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Research Article

Strong domination number of some operations on a graph

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Abstract: Let $G = (V(G), E(G))$ be a simple graph. A set $D \subseteq V(G)$ is a strong dominating set of G, if for every vertex $x \in V(G) \setminus D$ there is a vertex $y \in D$ with $xy \in E(G)$ and $\deg(x) \leq \deg(y)$. The strong domination number $\gamma_{st}(G)$ is defined as the minimum cardinality of a strong dominating set. In this paper, we examine the effects on $\gamma_{st}(G)$ when G is modified by operations on edge (or edges) of G.

Keywords: edge deletion, edge subdivision, edge contraction, strong domination number

AMS Subject classification: 05C15, 05C25

1. Introduction

A dominating set of a graph $G = (V(G), E(G)) = (V, E)$ is any subset D of V such that every vertex not in D is adjacent to at least one member of D. The minimum cardinality of all dominating sets of G is called the domination number of G and is denoted by $\gamma(G)$. This parameter has been extensively studied in the literature and there are hundreds of papers concerned with domination. For a detailed treatment of domination theory, the reader is referred to [7]. Also, the concept of domination and related invariants have been generalized in many ways.
The corona product $G \circ H$ of two graphs G and H is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the i-th vertex of G to every vertex in the i-th copy of H.

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A set $D \subseteq V(G)$ is a strong dominating set of G, if for every vertex $x \in V(G) \setminus D$ there is a vertex $y \in D$ with $xy \in E(G)$ and $\deg(x) \leq \deg(y)$. The strong domination number $\gamma_{st}(G)$ is defined as the minimum cardinality of a strong dominating set. A strong dominating set with cardinality $\gamma_{st}(G)$ is called a γ_{st}-set. The strong domination number was introduced in [9] and some upper bounds on this parameter were presented in [8]. Similar to strong domination number, a set $D \subset V$ is a weak dominating set of G if every vertex $v \in V \setminus D$ is adjacent to a vertex $u \in D$ such that $deg(v) \geq deg(u)$ (see [5, 10, 11]). The minimum cardinality of a weak dominating set of G is denoted by $\gamma_w(G)$. Boutrig and Chellali [5] proved that the relation $\gamma_w(G) + \frac{3}{\Delta+1}\gamma_{st}(G) \leq n$ holds for any connected graph of order $n \geq 3$.
Motivated by counting of the number of dominating sets of a graph and domination polynomial (see e.g. [1, 3]), recently, we have studied the number of the strong dominating sets for certain graphs [12].
Let e be an edge of a connected simple graph G. The graph obtained by removing an edge e from G is denoted by $G-e$. The edge subdivision operation for an edge $uv \in E$ is the deletion of $\{u, v\}$ from G and the addition of two edges uw and wv along with the new vertex w. A graph which has been derived from G by an edge subdivision operation for an edge e is denoted by G_e. The k-subdivision of G, denoted by $G^{\frac{1}{k}}$, is constructed by replacing each edge v_iv_j of G with a path of length k. The contraction of an edge e with endpoints u, v in graph G is denoted by G/e and is the replacement of u and v with a single vertex such that edges incident to the new vertex are the edges other than e that were incident with u or v.

In the next section, we examine the effects on $\gamma_{st}(G)$ when G is modified by operations edge deletion, edge subdivision and edge contraction. Also we study the strong domination number of k-subdivision of G in Section 3.

2. Strong domination number of some operations on a graph

In this section, we study the relations between the strong domination number of $G, G-e, G_e$ and G/e. First we consider the edge deletion.

2.1. Edge deletion

We begin with the following result.

Theorem 1. *Let $G = (V, E)$ be a connected graph of order at least three (or the components of the graph are not isomorphic to K_2), and $e = uv \in E$. Then,*

$$\gamma_{st}(G) - 1 \leq \gamma_{st}(G-e) \leq \gamma_{st}(G) + \deg(u) + \deg(v) - 2.$$

Proof. First we find the upper bound for $\gamma_{st}(G-e)$. Suppose that D is a strong dominating set of G. Both vertices u and v are in D and u has the same degree with some of its neighbours (except v) and strong dominates them, and the same for v.

