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Research Article

On graphs with integer Sombor indices

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Abstract: Sombor index of a graph G is defined by $SO(G) = \sum_{uv\in E(G)} \sqrt{d_G^2(u) + d_G^2(v)}$, where $d_G(v)$ is the degree of the vertex v of G. An r-degree graph is a graph whose degree sequence includes exactly r distinctive numbers. In this article, we study r-degree connected graphs with integer Sombor index for $r \in \{5, 6, 7\}$. We show that: if G is a 5-degree connected graph of order n with integer Sombor index then $n \geq 50$ and the equality occurs if only if G is a bipartite graph of size 420 with $SO(G) = 14830$; if G is a 6-degree connected graph of order n with integer Sombor index then $n \geq 75$ and the equality is established only for the bipartite graph of size 1080; and if G is a 7-degree connected graph of order n with integer Sombor index then $n \geq 101$ and the equality is established only for the bipartite graph of size 1680.

Keywords: Integer Sombor index; Bipartite graphs; r-degree

AMS Subject classification: 05C07, 05C09

1. Introduction

Throughout this paper G is a simple undirected connected graph of order $n = |V(G)|$ and size $m = |E(G)|$. For any vertex $x \in V(G)$, the *degree* of a vertex x, denoted by $d_G(x)$ (or just d_x), is the number of vertices adjacent to x. We denote by $\Delta(G)$ and $\delta(G)$ the *maximum degree* and *minimum degree* amongst the vertices of G, respectively. A vertex of degree one is called a *pendant* vertex, and an edge incident with a pendant vertex is called a pendant edge. For a positive integer k, a graph G is called k-regular, if every vertex of G has degree k. A *bipartite graph* is a graph whose vertex

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set can be partitioned into two sets, namely partite sets, in such a way that no edge has its both end-vertices in the same partite set. A *complete bipartite* graph is one that every vertex of each partite set is adjacent to all vertices in the other partite set. We denote by $K_{r,s}$ the complete partite set in which one partite set has cardinality r and the other partite set has cardinality s. In particular, $K_{1,r-1}$ is called the *star* of order r. We denote the complete graph, the empty graph, a path and a cycle, all of order n, by K_n, K'_n, P_n and C_n, respectively. For a subgraph H of G, we mean by $G \setminus H$ the graph obtained by removing the edges of H from G.
Topological indices are a numerical quantity computed from the molecular graph of a chemical compound. One of most recent index, namely Sombor index, is introduced by Gutman [7] as a new vertex-degree-based molecular structure, and received much attention in both Mathematics and Chemistry. The Sombor index of a graph G, denoted by $SO(G)$, is defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2} \tag{1}$$

where d_u is the vertex degree of u. For example, for every $n \geq 1$ the Sombor index of the cycle graph C_n is $2n\sqrt{2}$ and the Sombor index of the path P_n is $2\sqrt{5}+2(n-3)\sqrt{2}$. Due to chemical applications of the Sombor index, it is followed by a large number of mathematical studies, most of which dealing with bounds and characterizations of graphs, see for example, [1–4, 8–10, 13–15]. The study of graphs whose Sombor index is integer has taken much interest in recent years, see for example, [5, 11, 12]. It is claimed in [5] that $SO(G)$ is an integer if and only if G is a bipartite semi-regular and its degrees δ and Δ appear as non-maximal elements in some Pythagorean triple". Oboudi [11] showed that the "only if" part of the above claim is not true. Oboudi [11] constructed infinite number of connected bipartite graphs such that in their degree sequence there are three or four distinct numbers. In [12] he studied r-degree graphs whose Sombor index is integer. He characterized all 1-degree and 2-degree graphs with integer Sombor index. He showed that if G is a 3-degree connected graph of order n with integer Sombor index then $n \geq 25$ and if G is a 4-degree connected graph of order n with integer Sombor index, then $n \geq 30$.
In this paper, we continue the study of graphs with integer Sombor index for 5-degree, 6-degree and 7-degree connected graphs. We show that if G is a 5-degree connected graph of order n with integer Sombor index, then $n \geq 50$ and the equality occurs if only if G is a bipartite graph of size 420 with $SO(G) = 14830$. We also show that if G is a 6-degree connected graph of order n with integer Sombor index, then $n \geq 75$ and the equality is established only for the bipartite graph of size 1080. We next show that if G is a 7-degree connected graph of order n with integer Sombor index, then $n \geq 101$ and the equality is established only for the bipartite graph of size 1680.
We make use of the following.

Theorem 1. *[6] Let $n_1, \ldots, n_k$ be positive integers. Then $\sqrt{n_1} + \cdots + \sqrt{n_k}$ is integer if and only if all $n_1, \ldots, n_k$ are squares.*

2. Five-degree graphs with integer Sombor index

In this section we present a lower bound for the order of 5-degree graphs whose Sombor index is integer, and characterize 5-degree graphs with integer Sombor index and minimum order. Let G be the graph depicted in Figure 1. It can be seen that G is a bipartite graph of size 420 and Sombor index 14830.

Theorem 2. *If G is a 5-degree graph of order n with integer Sombor index, then $n \geq 50$, with equality if and only if G is the graph depicted in Figure 1.*

Proof. Let a, b, c, d and e be the five distinct values of the degree sequence of G. Clearly $n > \max\{a, b, c, d, e\}$. Let V_1, V_2, V_3, V_4 and V_5 be the sets of all vertices of G of degrees a, b, c, d and e, respectively. If there are two adjacent vertices in one of these five sets then they produce a $\sqrt{2}$ in $SO(G)$, which is a contradiction. Thus V_i is an independent set for $i = 1, 2, 3, 4, 5$. Set $S = \{u_1, u_2, u_3, u_4, u_5\}$ and let $\Lambda(G)$ be the graph on S such that u_i and u_j are adjacent if and only if there is at least one edge in G between the parts V_i and V_j . Since G is connected, $\Lambda(G)$ is also connected. Thus $\Lambda(G)$ is one of the graphs

$$\{K_{1,4},\ K_{2,3},\ P_5,\ K_5,\ K_5\backslash e,\ K_5\backslash 2e,\ K_5\backslash 3e,\ K_5\backslash 4e,\ C_5,\ H_5,\ H_5\backslash e,\ H_5\backslash 2e,\ R_5\}, \tag{2}$$

where H_5 is a graph obtained by adding a pendent edge to one of the vertices of K_4. In addition, R_5 is the graph constructed by adding two pendent edges to one or two of the vertices of K_3 and $K_5\backslash ie$ is the graph constructed by removing (i) edges $(i = 1, 2, 3, 4)$ from K_5, and in $H_5\backslash ie (i = 1, 2)$, e is not a pendant edge. We proceed according to each possibility of $\Lambda(G)$.

