta - 1$ in Theorems 5 gives the following characterization.

Theorem 9. *Let δ and Δ be two positive integers with $2 \leq \delta \leq \Delta$. A connected graph $G = (V, E)$ of maximum degree Δ and minimum degree δ satisfies $\beta_{\delta-1}(G) = \beta_\Delta(G)$ if and only if it belongs to $\mathcal{H}(\Delta, \Delta)$.*

Proof. Since the minimum degree of graphs of $\mathcal{G}(j, \Delta)$ is at most j, there is no graph of minimum degree δ in $\mathcal{G}(\delta - 1, \Delta)$. Therefore by Theorem 5, $\beta_{\delta-1}(G) = \beta_\Delta(G)$ if and only if $\delta - 1 = \Delta - 1$, i. e., G is regular, and $G \in \mathcal{L}(\Delta)$. This is equivalent to $G \in \mathcal{H}(\Delta, \Delta)$. □

For example, the $(n-2)$-regular graph H_n considered after Proposition 1 satisfies $\beta_{n-3}(H_n) = \beta_{n-2}(H_n) = n - 2$. The $n/2$ partitions (X, Y) obtained by taking for Y a set of two non-adjacent vertices are good and show that $H_n \in \mathcal{H}(\Delta, \Delta)$. Note that $\gamma(H_n) = \gamma_2(H_n) = 2$.

Corollary 6. *Let G be a connected graph with maximum degree Δ and minimum degree $\delta \geq 2$. Then $\beta_{\delta-1}(G) = \beta_\Delta(G)$ if and only if G is regular and $\gamma(G) = \gamma_2(G)$.*

Proof. Suppose $\beta_{\delta-1}(G) = \beta_\Delta(G)$. Then G is regular by Theorem 9 and $\gamma(G) = \gamma_2(G)$ to satisfy the property (b) of the equality case in Theorem 4.
Conversely, suppose G is regular and $\gamma(G) = \gamma_2(G)$. By Theorem 3, $n = \gamma(G) + \beta_\Delta(G) = \gamma_2(G) + \beta_{\Delta-1}(G)$ and thus $\beta_{\Delta-1}(G) = \beta_\Delta(G)$. □

Recall that the *Cartesian product* of two graphs G_1 and G_2 is the graph $G_1 \Box G_2$ with vertex set $V(G_1) \times V(G_2)$ and vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$.

The characterization of graphs G such that $\gamma(G) = \gamma_2(G)$ is only known in some particular classes of graphs. For instance it is proved in [5] that the unique regular claw-free graphs G such that $\gamma(G) = \gamma_2(G)$ are the graph H_n and the Cartesian product of two complete graphs of the same order. This gives the following corollary.

Corollary 7. *Let G be a connected Δ-regular claw-free graph of order n. Then $\beta_{\Delta-1}(G) = \beta_\Delta(G)$ if and only if $G = H_n$ or $G = K_p \Box K_p$ with $p \geq 2$ and $p^2 = n$.*

We finish with some properties of graphs G satisfying $\beta_{\delta-1}(G) = \beta_\Delta(G)$. The first one is a consequence of Corollary 2.

Corollary 8. *Let G is a connected graph of maximum degree $\Delta \leq n-2$ and minimum degree $\delta \geq 2$. If $\beta_{\delta-1}(G) = \beta_\Delta(G)$, then G contains at least $\gamma(G)-1$ induced C_4.*

Proposition 7. *Let G be a connected graph with maximum degree Δ and minimum degree $\delta \geq 3$. If $\beta_{\delta-1}(G) = \beta_\Delta(G)$, then $\gamma(G) \geq \frac{2n}{\Delta+2}$ and $\Delta \geq 4$. In particular, $\beta_2(G) < \beta_3(G)$ for every cubic graph.*

Proof. Let S be a $\beta_{\delta-1}(G)$-set. By Theorem 9, the number $m(S, V-S)$ of edges of G between S and $V-S$ satisfies $2|S| \leq m(S, V-S) = \Delta|V-S|$, and we obtain $\beta_{\delta-1}(G) = |S| \leq \frac{\Delta n}{\Delta+2}$. From Theorem 3 we get $n - \gamma(G) \leq \beta_\Delta(G) = \beta_{\delta-1}(G) \leq \frac{\Delta n}{\Delta+2}$ and thus $\gamma(G) \geq \frac{2n}{\Delta+2}$. By a result of Reed [7], $\gamma(G) \leq \frac{3n}{8}$ if $\delta(G) \geq 3$. Hence $\Delta \geq \lceil \frac{10}{3} \rceil = 4$. □

Lemma 1. *Let $G = (V, E)$ be a graph with maximum degree Δ. Then G has a maximum Δ-independent set S containing a vertex with exactly $\Delta - 1$ neighbors in S.*

Proof. Let S be a $\beta_\Delta(G)$-set. If S contains a vertex of degree $\Delta - 1$ in S, we are done. Otherwise, $d_S(x) \leq \Delta - 2$ for all x in S. Let $y \in V - S$ and u a neighbor of y in S. Since $S \cup \{y\}$ is not Δ-independent, $d_S(y) = \Delta$. Hence the set $S' = (S - \{u\}) \cup \{y\}$ is a $\beta_\Delta(G)$-independent set containing vertex y of degree $\Delta - 1$ in S'. □

Proposition 8. *Let G be a connected graph with maximum degree Δ and minimum degree $\delta \geq 2$. If $\beta_{\delta-1}(G) = \beta_\Delta(G)$, then G has at least two maximum Δ-independent sets.*

Proof. By Theorem 9, $G \in \mathcal{H}(\Delta, \Delta)$. Let (X, Y) be a good partition of G. The set X is a $\beta_\Delta(G)$-independent set of G of maximum degree at most $\Delta - 2$. By the previous lemma, there exists at least one other $\beta_\Delta(G)$-independent set. □

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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