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Short Note

Some properties of star-perfect graphs

Sanghita Ghosh*, **Ravindra Gundgurti**†, **Abraham Vettiyankal**‡

Department of Mathematics, CHRIST(Deemed to be University), Bengaluru, India
*sanghita.ghosh@res.christuniversity.in
†gravindra49@gmail.com
‡frabraham@christuniversity.in

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Abstract: For a finite simple graph $G = (V, E)$, $\theta_s(G)$ denotes the minimum number of induced stars contained in G such that the union of their vertex sets is $V(G)$, and $\alpha_s(G)$ denotes the maximum number of vertices in G such that no two are contained in the same induced star of G. We call the graph G star-perfect if $\alpha_s(H) = \theta_s(H)$, for every induced subgraph H of G. We prove here that no cycle in a star-perfect graph has crossing chords and star-perfect graphs are planar. Also we present a few properties of star perfect graphs.

Keywords: star-perfect graphs, crossing chords, planar graph, total graphs.

AMS Subject classification: 05C17

1. Introduction

Berge's perfect graphs [1] have inspired various researchers to study many variations of perfect graphs [3]. The importance of perfect graphs is both theoretical (as it has applications in coloring problems) and practical (applications in communication theory and operations research, municipal services optimization problems, maintainance of temperatures of chemicals, etc).

All our graphs are finite and simple. We follow West [9] for basic terminology and notation. If G is a graph, then $V(G)$ and $E(G)$ are its vertex set and edge set respectively. For $k \geq 3$, define a k-cycle to be a cycle of length at least k. A chord of a cycle (or a path) C is an edge between two vertices of C that is not an edge of C itself. Two chords x_1y_1 and x_2y_2 of C are crossing chords of C if their four

* *Corresponding Author*

endpoints come in the order x_1, x_2, y_1, y_2 along C. The length of a path or a cycle is the number of edges in it. If $D \subseteq V(G)$, the subgraph of G induced by D is obtained by deleting the vertices of $V(G) - D$ and is denoted by $G[D]$. Given a graph H, we say that a graph G is H-free, if G does not contain an induced subgraph isomorphic to H. Given a family $\{H_1, H_2, ...\}$ of graphs, we say that G is $(H_1, H_2, ...)$-free if G is H_i - free, for every $i \geq 1$.

2. Star-Perfect Graphs

An induced *star* is a graph on $t+1$ vertices, $t \geq 0$, say $v_0, v_1, v_2, \ldots, v_t$ such that v_0 is adjacent to every vertex in $W = \{v_1, v_2, \ldots, v_t\}$ and no two vertices in W are adjacent. The vertex v_0 is called *central vertex* of the star and we say that the star is *centered* at v_0. Note that K_1 and K_2 are stars.

Let G be a graph. A *star-cover* of G, denoted by $\mathcal{S}$, is a family of induced stars contained in G such that every vertex of G is in one of the induced stars. The *star-covering number* of G is the minimum cardinality of a star-cover and is denoted by $\theta_s(G)$. A θ_s*-cover* of G is a star-cover of G containing $\theta_s(G)$ stars.

A set $T \subseteq V(G)$ is a *star-independent* set of G if no two vertices of T are contained in the same induced star in G (equivalently, any two vertices in T are at a distance at least 3). The maximum cardinality of a star-independent set is called the *star-independence number* of G and is denoted by $\alpha_s(G)$. An α_s- *independent set* of G is a star-independent set with $\alpha_s(G)$ vertices.

We call the graph G star-perfect if $\alpha_s(H) = \theta_s(H)$, for every induced subgraph H of G.

In [8], we characterized star-perfect graphs as follows: A graph G is star-perfect if and only if G is $(C_3, C_{3k+1}, C_{3k+2})$-free, $k \geq 1$. The theorem is a min-max theorem for star-perfect graphs.

The minimum parameter $\theta_s(G)$ has been extensively studied in various contexts with different terminology. For example, star-independent sets are studied in [2], where a star-independent set in G corresponds to a color class in a $(k, 3)$-coloring of G.

We refer to the following theorems from [8], which is implicit in proving the Theorem 3 on crossing chords.

Theorem 1. *[8] A graph G is star-perfect if and only if G is $(C_3, C_{3k+1}, C_{3k+2})$-free, $k \geq 1$.*

We need the following definitions and theorems in proving planarity of star-perfect graphs. In 1930, Kuratowski published the theorem giving a necessary and sufficient condition for planarity.