Case 1. $\Lambda(G) = K_{1,4}$. Without loss of generality, suppose that $|V_1| \geqslant |V_i| (i = 2, \ldots, 6)$ in $\Lambda(G)$, and note that $V_2 \cup V_3 \cup V_4 \cup V_5$ is an independent set in G, and all $a^2 + b^2$, $a^2 + c^2$, $a^2 + d^2$ and $a^2 + e^2$ are squares, since $\Lambda(G) = K_{1,4}$. Thus, G is a bipartite graph with partite sets V_1 and $V_2 \cup V_3 \cup V_4 \cup V_5$. Focusing on those a, b, c, d, e where $1 \leq a, b, c, d, e \leq 100$, by using $MATLAB$, we obtain that if $a^2 + b^2, a^2 + c^2, a^2 + d^2, a^2 + e^2$ are squares such that $b > c > d > e$, then (a, b, c, d, e) is one of the following: (12, 35, 16, 9, 5), (20, 99, 48, 21,15), (24, 32, 18, 10, 7), (24, 45, 32, 18, 10), (24, 70,45, 32, 18), (36, 77, 48, 27, 15), (40, 75, 42, 30, 9), (40, 96, 75, 42, 30), (48, 55, 36, 20, 14), (48, 64, 55, 36, 20), (48, 90, 64, 55, 36), (60, 45, 32, 25, 11), (60, 63, 45, 32, 25), (60, 80, 63, 45, 32), (60, 91, 80, 63, 45), (72, 62, 54, 30, 21), (72, 96, 65, 54, 30), (80, 84, 60, 39, 18), (84, 80, 63, 35, 13).
Observe that:

$$|V_1| \geq \max\{b, c, d, e\} \textit{ and } |V_2| + |V_3| + |V_4| + |V_5| \geq a. \tag{3}$$

By enumerating the edges of G we have:

$$a|V_1| = b|V_2| + c|V_3| + d|V4| + e|V_5|. \tag{4}$$

In addition

$$n = |V_1| + |V_2| + |V_3| + |V_4| + |V_5| \geq a + \max\{b, c, d, e\}. \tag{5}$$

We now show that $n \geq 50$. Clearly we only need to consider the case that $1 \leq a, b, c, d, e \leq 49$. It can be seen that for (a, b, c, d, e) with $b > c > d > e$ where all of $a^2 + b^2$, $a^2 + c^2$, $a^2 + d^2$ and $a^2 + e^2$ are squares, from all possibilities denoted above, there are only the following three possibilities for (a, b, c, d, e):
$(a, b, c, d, e) = (12, 35, 16, 9, 5)$ *or* $(24, 32, 18, 10, 7)$ *or* $(24, 45, 32, 18, 10)$.
By (4), if $(a, b, c, d, e) = (12, 35, 16, 9, 5)$ then $n \geq 50$, if $(a, b, c, d, e) = (24, 32, 18, 10, 7)$ then, $n \geq 56$, and if $(a, b, c, d, e) = (24, 45, 32, 18, 10)$ then $n \geq 69$, as desired.

Case 2. $\Lambda(G) = P_5$. Without loss of generality, assume that $\Lambda(G) = V_1 - V_2 - V_3 - V_4 - V_5$, that is, the edges of G are between V_1 and V_2, V_2 and V_3, V_3 and V_4 and V_4 and V_5. Since $SO(G)$ is an integer, by Theorem 1 we obtain $a^2 + b^2$, $b^2 + c^2$, $c^2 + d^2$ and $d^2 + e^2$ are squares. Focusing on those a, b, c, d, e where $1 \leq a, b, c, d, e \leq 100$, by using $MATLAB$ we obtain that if $a^2 + b^2$, $b^2 + c^2$, $c^2 + d^2$ and $d^2 + e^2$ are squares then (a, b, c, d, e) is one of the following: (5, 12, 9, 40, 30), (5, 12, 9, 40, 42), (5, 12, 16, 30, 40), (6, 8, 15, 20, 21), (6, 8, 15, 20, 48), (6, 8, 15, 36, 27), (6, 8, 15, 36, 48), (7, 24, 45, 28, 21),(8, 15, 20, 21, 28), (8, 15, 20, 48, 14), (8, 15, 20, 48, 36), (8, 15, 20, 48, 55), (8, 15, 36, 48, 14), (8, 15, 36, 48, 20), (9, 12, 16, 30, 40), (9, 40, 30, 16, 12), (10, 24, 45, 28, 21), (12, 9, 40, 30, 16), (12, 16, 30, 40, 9), (12, 16, 30, 40, 42), (14, 48, 20, 15, 36), (14, 48, 20, 21, 28), (14, 48, 36, 15, 8),(14, 48, 20, 15, 8), (15, 20, 21, 28, 45), (15, 20, 48, 36, 27), (15, 36, 48, 20, 21), (16, 12, 9, 40, 30), (16, 12, 9, 40, 42), (16, 30, 40, 9, 12), (18, 24, 45, 28, 21), (20, 15, 36, 48, 14), (20, 21, 28, 45, 24), (20, 48, 36, 15, 8), (21, 20, 15, 8, 6), (21, 20, 15, 36, 27), (21, 28, 45, 24, 7), (21, 28, 45, 24, 10), and $(a, b, c, d, e)'$s with $a \geq 21$ and $b + c + d > 50$.
Considering the structure of G and $\Lambda(G)$, we find that:

$$|V_2| \geq a, \ |V_4| \geq e, \ |V_1| + |V_3| \geq b, \ |V_2| + |V_4| \geq c \ \textit{and} \ |V_3| + |V_5| \geq d. \tag{6}$$