Suppose that u' is adjacent to u, $u' \neq v$, $\deg(u) = \deg(u')$, and u' is strong dominated only by u. Then, by removing e, there is no vertex that strong dominates u'. So, we remove u from D and put all of its neighbours in D. Now, u is strong dominated by at least u'. We have the same argument for v too. So, $D' = (D \cup N(u) \cup N(v)) \setminus \{u, v\}$, is a strong dominating of $G - e$. If we can keep u in our strong dominating set to strong dominate at least one vertex (say u''), but condition for v be the same as before, then we consider

$$D'' = (D \cup N(u) \cup N(v)) \setminus \{u'', v\},$$

and we are done. If we can keep u in our strong dominating set to strong dominate at least one vertex (say u'''), and keep v in our strong dominating set to strong dominate at least one vertex (say v'''), then we consider

$$D''' = (D \cup N(u) \cup N(v)) \setminus \{u''', v'''\},$$

and we have a strong dominating set. Hence, in all cases, we have

$$\gamma_{st}(G - e) \leq \gamma_{st}(G) + \deg(u) + \deg(v) - 2.$$

Note that if $u \in D$ and $v \notin D$, then after removing e, the set $D \cup \{v\}$ is strong dominating set of $G - e$ and the inequality holds for this condition too. If $u, v \notin D$, then after removing e, they are strong dominated by the same vertices as before.
Now, we find a lower bound for $\gamma_{st}(G - e)$. First we remove e and find a strong dominating set for $G - e$. Suppose that this set is S. We have the following cases:

(i) $u, v \in S$. In this case, adding edge e does not make any difference and S is a strong dominating set of G too. So $\gamma_{st}(G) \leq \gamma_{st}(G - e)$.

(ii) $u \in S$ and $v \notin S$. In this case, after adding edge e, let $S' = S \cup \{v\}$. The set S' is a strong dominating set of G, and $\gamma_{st}(G) \leq \gamma_{st}(G - e) + 1$.

(iii) $u, v \notin S$. Without loss of generality, suppose that $\deg(u) \leq \deg(v)$. After adding edge e, let $S'' = S \cup \{v\}$. Then, u is strong dominated by v and all other vertices in $V(G) \setminus S'$ are strong dominated as before. Hence, S'' is a strong dominating set of G, and $\gamma_{st}(G) \leq \gamma_{st}(G - e) + 1$.

Therefore in all cases we have $\gamma_{st}(G - e) \geq \gamma_{st}(G) - 1$, and we have the result. □

Remark 1. Bounds in Theorem 1 are tight. For the upper bound, consider G as shown in Figure 1. The set of black vertices is a strong dominating set of G (say D). If we remove edge e, then for example, for the vertex v_1, we have $\deg(v) < \deg(v_1)$, and v does not strong dominate v_1 any more. Since all of the neighbours of v_1 have less degree, so we should have it in our strong dominating set. So, by the same argument for all vertices,

$$D' = (D \cup \{v_1, v_2, v_3, v_4, u_1, u_2, u_3\}) \setminus \{v, u\}$$

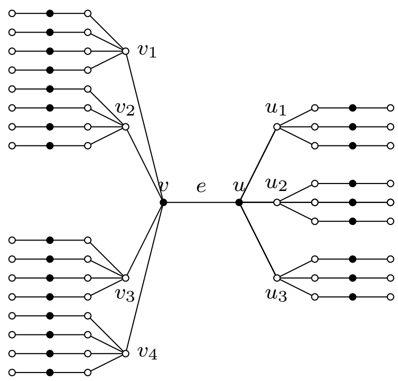

Figure 1. **The graph** G

is a strong dominating set for $G - e$, and we are done. For the lower bound, consider H as shown in Figure 2. One can easily check that $S = \{v_1, v_2, v_3, u_1, u_2, u_3, u_4\}$ is a strong dominating set for $H - e$, and $S' = \{u, v_1, v_2, v_3, u_1, u_2, u_3, u_4\}$ is a strong dominating set for H, as desired.

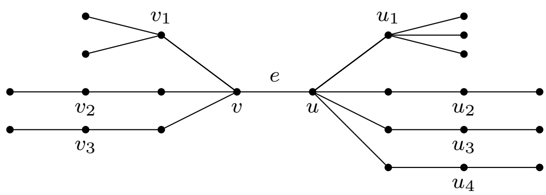

Figure 2. **The graph** H

Remark 2. It is easy to see that if P_n and C_n are the path and the cycle of order $n \geq 3$, respectively, then $\gamma_{st}(P_n) = \gamma_{st}(C_n) = \lceil \frac{n}{3} \rceil$. So the path P_n (if $n \not\equiv 1 \pmod 3$ and e is an edge incident with leaves), is another example for the tightness of the upper bound in Theorem 1. Note that we do not have equalities of Theorem 1 for the cycles.