Definition 1. A planar graph is a graph that can be embedded in the plane, that is, it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

Definition 2. A subdivision of an edge is the operation where the edge is replaced by a path of length 2, the internal vertex added to the original graph. A subdivision of a graph G is a graph achieved by a sequence of edge-subdivisions on G.

Theorem 2 (Kuratowski's Theorem 1930). *A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

3. Results and Discussions

3.1. Properties of Star-Perfect Graphs

Throughout the paper, we fix a clockwise orientation to a cycle C in G.

Theorem 3. *If a graph G is star-perfect, then no cycle of G has crossing chords.*

Proof. If not, let C be a smallest cycle which has two crossing chords (See Figure 1).

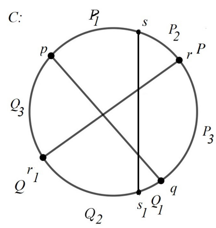

Figure 1. **Smallest cycle C which has two crossing chords in G**

We need the following Lemma to prove the Theorem 3.

Lemma 1. *If C' is a cycle in G, then C' has an induced cycle of the form $3k, k \geq 2$.*

Proof. Since C' is a cycle in star-perfect graph G, by Theorem 1 G is $(C_3, C_{3k+1}, C_{3k+2})$-free, $k \geq 1$. Let the symbols P, Q, p, s, r, q, r_1 and s_1 be as shown in Figure 1. If C' is not induced then there exists $r \in P$ and $r_1 \in Q$ such that rr_1 is a chord in the cycle. We can choose r and r_1 such that there is no chord joining a vertex between r and q in P and a vertex r_1 and p in Q. Then $rPqpQr_1r$ is an induced cycle of the form $3k, k \geq 2$, by Theorem 1.

Proof of Theorem 3 continued.
By the choice of C, the cycles $pPqp, r_1QpPrr_1, qQpq$ and rQr_1r are induced cycles of the form $3k, k \geq 2$. If C is induced then we are done. If not, then by Lemma 1 C has an induced cycle $C'' = sPrr_1Qs_1s$ of the form $3k, k \geq 2$. Let the portion

pPs be denoted by P_1. Let $pPq, sPr, rPq, qQs_1, s_1Qr_1$ and r_1Q_1P be denoted by $P_1, P_2, P_3, Q_1, Q_2, Q_3$ respectively (Figure 1).
We observe $|P_1 \cup P_2 \cup P_3| = |P_1| + |P_2| + |P_3| - (|P_1 \cap P_2| + |P_2 \cap P_3|) = |P_1| + |P_2| + |P_3| - 2$, since $|P_1 \cap P_2| = |P_2 \cap P_3| = 1$ and similarly for other triplets $\{P_2, P_3, Q_1\}$, $\{P_3, Q_1, Q_2\}$, $\{Q_1, Q_2, Q_3\}$, $\{Q_2, Q_3, P_1\}$ and $\{Q_3, P_1, P_2\}$. All the five triplets are induced cycles of the form $3k, k \geq 2$. (A)
We consider the following cases.

Case 1. P_1 is of the form $3k$.

Subcase 1(a). Q_3 is of the form $3k$. Then P_3, Q_2 and P_2 are of the form $3k, 3k+2$ and $3k+1$ respectively, using (A) and the fact that the cycle formed by P_3, Q_3 and P_2, Q_2 are of the form $3k, k \geq 2$. This by (A) implies that $P_1 \cup P_2 \cup P_3$ is of the form $3k+(3k+2)+(3k+1)-2$, equivalent to $3k+1$, a contradiction to our assumption that G is star-perfect.

Subcase 1(b). Q_3 is of the form $3k+1$. Then P_3, Q_2 and P_2 are of the form $3k+2, 3k+1$ and $3k+2$ respectively. This by (A) implies that $P_1 \cup P_2 \cup P_3$ is of the form $(3k) + (3k+2) + (3k+2) - 2$, equivalent to $3k+2$, a contradiction to our assumption that G is star-perfect.

Subcase 1(c). Q_3 is of the form $3k+2$. Then P_3, Q_2 and P_2 are of the form $3k+1, 3k$ and $3k$ respectively. This by (A) implies that $P_1 \cup P_2 \cup P_3$ is of the form $(3k) + (3k) + (3k+1) - 2$, equivalent to $3k+2$, a contradiction to our assumption that G is star-perfect.