By enumerating the edges of G, we find that:

$$a|V_1| + c|V_3| + e|V_5| = b|V_2| + d|V_4|. \tag{7}$$

On the other hand

$$n = |V_1| + |V_2| + |V_3| + |V_4| + |V_5| \geq b + c + d. \tag{8}$$

We now show that $n \geq 50$. Clearly we only need to consider the case that $1 \leq a, b, c, d, e \leq 49$. From (6), (7) and (8), we obtain that:
If $a > c$ then (a, b, c, d, e) is one of the following: $(16, 12, 9, 40, 30)$, $(16, 12, 9, 40, 42)$, $(21, 20, 15, 8, 6)$,
$(21, 20, 15, 36, 27)$ or $(21, 20, 15, 36, 48)$.
If $a > d$ then (a, b, c, d, e) is one of the following: $(16, 30, 40, 9, 12)$, $(20, 48, 36, 15, 8)$ or $(21, 20, 15, 8, 6)$.
If $a > e$ then (a, b, c, d, e) is one of the $(12, 16, 30, 40, 9)$, $(14, 48, 20, 15, 8)$, $(14, 48, 36, 15, 8)$, $(16, 30, 40, 9, 12)$, $(20, 15, 36, 48, 14)$ or $(21, 20, 15, 8, 6)$.

Now it can be seen that in $(a, b, c, d, e) = (16, 30, 40, 9, 12)$ by (8) we arrive $n \geq 79$, in $(a, b, c, d, e) = (12, 16, 30, 40, 9)$ we arrive $n \geq 86$, and in $(a, b, c, d, e) = (21, 20, 15, 8, 6)$, by (7) and (8), there is no possibility for $|V_1|, |V_2|, |V_3|, |V_4|$ and $|V_5|$ such that $|V_1| + |V_2| + |V_3| + |V_4| + |V_5| \leq 62$. We deduce that $n \geq 63 > 50$.

Case 3. $\Lambda(G) = K_{2,3}$. Without loss of generality, assume that there $\{V_1, V_2\}$ and $\{V_3, V_4, V_5\}$ are partite sets of $\Lambda(G)$. Thus, there is at least one edge between V_j $(j = 1, 2)$ and any parts V_i , for $i = \{3, 4, 5\}$. Since $SO(G)$ is an integer, by Theorem 1 we obtain that all of $a^2 + c^2$, $a^2 + d^2$, $a^2 + e^2$, $b^2 + c^2$, $b^2 + d^2$ and $b^2 + e^2$ are squares. Then as before, using $MATLAB$ we obtain that there is no possibility for (a, b, c, d, e) such that $1 \leq a, b, c, d, e \leq 50$. Consequently, $n > 50$.

Case 4. $\Lambda(G) \in \{ K_5, K_5\backslash e, K_5\backslash 2e, K_5\backslash 3e, K_5\backslash 4e, C_5, H_5, H_5\backslash e, H_5\backslash 2e, R_5\}$. Then $\Lambda(G)$ has a triangle. Without loss of generality, assume that there are some edges between the parts $V_3V_4V_5$ is a triangle of $\Lambda(G)$. Since $SO(G)$ is an integer, by Theorem 1, all of $c^2 + d^2, c^2 + e^2$, and $d^2 + e^2$ are squares. Now using $MATLAB$ we obtain that for $c \leq 1000$, there are ten possibilities for (c, d, e) which are:

$$\begin{gathered}(240, 117, 44), (275, 252, 240), (480, 234, 88), (550, 504, 480), (693, 480, 140), \\ (720, 132, 85), (720, 351, 132), (792, 231, 160), (825, 756, 720), (960, 468, 176).\end{gathered} \tag{9}$$

all of which confirm that $n \geq 241 > 50$.

We conclude that $n \geq 50$. Now we show there is exactly one graph of order 50 whose Sombor index is minimum. Following the above cases, we obtain that the only possibility occurs in the Case 1. (for $\Lambda(G) = K_{1,4}$) that is $(a, b, c, d, e) = (12, 35, 16, 9, 5)$, where $|V_1| = 35$, $|V_2| = 11$, $|V_3| = 1$, $|V_4| = 1$ and $|V_5| = 2$. Let $V_1 = \{a_1, \ldots, a_{35}\}$, $V_2 = \{b_1, \ldots, b_{11}\}$, $V_3 = \{c\}$, $V_4 = \{d\}$ and $V_5 = \{e_1, e_2\}$. Since each vertex of V_2 is adjacent to some vertices in V_1 and every vertex of V_2 has degree 35, from $|V_1| = 35$ we find that each vertex of V_2 is adjacent to all vertices of V_1. Similarly, each of vertices c and d are only adjacent to vertices of V_1. Thus, c is adjacent to 16 vertices of V_1 and d is adjacent to 9 vertices of V_1. Without loss of generality, assume that c is adjacent to $a_1, \ldots, a_{16}$ and d is adjacent to $a_{17}, \ldots, a_{25}$. Since every vertex of V_1 has degree 12, there is no edge between a_i and V_5 for $i = 1, 2, \ldots, 25$. Now the vertices of V_5 can only be adjacent to some vertices in $\{a_{26}, \ldots, a_{35}\}$. Since each of e_1 and e_2 has degree five, we may assume that e_1 is adjacent to $a_{26}, \ldots, a_{30}$ and e_2 is adjacent to $a_{31}, \ldots, a_{35}$. Consequently, G is the graph depicted in Figure 1. □