We close this subsection with the following theorem which is about the strong domination number of corona of two graphs $G_1 \circ G_2$ when it is modified by deletion of an edge.

Theorem 2. *If G_1 and G_2 are two graphs, then*

$$\gamma_{st}((G_1 \circ G_2) - e) = \begin{cases} \gamma_{st}(G_1 \circ G_2) & \text{if } e \in E(G_1) \text{ or } e \in E(G_2), \\ \gamma_{st}(G_1 \circ G_2) + 1 & \text{if } e = v_i v_j, v_i \in V(G_1), v_j \in V(G_2). \end{cases}$$

Proof. In the removing edge e of $G_1 \circ G_2$, we have three cases:

Case 1. $e \in E(G_1)$. Since the minimum strong dominating set of $G_1 \circ G_2$ is $V(G_1)$, so in this case, $\gamma_{st}((G_1 \circ G_2) - e) = \gamma_{st}(G_1 \circ G_2)$.

Case 2. $e \in E(G_2)$. In this case the minimum dominating set of $(G_1 \circ G_2) - e$, does not change and so $\gamma_{st}((G_1 \circ G_2) - e) = \gamma_{st}(G_1 \circ G_2)$.

Case 3. If $e = uv$, $u \in V(G_1), v \in V(G_2)$ or $v \in V(G_1), u \in V(G_2)$. In this case by removing the edge e, $V(G_2)$ are not dominated by the minimum strong dominating set of $G_1 \circ G_2$. Therefore $\gamma_{st}((G_1 \circ G_2) - e) = \gamma_{st}(G_1 \circ G_2) + 1$. □

2.2. Edge subdivision

In this subsection, we examine the effects on $\gamma_{st}(G)$ when G is modified by subdivision on an edge of G.

Theorem 3. *If $G = (V, E)$ is a graph, then*

$$\gamma_{st}(G) \leq \gamma_{st}(G_e) \leq \gamma_{st}(G) + 1.$$

Proof. First we find the upper bound for $\gamma_{st}(G_e)$. Suppose that v_e is the new vertex in G_e and also D is a γ_{st}-set of G. If D is a strong dominating set of G_e, too, then we have the result. Otherwise, since $\deg(v_e) = 2$, so the set $D' = D \cup \{v_e\}$ is a strong dominating set of G_e, and we are done. Now, we find the lower bound. Consider the graph G_e and let D_e be its strong dominating set. If $v_e \in D_e$, then it may strong dominate its neighbours or not. If it does, then since its degree is 2, its neighbours should have degree at most two. So for G, let strong dominating set be the old one by adding the neighbour of v_e with higher (or equal) degree and removing v_e, and hence $\gamma_{st}(G) \leq \gamma_{st}(G_e)$. If it does not, then removing that from our strong dominating set does not have effect on being strong dominating set for G. So $\gamma_{st}(G) \leq \gamma_{st}(G_e) - 1$. So, if $v_e \in D_e$, then $\gamma_{st}(G) \leq \gamma_{st}(G_e)$. If $v_e \notin D_e$, then one can easily check that D_e is a strong dominating set of G too. Therefore we have the result. □

Remark 3. The bounds in Theorem 3 are tight. For the upper bound, consider G as the cycle graph C_{3k} or the path graph P_{3k}. For the lower bound, consider G as the cycle graph C_{3k+1} or the path graph P_{3k+1}.

Remark 4. From Theorems 1 and 3, we see that for some graphs $\gamma_{st}(G - e) = \gamma_{st}(G_e)$. For example, the cycle graphs C_n (when $n \not\equiv 0 \pmod 3$), and the complete bipartite graph $K_{m,n}$ satisfy this equality. The characterization of these kind of graphs is an interesting problem which we propose it here:

Problem 1. Characterize graph G and edge e with $\gamma_{st}(G - e) = \gamma_{st}(G_e)$.

The following theorem gives a relation for the strong domination number of the corona product of two graphs when it is modified by subdivision of an edge.