Case 2. P_1 is of the form $3k+1$.

Subcase 2(a). Q_3 is of the form $3k$. Then P_3, Q_2 and P_2 are of the form $3k, 3k+1$ and $3k+2$ respectively. This by (A) implies that $P_1 \cup P_2 \cup P_3$ is of the form $(3k+1) + (3k+2) + (3k) - 2$, equivalent to $3k+1$, a contradiction to our assumption that G is star-perfect.

For Q_3 of the form $3k+1$ or $3k+2$ we have similar contradictions.

Case 3. P_1 is of the form $3k+2$.
For Q_3 of the form $3k, 3k+1$ or $3k+2$ we have similar contradictions.

These three cases contradict our assumption that C has crossing chords and hence the theorem.

□

3.2. Planarity of Star-Perfect Graphs

In this section we discuss the planarity of star-perfect graphs.

Theorem 4. *Star-perfect graphs are planar.*

Proof. Let G be a star-perfect graph. If G is not planar, then by Theorem 2(Kuratowski's Theorem) 1930 G has a subdivision of K_5 or $K_{3,3}$.
Let T be a $K_{3,3}$ with the bipartition $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$. Let u_1v_2 and u_2v_1 be two crossing chords in T. Let H be a subdivision of T without subdividing the edges u_1v_2 and u_2v_1. Let C be an induced cycle in H having the two crossing chords u_1v_2 and u_2v_1. Since G is star-perfect, then by Theorem 1, C is of length $3k, k \geq 2$ which has two crossing chords, contradiction to Theorem 3. Hence the theorem.
If G has a subdivision of K_5, then by similar arguments we arrive at a contradiction. This proves that the star-perfect graphs are planar. □

Definition 3. [7] A generalized theta graph $\Theta[P(l_1, l_2, \ldots, l_t)]$ is a graph obtained from internally disjoint paths $P(l_i), 1 \leq i \leq t$, and these paths share common end vertices x and y. So there are exactly two vertices of degree t and all other vertices are of degree 2.

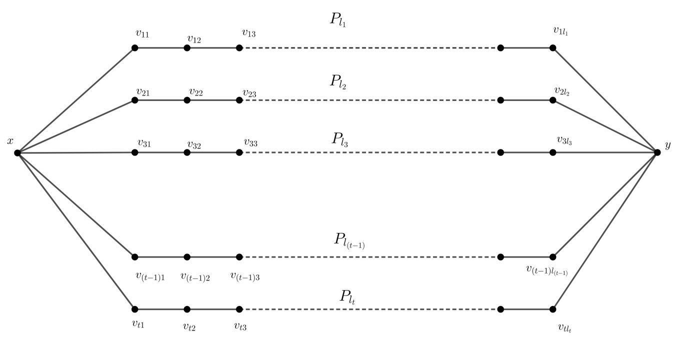

Figure 2. **Example of generalized theta graph** $\Theta[P(l_1, l_2, \ldots, l_t)]$

Theorem 5. *[7] A generalized theta graph* $\Theta[P(l_1, l_2, \ldots, l_t)]$ *is star-perfect if every* l_i *is of the form* $2(mod3)$*.*

Theorem 6. *If G is a star-perfect graph, then every block of G is a cycle of length $3k$ or a generalized theta graph,* $\Theta[P(l_1, l_2, \ldots, l_t)]$*, where every l_i is of the form* $2 \pmod 3$*.*

Proof. Let $G_1, G_2, \ldots, G_k$ be the blocks of G. Let $C_l = v_1v_2 \ldots v_kv_1$ be the smallest cycle in $G_l, 1 \leq l \leq k$. If C_l is not induced, C_l has a chord xy which divides C_l into two induced cycles C_1 and C_2, each of length $3k$. But then the length of C_l is $(3k-2)+(3k-2)+2$ equivalent to $3k-2$ which is of the form $3k+1$, a contradiction. Thus C_l is induced.
Now if $V(G_l) = V(C_l)$, then G_l is the cycle C_l. Therefore by Theorem 1, C_l is of length $3k$, which is the former part of the theorem.