3. Six-degree graphs with integer Sombor index

In this section we present a lower bound for the order of 6-degree graphs whose Sombor index is integer, and then give a description for all extremal graphs achieving the equality of the bound. We need to introduce the following families of graphs of order 75. Let $\mathcal{G}_{75}$ be the family of simple graphs of order 75 with 6 distinct degrees $(d_1, d_2, d_3, d_4, d_5, d_6) = (24, 45, 32, 18, 10, 7)$ such that the vertices of the same degrees are independent, there are 45 vertices of degree d_1, there are 15 vertices of degree d_2,

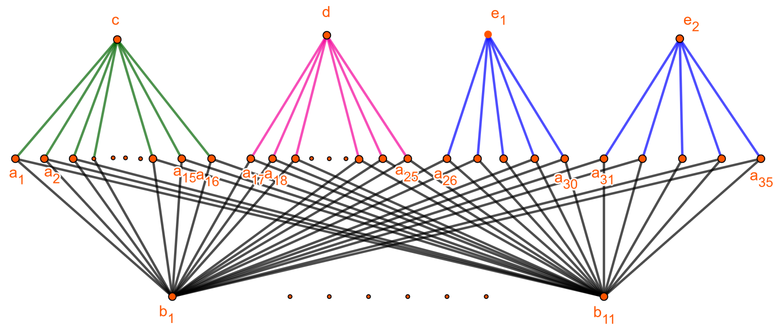

Figure 1. **A graph G of order 50, size 420 and Sombor index 14830.**

there are 11 vertices of degree d_3, there are 2 vertices of degree d_4, there is 1 vertex of degree d_5, there is 1 vertex of degree d_6, and every vertex of degree d_1 is adjacent to all vertices of degrees $d_2, \ldots, d_6$. Figure 2 illustrates a graph in $\mathcal{G}_{75}$.

Lemma 1. *If $G \in \mathcal{G}_{75}$, then $SO(G)$ is an integer.*

Proof. Let $G \in \mathcal{G}_{75}$. Then we may assume that $V(G) = \{x_1, x_2, \ldots, x_{75}\} = \bigcup_{i=1}^{6} V_i = \{x_1, \ldots, x_{45}\} \cup \{x_{46}, \ldots, x_{60}\} \cup \{x_{61}, \ldots, x_{71}\} \cup \{x_{72}, x_{73}\} \cup \{x_{74}\} \cup \{x_{75}\}$, where V_i is an independent set for each $i = 2, \ldots, 6$ and x_i is adjacent to x_j for all $i = 1, 2, \ldots, 45$ and $j = 46, \ldots, 75$. Since every vertex of V_2 has degree 45, from $|V_1| = 45$ we find that each vertex of V_2 is adjacent to all vertices of V_1. Similarly, each vertex of V_i $(i = 2, \ldots, 6)$ is adjacent to all vertices of V_1. Without loss of generality, assume that the adjacent vertices are as the following:
x_{46} is adjacent to $\{x_{13}, \ldots, x_{19}, x_{21}, \ldots, x_{45}\}$, x_{47} is adjacent to $\{x_{13}, \ldots, x_{20}, x_{22}, \ldots, x_{45}\}$, x_{48} is adjacent to $\{x_{13}, \ldots, x_{21}, x_{23}, \ldots, x_{45}\}$, x_{49} is adjacent to $\{x_1, \ldots, x_{12}, x_{14}, x_{15}, x_{17}, \ldots, x_{22}, x_{24}, \ldots, x_{32}, x_{40}, x_{43}, x_{45}\}$, x_{50} is adjacent to $\{x_1, \ldots, x_{13}, x_{15}, x_{16}, x_{18}, \ldots, x_{23}, x_{25}, \ldots, x_{32}, x_{41}, x_{43}, x_{45}\}$, x_{51} is adjacent to $\{x_1, \ldots, x_{14}, x_{16}, \ldots, x_{24}, x_{26}, \ldots, x_{32}, x_{42}, x_{45}\}$, x_{52} is adjacent to $\{x_1, \ldots, x_{17}, x_{19}, \ldots, x_{25}, x_{27}, \ldots, x_{32}, x_{44}, x_{45}\}$, x_{53} is adjacent to $\{x_1, \ldots, x_{18}, x_{20}, \ldots, x_{26}, x_{28}, \ldots, x_{32}, x_{44}, x_{45}\}$, x_{54} is adjacent to $\{x_1, \ldots, x_{27}, x_{29}, \ldots, x_{32}, x_{45}\}$, x_{55} is adjacent to $\{x_1, \ldots, x_{22}, x_{33}, \ldots, x_{44}\}$, x_{56} is adjacent to $\{x_1, \ldots, x_{12}, x_{21}, \ldots, x_{28}, x_{33}, \ldots, x_{44}\}$, x_{57} is adjacent to $\{x_1, \ldots, x_5, x_{33}, \ldots, x_{44}\}$, x_{58} is adjacent to $\{x_7, \ldots, x_{12}, x_{33}, \ldots, x_{44}\}$, x_{59} is adjacent to $\{x_{33}, \ldots, x_{42}\}$ and x_{60} is adjacent to $\{x_{33}, \ldots, x_{39}\}$. Now:

$$SO(G) = \sum_{x_i x_j \in E(G)} \sqrt{d_{x_i}^2 + d_{x_j}^2} = 15 \times 45\sqrt{(45)^2 + (24)^2} + 11 \times 32\sqrt{(32)^2 + (24)^2}$$
$$+ 2 \times 18\sqrt{(18)^2 + (24)^2} + 1 \times 10\sqrt{(10)^2 + (24)^2} + 1 \times 7\sqrt{(7)^2 + (24)^2} = 50020$$