Theorem 4. *If G_1 and G_2 are two graphs, then*

$$\gamma_{st}((G_1 \circ G_2)_e) = \begin{cases} \gamma_{st}(G_1 \circ G_2) & \text{if } e \in E(G_1), \\ \gamma_{st}(G_1 \circ G_2) + 1 & \text{if } e \in E(G_2) \text{ or } e = v_i v_j, v_i \in V(G_1), v_j \in V(G_2). \end{cases}$$

Proof. If $e \in E(G_1)$, since the minimum strong dominating set of $G_1 \circ G_2$ is $V(G_1)$, so by subdividing e, the minimum strong dominating set of $(G_1 \circ G_2)_e$ is also $V(G_1)$ and so $\gamma_{st}((G_1 \circ G_2)_e) = \gamma_{st}(G_1 \circ G_2)$. If $e \in E(G_2)$ or $e = v_i v_j, v_i \in V(G_1), v_j \in V(G_2)$, by subdividing edge e, one vertex of one copy of G_2 or vertex that added to $G_1 \circ G_2$, are not dominated by the minimum strong dominating set of $G_1 \circ G_2$. Therefore in this case $\gamma_{st}((G_1 \circ G_2)_e) = \gamma_{st}((G_1 \circ G_2) + 1$. □

2.3. Edge contraction

In this subsection, we examine the effects on $\gamma_{st}(G)$ when G is modified by contraction on an edge of G.

Theorem 5. *If $G = (V, E)$ is a graph which is not K_2, and $e = uv \in E$ is not a pendant edge, then,*

$$\gamma_{st}(G) - \deg(u) - \deg(v) + 3 \leq \gamma_{st}(G/e) \leq \gamma_{st}(G) + 1.$$

Proof. Suppose that w is the new vertex in G/e by contraction of e and replacement of that with u and v. First we find the upper bound for $\gamma_{st}(G/e)$. Suppose that D is a strong dominating set of G. If at least one of u and v be in D, then $D' = (D \cup \{w\}) \setminus \{u, v\}$ is a strong dominating set for G/e, since every vertices in $V(G) \setminus D$ are strong dominated by same vertices as before or possibly w. If $u, v \notin D$, then one can easily check that $D' = (D \cup \{w\})$ is a strong dominating set for G/e, and therefore $\gamma_{st}(G/e) \leq \gamma_{st}(G) + 1$. Now, we find the lower bound for $\gamma_{st}(G/e)$. First, we find a strong dominating set S for G/e. We have two cases:

(i) $w \notin S$. The set $S \cup \{u\}$ is a strong dominating set of G, if, without loss of generality, $deg(u) \geq deg(v)$, and we have $\gamma_{st}(G) \leq \gamma_{st}(G/e) + 1$.

(ii) $w \in S$. If every vertices in $V(G) \setminus S$ is strong dominated by vertices except w, then clearly $S' = (S \cup \{u, v\}) \setminus \{w\}$ is a strong dominating set for G and we have $\gamma_{st}(G) \leq \gamma_{st}(G/e) + 1$. Now suppose that there exists $w' \in N(w) \setminus S$ such that $\deg(w') \leq \deg(w)$ and w strong dominates the vertex w'. We have the following cases:

(1) For all vertices $x \in N(u)$, we have $\deg(x) \leq \deg(u)$, and for all vertices $y \in N(v)$, we have $\deg(y) \leq \deg(v)$. In this case, one can easily check that

$$S' = (S \cup \{u, v\}) \setminus \{w\}$$

is a strong dominating set for G, and we have $\gamma_{st}(G) \leq \gamma_{st}(G/e) + 1$.

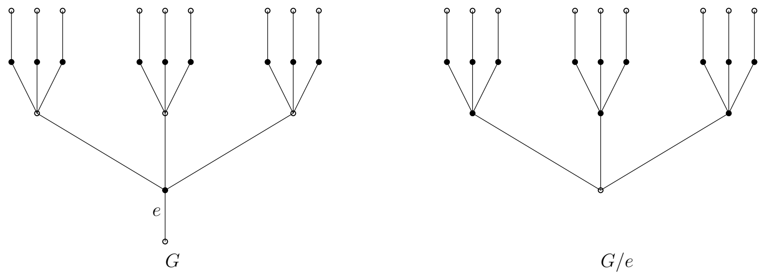

Figure 3. $\gamma_{st}(G) = 10$ **and** $\gamma_{st}(G/e) = 12$.

(2) For all vertices $x \in N(u)$, we have $\deg(x) \leq \deg(u)$, and there exists $y' \in N(v)$, such that $\deg(v) < \deg(y')$. In this case, let

$$S' = (S \cup N(v)) \setminus \{w\}.$$

Then v is strong dominated by y' and the rest of vertices in $V(G) \setminus S$ are strong dominated as before (and possibly by u). So S' is a strong dominating set, and hence $\gamma_{st}(G) \leq \gamma_{st}(G/e) + \deg(v)$.