If $V(G_l) \neq V(C_l)$, then there exists $v \in V(G_l) - V(C_l)$, such that $v \leftrightarrow v_1$, say. Since G_l is a block, there exists a cycle C' containing the edges vv_1, v_1v_2. Obviously C' is of length at least 3. Then $C_l \cup C'$ contains a cycle of length at least $k+1$, a contradiction to the choice of C_l. As seen earlier, C_l is a cycle of length $3k$, the former part of the corollary is proved. If C_l has a vertex u of degree ≥ 3, then C_l has at least a vertex w such that $d(w) \geq 3$. Since G_l is a block, there is a path connecting u and w in G_l. If $deg(u), deg(w) = k \geq 3$, then we will have k disjoint paths connecting u and w in G_l. Then by Theorem 5, each of the k disjoint paths $P(l_i), 1 \leq i \leq k$, with these paths share common end vertices u and w are of length $3k+2$ (excluding u and v). Then we have $\Theta[P(l_1, l_2, \ldots, l_k)]$, where every l_i is of the form $2(mod 3)$, which proves the later part of the theorem. □

Definition 4. The total graph G of a graph H is a graph such that the vertex set of G corresponds to the vertices and edges of H, called elements of H, and two vertices are adjacent in G if and only if their corresponding elements are adjacent or incident in H.

Theorem 7. *[6] A total graph G of a graph H is perfect if and only if every block of H is either K_2 or K_3.*

Theorem 8. *A perfect total graph G of a graph H is star-perfect if and only if G is a tree.*

Proof. If G is perfect, then by Theorem 7, every block of H is K_2 or K_3. Suppose G is star-perfect. Then by Theorem 1, G is $(C_3, C_{3k+1}, C_{3k+2})$-free, $k \geq 1$. This means that the only induced cycles in G are $C_{3k}, k \geq 2$. This implies that G has an induced cycle of length ≥ 4. If such an induced cycle in G is $C = v_1e_1v_2e_2 \ldots v_ne_nv_1$, then using the arguments in the proof of Theorem 7, we show that G contains an induced odd cycle of length at least 5. If n is odd then $G[\{e_1, e_2, \ldots, e_n\}] = C_{2m+1}, m \geq 2$, contradicting the perfectness of G. Therefore let n be even. Then $H[\{v_1, v_2, \ldots, v_n\}] = K_4$ or $K_4 - e$ or $C_{2m}, m \geq 2$. In all these cases clearly $G[\{e_n, e_1, e_2, v_3, v_4, \ldots, v_n\}] = C_{2m+1}, m \geq 2$. This violates that G is perfect, thereby proving the assertion. Therefor G has no induced cycle of length ≥ 3 and hence G is a tree.
Conversely, every tree is star-perfect, as proved in [8]. Hence the theorem. □

Note 9. It is interesting to note that not all total perfect graphs are star-perfect, for example C_3. Also not all star-perfect graphs are perfect, for example C_6.

3.3. β- star perfection

In 1972, Lovász[5] proved that a graph G is perfect if and only if $\omega(H)\alpha(H) \geq |V(H)|$, for every induced subgraph H of G. In the following theorem we obtain a result similar to this result of Lovász. The property "$\omega(H)\alpha(H) \geq |V(H)|$, for all induced H" was suggested by Fulkerson, and is called β-perfection. This result is also mentioned in [9] and [4].

Definition 5. The *star-density* of graph G, denoted by $\omega_s(G)$, is the maximum size of the induced star subgraph of G.

We define *β-star perfection* as the property "$\omega_s(H)\alpha_s(H) \geq |V(H)|$, for all induced H".

Theorem 10. *For any graph G, the following statements are equivalent.*

(i) G is star-perfect.

(ii) G has β-star perfection property.

(iii) G is $(C_3, C_{3k+1}, C_{3k+2})$-free, $k \geq 1$.

Proof. To prove $(i) \implies (ii)$, let $S_1, S_2, \ldots, S_t, t \geq 1$ be a star-cover of G, obviously $|S_i| \leq \omega_s(G), 1 \leq i \leq t$. Then $\omega_s(G)\theta_s(G) \geq |V(G)|$ which implies $\omega_s(G)\alpha_s(G) \geq |V(G)|$ (since G is star-perfect, $\alpha_s(G) = \theta_s(G)$).
To prove $(ii) \implies (iii)$, let G be a graph with an induced H where H is C_3 or C_{3k+1} or C_{3k+2}, $k \geq 1$. Then clearly $\alpha_s(H)\omega_s(H) \not\geq |V(H)|$.
By Theorem 1, obviously (iii) implies (i). □

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Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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