which is an integer. See the Figure 2 (above). □

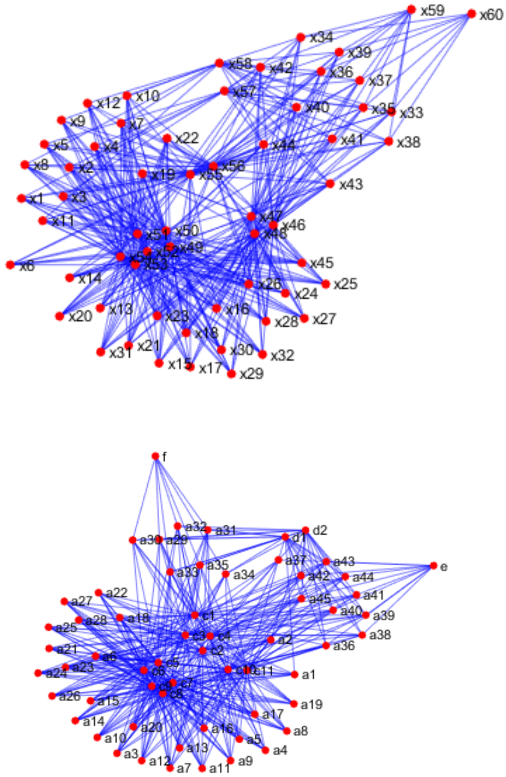

Figure 2. **Two graphs from the family $\mathcal{G}_{75}$.**

Theorem 3. *If G is a 6-degree graph of order n with integer Sombor index, then $n \geq 75$, with equality if and only if $G \in \mathcal{G}_{75}$.*

Proof. Let a, b, c, d, e and f be the six distinct values of the degree sequence of G. Clearly $n > \max\{a, b, c, d, e, f\}$. Let V_1, V_2, V_3, V_4, V_5 and V_6 be the sets of all vertices of G of degrees a, b, c, d, e and f, respectively. If there are two adjacent vertices in one of these six sets then they produce a $\sqrt{2}$ in $SO(G)$, which is a contradiction. Thus V_i is an independent set for $i = 1, 2, \ldots, 6$. Let $S = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $\Lambda(G)$ be the graph on S such that u_i and u_j are adjacent if and only if there is at least one edge in G between the parts V_i and V_j. Since G is connected, $\Lambda(G)$ is also connected. Thus $\Lambda(G)$ is one of the next graphs:

$$K_{1,5},\ K_{2,4},\ K_{3,3},\ K_6,\ K_6\backslash e,\ K_6\backslash 2e,\ \ldots,\ K_6\backslash 9e,\ C_6,\ P_6, H_6,\ H_6\backslash e, H_6\backslash 2e, \ldots, H_6\backslash 5e,\ R_6,\ R_6\backslash e,\ R_6\backslash 2e,\ Z_6, \tag{10}$$

where H_6 is the graph obtained by adding a pendent edge to one of the vertices of K_5, R_6 is a graph constructed by adding two pendent edges to one or two of the vertices of K_4, Z_6 is a graph obtained by adding three pendent edges to the vertices of K_3, $K_6\backslash ie$ is the graph constructed by removing (i) edges $(i = 1, 2, \ldots, 9)$ from K_6, and in $H_6\backslash ie$ and $R_6\backslash ie$, e is not a pendant edge. We proceed according to each possibility of $\Lambda(G)$.

Case 1. $\Lambda(G) = K_{1,5}$.

Without loss of generality, assume that $|V_1| \geqslant |V_i| (i = 2, \ldots, 6)$ in $\Lambda(G)$, and note that $V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6$ is an independent set. Clearly, there is at least one edge between V_1 and any parts V_i, for $i = 2, \ldots, 6$. Assume that $b > c > d > e > f$. Since $SO(G)$ is integer, by Theorem 1, all of a^2+b^2, a^2+c^2, a^2+d^2, a^2+e^2 and a^2+f^2 are squares. Hence G is a bipartite graph with partite sets V_1 and $V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6$. Focusing on those a, b, c, d, e where $1 \le a, b, c, d, e \le 150$, by using $MATLAB$, we obtain that if $a^2 + b^2$, $a^2 + c^2$, $a^2 + d^2$, $a^2 + e^2$ and $a^2 + f^2$ are squares such that $b > c > d > e > f$, then (a, b, c, d, e, f) is one of the following: (24, 45, 32, 18, 10, 7), (24, 70, 45, 32, 18, 10), (24, 143, 70, 45, 32, 18), (36, 105, 77, 48, 27, 15), (36, 160, 105, 77, 48, 27), (40, 96, 75, 42, 30, 9), (40, 198, 96, 75, 42, 30), (45, 200, 108, 60, 28, 24), (48, 64, 55, 36, 20, 14), (48, 90, 64, 55, 36, 20), (48, 140, 90, 64, 55, 36), (48, 189, 140, 90, 64, 55), (56, 192, 105, 90, 42, 33), (60, 63, 45, 32, 25, 11), (60, 80, 63, 45, 32, 25), (60, 91, 80, 63, 45, 32), (60, 144, 91, 80, 63, 45), (72, 96, 65, 54, 30, 21), (72, 135, 96, 65, 54, 30), (80, 150, 84, 60, 39, 18), (84, 112, 80, 63, 35, 13), (84, 135, 112, 80, 63, 35), (96, 128, 110, 72, 40, 28), (105, 140, 100, 88, 56, 36), (120, 64, 50, 35, 27, 22), (120, 90, 64, 50, 35, 27), (120, 119, 90, 64, 50, 35), (120, 126, 119, 90, 64, 50), (144, 130, 108, 60, 42, 17).

By considering the structure of G and $\Lambda(G)$ we find that:

$$|V_1| \geq \max\{b, c, d, e, f\} \ and \ |V_2| + |V_3| + |V_4| + |V_5| + |V_6| \geq a. \tag{11}$$

By enumerating the edges of G we find that:

$$a|V_1| = b|V_2| + c|V_3| + d|V4| + e|V_5| + f|V_6| \tag{12}$$

In addition by equation (11) we have:

$$n = |V_1| + |V_2| + |V_3| + |V_4| + V_5| + |V_6| \geq a + \max\{b, c, d, e, f\} = a + b \tag{13}$$

Focusing on those (a, b, c, d, e, f) where $1 \le a, b, c, d, e, f \le 74$, we can easily see that if $b > c > d > e > f$ and all of $a^2 + b^2$, $a^2 + c^2$, $a^2 + d^2$, $a^2 + e^2$ and $a^2 + f^2$ are squares, then from all possibilities denoted above, we obtain that (a, b, c, d, e, f) is $(24, 45, 32, 18, 10, 7)$, $(48, 64, 55, 36, 20, 14)$ or $(60, 63, 45, 32, 25, 11)$.