(3) There exists $x' \in N(u) - \{v\}$, such that $\deg(u) \leq \deg(x')$, and there exists $y' \in N(v) - \{u\}$, such that $\deg(v) \leq \deg(y')$. In this case, let

$$S' = (S \cup (N(u) \setminus \{v\}) \cup (N(v) \setminus \{u\})) \setminus \{w\}.$$

Then u is strong dominated by x', v is strong dominated by y', and the rest of vertices in $V(G) \setminus S$ are strong dominated as before. Hence $\gamma_{st}(G) \leq \gamma_{st}(G/e) + \deg(u) + \deg(v) - 3$.

Hence in any case, $\gamma_{st}(G/e) \geq \gamma_{st}(G) - \deg(u) - \deg(v) + 3$.

Therefore we have the result. □

Remark 5. The condition "e is not a pendant edge" is necessary in Theorem 5. For example, consider Figure 3. The set of black vertices of G and G/e are strong dominating sets and so $\gamma_{st}(G) = 10$ and $\gamma_{st}(G/e) = 12$.

Remark 6. Bounds in Theorem 5 are tight. For the upper bound, consider Figure 4. The set of black vertices of G and G/e are strong dominating sets and we are done. For the lower bound, consider Figure 5. One can easily check that the set of black vertices of H and H/e are strong dominating sets, as desired. Also, for the cycles C_{3k+1} we have equality in the left inequality of Theorem 5.

Figure 4. **Graphs** G **and** G/e

Figure 5. **Graphs** H **and** H/e

As an immediate result of Theorems 1, 3, and 5, we have:

Corollary 1. *Suppose that* e *is not a pendant edge. If* $\alpha = \gamma_{st}(G-e)+\gamma_{st}(G_e)+\gamma_{st}(G/e)$, *and* $\beta = \deg(u) + \deg(v)$*, then,*

$$\frac{\alpha-\beta}{3} \leq \gamma_{st}(G) \leq \frac{\alpha+\beta-2}{3}.$$

3. Strong domination number of k-subdivision of a graph

The k-subdivision of G, denoted by $G^{\frac{1}{k}}$, is constructed by replacing each edge v_iv_j of G with a path of length k, say $P^{\{v_i,v_j\}}$. These k-paths are called *superedges*, any new vertex is an internal vertex, and is denoted by $x_l^{\{v_i,v_j\}}$ if it belongs to the superedge $P_{\{v_i,v_j\}}$, $i<j$ with distance l from the vertex v_i, where $l \in \{1,2,\ldots,k-1\}$ (see for example Figure 6). Note that for $k=1$, we have $G^{1/1}=G^1=G$, and if G has n vertices and m edges, then the graph $G^{\frac{1}{k}}$ has $n+(k-1)m$ vertices and km edges. Some results about subdivision of a graph can be found in [2, 4, 6]. In this section, we study the strong domination number of k-subdivision of a graph. First, we consider the graphs with minimum degree at least 3.

Theorem 6. *Let* G *be a graph of order* n*, size* m*, and* $\delta(G) \geq 3$*. Then for* $k \geq 2$*,*

$$\gamma_{st}(G^{\frac{1}{k}}) = \begin{cases} n & \text{if } k=2,3, \\ n+m\left\lceil\frac{k-3}{3}\right\rceil & \text{otherwise.} \end{cases}$$

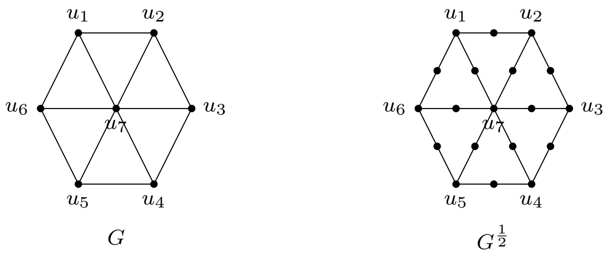

Figure 6. **Graphs G and $G^{\frac{1}{2}}$**

Proof. Suppose that $v_iv_j \in E(G)$. First, let $k = 2$. Then, $P^{\{v_i,v_j\}}$ consists of vertices v_i, $x_1^{\{v_i,v_j\}}$, and v_j. Since $\deg(x_1^{\{v_i,v_j\}}) = 2$ and $\delta(G) \geq 3$, then we should have v_i and v_j in strong dominating set of $G^{\frac{1}{k}}$. Hence, $\gamma_{st}(G^{\frac{1}{2}}) = n$. By the same argument, we have $\gamma_{st}(G^{\frac{1}{3}}) = n$, too. Now consider the graph $G^{\frac{1}{k}}$, where $k \geq 4$. Then, $P^{\{v_i,v_j\}}$ consists of vertices $v_i, x_1^{\{v_i,v_j\}}, x_2^{\{v_i,v_j\}}, \ldots, x_{k-1}^{\{v_i,v_j\}}, v_j$. By the same argument as cases $k = 2, 3$, we need v_i and v_j in our strong dominating set, and they strong dominate vertices $x_1^{\{v_i,v_j\}}$ and $x_{k-1}^{\{v_i,v_j\}}$, respectively. Now, for the rest of vertices, we have a path of order $k-3$, and since we need $\lceil \frac{k-3}{3} \rceil$ vertices among them to have a strong dominating set for this path, then the proof is complete. □

By the same argument as proof of Theorem 6, we have the upper bound in case $\delta(G) \geq 2$.

Theorem 7. *Let G be a graph of order n, size m, and $\delta(G) \geq 2$. Then,*

$$\gamma_{st}(G^{\frac{1}{k}}) \leq \begin{cases} n & \text{if } k = 2, 3, \\ n + m \lceil \frac{k-3}{3} \rceil & \text{otherwise.} \end{cases}$$

The following example shows that for some graphs and some $k \in \mathbb{N} \setminus \{1\}$, the equality holds, and for some it does not.

Example 1. Let $G = C_5$. Then one can easily check that $\gamma_{st}(G^{\frac{1}{2}}) = 4 < 5$, and $\gamma_{st}(G^{\frac{1}{k}}) < n(1 + \lceil \frac{k-3}{3} \rceil)$, where $k \in \mathbb{N} \setminus \{1, 2, 3t \mid t \in \mathbb{N}\}$. But, $\gamma_{st}(G^{\frac{1}{3r}}) = nr$, where $r \in \mathbb{N}$, as desired.

Now, we consider graphs with pendant vertices and find an upper bound for $\gamma_{st}(G^{\frac{1}{k}})$.

Theorem 8. *Let G be a graph of order n, size m, and t pendant vertices, where $1 \leq t \leq n-1$. Then,*

$$\gamma_{st}(G^{\frac{1}{k}}) \leq \begin{cases} n & \text{if } k = 2, 3, \\ n + t \lceil \frac{k-4}{3} \rceil + (m-t) \lceil \frac{k-3}{3} \rceil & \text{otherwise.} \end{cases}$$

Proof. Suppose that $v_iv_j \in E(G)$, and v_i is a pendant vertex. First, let $k = 2$. Then, $P^{\{v_i,v_j\}}$ consists of vertices v_i, $x_1^{\{v_i,v_j\}}$, and v_j. Since $\deg(x_1^{\{v_i,v_j\}}) = 2$ and $\deg(v_i) = 1$, then we should have $x_1^{\{v_i,v_j\}}$ in our strong dominating set. So the set S containing these vertices and non-pendant vertices of G, is a strong dominating set and we are done. By the same argument, we have $\gamma_{st}(G^{\frac{1}{3}}) \leq n$, too. Now consider the graph $G^{\frac{1}{k}}$, where $k \geq 4$. The superedge $P^{\{v_i,v_j\}}$ consists of vertices $v_i, x_1^{\{v_i,v_j\}}, x_2^{\{v_i,v_j\}}, \dots, x_{k-1}^{\{v_i,v_j\}}, v_j$. By the same argument as cases $k = 2, 3$, we pick $x_1^{\{v_i,v_j\}}$ and v_j in our strong dominating set, and they strong dominate vertices v_i and $x_{k-1}^{\{v_i,v_j\}}$, respectively. Now, for the rest of vertices of $P^{\{v_i,v_j\}}$, we have a path graph of order $k-4$, and since we need $\lceil \frac{k-4}{3} \rceil$ vertices among them to have a strong dominating set for this path, then by adding cases when we do not have a pendant vertex as endpoint of an edge (same argument as proof of Theorem 6), we have the result. □

Remark 7. The upper bound in the Theorem 8 is tight, if $k \equiv 0 \pmod 3$. It suffices to consider G as the path graph P_4.

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