If $(a, b, c, d, e, f) = (48, 64, 55, 36, 20, 14)$, then by (13), $n \geq 110$, if $(a, b, c, d, e, f) = (60, 63, 45, 32, 25, 11)$ then $n \geq 123$, and if $(a, b, c, d, e, f) = (24, 45, 32, 18, 10, 7)$ then $n \geq 75$.

Case 2. $\Lambda(G) = P_6$.

Without loss of generality, suppose that $\Lambda(G) = V_1 - V_2 - V_3 - V_4 - V_5 - V_6$. Since $SO(G)$ is integer, by Theorem 1

$$a^2+b^2, b^2+c^2, c^2+d^2, d^2+e^2, e^2+f^2 \tag{14}$$

are squares. Focusing on those a,b,c,d,e,f where $1 \leq a,b,c,d,e,f \leq 74$, by using $MATLAB$, we obtain that (a,b,c,d,e,f) have the following forms: (5, 12, 9, 40, 30, 16), (5, 12, 9, 40, 30, 72), (5, 12, 9, 40, 42, 56), (5, 12, 16, 30, 40, 9),(5, 12, 16, 30, 40, 42), (5, 12, 16, 63, 60, 11), (6, 8, 15, 20, 21, 28), (6, 8, 15, 20, 21, 72), (6, 8, 15, 20, 48, 14), (6, 8, 15, 20, 48, 36), (6, 8, 15, 20, 48, 55), (6, 8, 15, 20, 48, 64), (7, 24, 45, 28, 21, 20), (7, 24, 45, 28, 21, 72), (7, 24, 45, 60, 63, 16), (8, 15, 20, 21, 28, 45), (9, 12, 16, 30, 40, 42), (9, 40, 30, 16, 12, 5), (10, 24, 32, 60, 45, 28), (10, 24, 32, 60, 63, 16), (11, 60, 63, 16, 12, 5), (11, 60, 63, 16, 12, 9), (12, 9, 40, 30, 16, 63), (12, 9, 40, 42, 56, 33), (12, 16, 30, 40, 42, 56), (14, 48, 20, 15, 8, 6), (15, 20, 21, 28, 45, 60), (16, 12, 9, 40, 42, 56), (16, 30, 40, 9, 12, 5), (16, 63, 60, 45, 24, 7), (18, 24, 32, 60, 45, 28), (18, 24, 45, 60, 63, 16), (20, 21, 28, 45, 24, 7), (20, 21, 28, 45, 24, 10), (21, 20, 48, 36, 15, 8), (21, 20, 15, 36, 48, 14), (24, 32, 60, 63, 16, 12), (24, 45, 28, 21, 20, 15), (24, 45, 28, 21, 72, 30) and (a,b,c,d,e,f)'s those where $a > 24, a+b+e+f > 75$. Considering the structure of G and $\Lambda(G)$ we find that:

$$|V_2| \geq a,\ |V_5| \geq f,\ |V_1|+|V_3| \geq b,\ |V_2|+|V_4| \geq c \text{ and } |V_3|+|V_5| \geq d \text{ and } |V_4|+|V_6| \geq e. \tag{15}$$

By enumerating the edges of G, we find that:

$$a|V_1| + c|V_3| + e|V_5| = b|V_2| + d|V_4| + f|V_6|. \tag{16}$$

On the other hand

$$n = |V_1|+|V_2|+|V_3|+|V_4|+|V_5|+|V_6| \geq a+b+e+f. \tag{17}$$

Now we demonstrate $n \geq 75$. Clearly we only need to consider the case that $1 \leq a,b,c,d,e,f \leq 74$. From (15), (16), (17), we obtain that:
if $a > c$ then $(a,b,c,d,e,f) =$ (16, 12, 9, 40, 42, 56) or (21, 20, 15, 36, 48, 14)
or if $a > d$ then: $(a,b,c,d,e,f) =$ (16, 30, 40, 9, 12, 5) or (24, 45, 28, 21, 20, 15)
or if $a > e$ then: $(a,b,c,d,e,f) =$ (24, 32, 60, 63, 16, 12) or (14, 48, 20, 15, 8, 6)
or if $a > f$ then: $(a,b,c,d,e,f) =$ (21, 20, 48, 36, 15, 8) or (9, 40, 30, 16, 12, 5) or (16, 63, 60, 45, 24, 7).
Now it can be seen that in $(a,b,c,d,e,f) = (5,12,9,40,30,16)$ or in $(a,b,c,d,e,f) = (6,8,15,20,21,28)$ or in $(a,b,c,d,e,f) = (21,20,48,36,15,8)$ by (15), there is no answer for $n \leq 150$, $n \leq 127$ and $n \geq 130$, respectively.
If $(a,b,c,d,e,f) = (16,12,9,40,42,56)$, then by (15) we arrive $n \geq 100$, in $(a,b,c,d,e,f) = (24,45,28,21,\ 20,\ 15)$ we arrive $n \geq 104$, in $(a,b,c,d,e,f) = (16,30,40,9,12,5)$, by (15), (16) there is no possibility for $|V_1|$, $|V_2|$, $|V_3|$, $|V_4|$, $|V_5|$ and $|V_6|$ such that $|V_1|+|V_2|+|V_3|+|V_4|+|V_5|+|V_6| \leq 92$. In $(a,b,c,d,e,f) =$

$(14, 48, 20, 15, 8, 6)$, by using (15) and (16), we have $n \geq 76$. Then we deduce that $n \geq 76 > 75$.

Case 3. $\Lambda(G) = K_{2,4}$.
Without loss of generality, suppose that there $\{V_1, V_2\}$ with $\{V_3, V_4, V_5, V_6\}$ are partite sets of $\Lambda(G)$. Thus, there is at least one edge betweenV_j $(j = 1, 2)$ and any parts V_i , for $i = \{3, 4, 5, 6\}$. Since SO(G) is an integer, by Theorem 1, we obtain that all of $a^2+c^2, a^2+d^2, a^2+e^2, a^2+f^2, b^2+c^2, b^2+d^2, b^2+e^2$ and b^2+f^2 are squares. Then as before, using $MATLAB$ we obtain that there is no possibility for (a, b, c, d, e, f) such that $1 \leq a, b, c, d, e, f \leq 75$. Consequently, $n > 75$.

Case 4. $\Lambda(G) = K_{3,3}$.
Without loss of generality, suppose $\{V_1, V_2, V_3\}$ and $\{V_4, V_5, V_6\}$ are partite sets of $\Lambda(G)$. Thus, there is at least one edge between V_j $(j = 1, 2, 3)$ and any parts V_i , for $i = \{4, 5, 6\}$. Since $SO(G)$ is integer, by Theorem 1, $a^2+d^2, a^2+e^2, a^2+f^2, b^2+d^2, b^2+e^2, b^2+f^2, c^2+d^2, c^2+e^2$ and c^2+f^2 are squares. Similarity to the case 3, by using $MATLAB$ we obtain that there is no possibility for (a, b, c, d, e, f) such that $1 \leq a, b, c, d, e, f \leq 75$. Consequently, $n > 75$.

Case 5. $\Lambda(G) \in \{K_6,\ K_6\backslash e,\ K_6\backslash 2e,\ K_6\backslash 3e,\ K_6\backslash 4e,\ K_6\backslash 5e,\ \ldots,\ K_6\backslash 9e,\ C_6,\ P_6,\ H_6, H_6\backslash e,\ \ldots,\ H_6\backslash 5e,\ R_6,\ R_6\backslash e,\ R_6\backslash 2e,\ Z_6\}$.
Then $\Lambda(G)$ has a triangle. Without loss of generality, assume that there are some edges between the parts $V_4V_5V_6$ is a triangle of $\Lambda(G)$. Since $SO(G)$ is integer, by Theorem 1, all of d^2+e^2, e^2+f^2, and d^2+f^2 are squares. According to the (9) we have $n \geq 241 > 75$. We conclude that $n \geq 75$.
We next prove the equality part. Assume that G is a 6-degree graph of order $n = 75$. Following the above cases, we obtain that the possibility occurs in the Case 1. (For $\Lambda(G) = K_{1,5}$) that is $(a, b, c, d, e, f) = (24, 45, 32, 18, 10, 7)$, where $|V_1| = 45, |V_2| = 18, |V_3| = 6, |V_4| = 3, |V_5| = 1, |V_6| = 2$ or $|V_1| = 45, |V_2| = 15, |V_3| = 11, |V_4| = 2, |V_5| = 1, |V_6| = 1$.
For the first case we have $SO(G) = 51220$ and second case we have $SO(G) = 50020$. Thus we model the second case. In this case, G will be a graph from the family of $\mathcal{G}_{75}$, so according to the Lemma 1 $SO(G)$ is integer.
Now, let $V_1 = \{a_1, \ldots, a_{45}\}$, $V_2 = \{b_1, \ldots, b_{15}\}$, $V_3 = \{c_1, \ldots, c_{11}\}$, $V_4 = \{d_1, d_2\}$, $V_5 = \{e\}$ and $V_6 = \{f\}$. Since each vertex of V_2 is adjacent to some vertices in V_1 and every vertex of V_2 has degree 45, from $|V_1| = 45$ we find that each vertex of V_2 is adjacent to all vertices of V_1. Similarly, each of vertices c_i, d_i, e and f are only adjacent to vertices of V_1. Thus we may assume that c_i $(i = 1, \ldots, 11)$ is adjacent to 32 vertices of V_1 as:
c_1 is adjacent to $\{a_{14}, \ldots, a_{45}\}$, c_2 is adjacent to $\{a_1, \ldots, a_{13}, a_{22}, \ldots, a_{32}, a_{37}, \ldots, a_{44}\}$, c_3 is adjacent to $\{a_{10}, \ldots, a_{32}, a_{37}, \ldots, a_{45}\}$, c_4 is adjacent to $\{a_1, \ldots, a_9, a_{19}, \ldots, a_{32}, a_{37}, \ldots, a_{45}\}$, c_5 is adjacent to $\{a_1, \ldots, a_{18}, a_{20}, \ldots, a_{32}, a_{45}\}$, c_6 is adjacent to $\{a_1, \ldots, a_{32}\}$, c_7 is adjacent to $\{a_1, \ldots, a_{28}, a_{33}, \ldots, a_{36}\}$, c_8 is adjacent to $\{a_1, \ldots, a_{28}, a_{33}, \ldots, a_{36}\}$, c_9 is adjacent to $\{a_1, \ldots, a_{28}, a_{33}, \ldots, a_{36}\}$, c_{10} is adjacent to $\{a_3, \ldots, a_{21}, a_{33}, \ldots, a_{45}\}$, c_{11} is adjacent to $\{a_1, \ldots, a_{19}, a_{33}, \ldots, a_{45}\}$. Also we may assume that d_1 is adjacent to

18 vertices in $\{a_1, a_{29}, \ldots, a_{45}\}$, d_2 is adjacent to to 18 vertices in $\{a_2, a_{29}, \ldots, a_{45}\}$, e is adjacent to 10 vertices in $\{a_{36}, \ldots, a_{45}\}$ and f is adjacent to 7 vertices in $\{a_{29}, \ldots, a_{35}\}$. Consequently, $G \in \mathcal{G}_{75}$. The converse follows from Lemma 1. See the Figure 2 (below). □

4. Seven-degree graphs with integer Sombor index

In this section we present a lower bound for the order of 7-degree graphs whose Sombor index is integer, and then give a description for all extremal graphs achieving the equality of the bound. At first we introduce the following families of graphs of order 101. Let $\mathcal{G}_{101}$ be the family of simple graphs of order 101 with 7 distinct degrees $(d_1, d_2, d_3, d_4, d_5, d_6, d_7) = (24, 70, 45, 32, 18, 10, 7)$ such that the vertices of the same degrees are independent, there are 70 vertices of degree d_1, 17 vertices of degree d_2, 9 vertices of degree d_3, 2 vertices of degree d_5, there is 1 vertex of degree d_4, 1 vertex of degree d_6, 1 vertex of degree d_7 and every vertex of degree d_1 is adjacent to all vertices of degrees $d_2, \ldots, d_7$. Figure 3 illustrates a graph in $\mathcal{G}_{101}$.

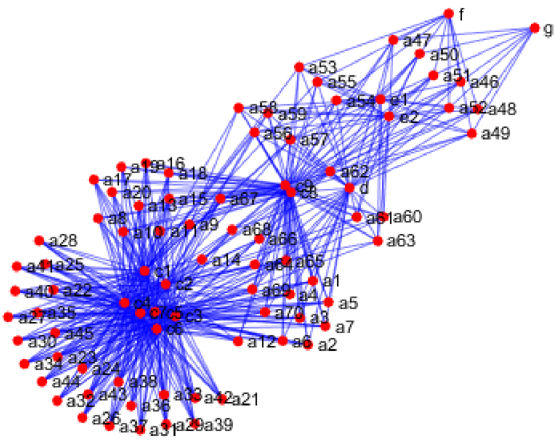

Figure 3. A graph G from the family of $\mathcal{G}_{101}$.

Lemma 2. *If $G \in \mathcal{G}_{101}$, then $SO(G)$ is an integer.*

Proof. Let $G \in \mathcal{G}_{101}$. Then we may assume that $V(G) = \bigcup_{i=1}^{7}(V_i) = \{a_1, \ldots, a_{70}\} \cup \{b_1, \ldots, b_{17}\} \cup \{c_1, \ldots, c_9\} \cup \{d\} \cup \{e_1, e_2\} \cup \{f\} \cup \{g\}$, where V_i is an independent set for each $i = 2, \ldots, 7$, and a_i is adjacent to all vertices of V_i $(i = 2, \ldots, 7)$. Since every vertex of V_2 has degree 70, from $|V_1| = 70$ we find that each vertex of V_2 is adjacent to all vertices of V_1. Similarly, each vertex of V_i $(i = 3, \ldots, 7)$ is adjacent to all vertices of V_1, and without loss of generality, we may assume that the adjacent vertices are as follows: c_1 is adjacent to $\{a_8, \dots, a_{45}, a_{53}, \dots, a_{59}\}$, c_2 is adjacent to $\{a_{16}, \dots, a_{45}, a_{56}, \dots, a_{70}\}$, c_3 is adjacent to $\{a_1, \dots, a_{15}, a_{21}, \dots, a_{45}, a_{60}, \dots, a_{64}\}$, c_4 is adjacent to $\{a_8, \dots, a_{45}, a_{64}, \dots, a_{70}\}$, c_5 is adjacent to $\{a_1, \dots, a_7, a_{15}, \dots, a_{45}, a_{64}, \dots, a_{70}\}$, c_6 is adjacent to $\{a_1, \dots, a_{14}, a_{21}, \dots, a_{45}, a_{65}, \dots, a_{70}\}$, c_7 is adjacent to $\{a_1, \dots, a_{45}\}$, c_8, c_9 are adjacent to $\{a_1, \dots, a_{20}, a_{46}, \dots, a_{70}\}$, d is adjacent to 32 vertices in $\{a_1, \dots, a_8, a_{46}, \dots, a_{70}\}$, e_1, e_2 are adjacent to 18 vertices in $\{a_{46}, \dots, a_{63}\}$, f is adjacent to 10 vertices in $\{a_{46}, \dots, a_{55}\}$ and g is adjacent to 7 vertices in $\{a_{46}, \dots, a_{52}\}$. Now

$$
\begin{aligned}
SO(G) &= \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2} \\
&= 17 \times 70\sqrt{(70)^2 + (24)^2} + 9 \times 45\sqrt{(45)^2 + (24)^2} + 1 \times 32\sqrt{(32)^2 + (24)^2} \\
&+ 2 \times 18\sqrt{(18)^2 + (24)^2} + 1 \times 10\sqrt{(10)^2 + (24)^2} + 1 \times 7\sqrt{(7)^2 + (24)^2} = 111510.
\end{aligned}
$$

Since G was an arbitrary graph of family $\mathcal{G}_{101}$, then the Sombor index of this family is an integer. See the Figure 3. □

Theorem 4. *If G is a 7-degree graph of order n with integer Sombor index, then $n \geq 101$, with equality if and only if $G \in \mathcal{G}_{101}$.*

Proof. The proof is similar to the Theorem 3, and note that extremality occurs for $\Lambda(G) = K_{1,6}$, that is, $(a, b, c, d, e, f, g) = (24, 70, 45, 32, 18, 10, 7)$, $|V_1| = 70, |V_2| = 17, |V_3| = 9, |V_4| = 1, |V_5| = 2, |V_6| = 1$ and $|V_7| = 1$. Thus $n = 101$, and one can see that $G \in \mathcal{G}_{101}$. Therefore, according to the Lemma 2, the Sombor index of these graphs is an integer. □

5. Conclusion

In this paper we studied connected 5, 6 and 7-degree graphs whose Sombor indices are integer. We found a lower bound of the orders of these graphs and described minimum order graphs. This method can be applied on more r-degree graphs ($r \geq 8$) with integer Sombor indices.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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