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Research Article

On the complement of the intersection graph of subgroups of a group

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Abstract: The complement of the intersection graph of subgroups of a group G, denoted by $\mathscr{I}^c(G)$, is the graph whose vertex set is the set of all nontrivial proper subgroups of G and its two distinct vertices H and K are adjacent if and only if $H \cap K = 1$, where 1 denotes the trivial subgroup of G. In this paper, we classify all finite groups whose complement of the intersection graph of subgroups is one of totally disconnected, bipartite, complete bipartite, tree, star graph or C_3-free. Also we characterize all the finite groups whose complement of the intersection graph of subgroups is planar.

Keywords: complement of intersection graph of subgroups, bipartite graph, planar graph.

AMS Subject classification: 05C25, 05C10

1. Introduction

Bosak [3] initiated the study of the intersection graph of subsemigroups of a semigroup. Subsequently, Csákány and Pollák [5] defined the intersection graph of subgroups of a group. Over the past several years, many significant results on this graph have been established by several researchers; see, for instance [1, 2, 7, 9, 10, 12–17, 19]. Let G be a group. The intersection graph of subgroups of a group G, denoted by $\mathscr{I}(G)$, is

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the graph whose vertex set is the set of all nontrivial proper subgroups of G and its distinct vertices H and K are adjacent if and only if $H \cap K \neq 1$, where 1 denotes the trivial subgroup of G.

The complement of the intersection graph of subgroups of a group G, denoted by $\mathscr{I}^c(G)$, is the graph whose vertex set is the set of all nontrivial proper subgroups of G and its distinct vertices H and K are adjacent if and only if $H \cap K = 1$. This graph was considered firstly by Visveswaran and Vadhel in [18]. Therein, they have studied the connectedness, diameter, girth, clique number, chromatic number and completeness of this graph.

We use the standard terminology of graphs following [8]. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. G is said to be bipartite if $V(G)$ can be partition into two subsets V_1 and V_2 such that every edge join a vertex of V_1 to a vertex of V_2. A complete bipartite graph is a bipartite graph in which every vertex in one partition is adjacent with all the vertices in the other partition and is denoted by K_{m_1,m_2}, where $m_i = |V_i|$, $i = 1, 2$. In particular, $K_{1,m}$ is a star. The complete graph and the cycle graph on n vertices are denoted by K_n and C_n, respectively. A graph whose edge set is empty is called totally disconnected. A connected graph with out a cycle is called a tree. A graph is said to be planar if there is a plane embedding of this graph. The complement of a graph G is denoted by $\overline{G}$. For given two graphs G_1 and G_2, their join and union are denoted by $G_1 + G_2$ and $G_1 \cup G_2$ respectively. The generalized quaternion group of order $2^\alpha (\alpha \geq 3)$ is given by $Q_{2^\alpha} = \langle a, b \mid a^{2^{\alpha-2}} = b^4 = e, a^{2^{\alpha-3}} = b^2, bab^{-1} = a^{-1} \rangle$. The multiplicative order of a nonzero element $x \in \mathbb{Z}_n$ is denoted by $ord_n(x)$.

This paper is organized as follows. In Section 2, we classify all finite groups whose complement of the intersection graph of subgroups is one of bipartite, complete bipartite, tree, star graph, totally disconnected and C_3-free. In Section 3, we characterize all finite groups whose complement of intersection graph of subgroups is planar.

2. Groups with specified complement of intersection graph of subgroups

Theorem 1. *Let G be a finite group. Then $\mathscr{I}^c(G)$ is totally disconnected if and only if G is isomorphic to either $\mathbb{Z}_{p^\alpha} (\alpha \geq 1)$ or $Q_{2^\alpha} (\alpha \geq 3)$, where p is a prime number.*

Proof. Suppose $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where p_is are distinct primes and $\alpha_i \geq 1$ for all i. If $k \geq 2$, then G has at least two subgroups of prime order and so they are adjacent in $\mathscr{I}^c(G)$. Now we assume that $k = 1$. Suppose G has at least two subgroups of prime order, then they are adjacent in $\mathscr{I}^c(G)$ and so G must be isomorphic to either $\mathbb{Z}_{p^\alpha} (\alpha \geq 1)$ or $Q_{2^\alpha} (\alpha \geq 3)$. In either case, G has a unique subgroup of prime order and so all their subgroups intersect non-trivially. It follows that $\mathscr{I}^c(G)$ is totally disconnected. □

Theorem 2. *Let G be a finite group. Then the following are equivalent.*

(1) *G is isomorphic to one of $\mathbb{Z}_{p^\alpha}(\alpha \geq 1)$, $Q_{2^\alpha}(\alpha \geq 3)$ or $|G| = p^\alpha q^\beta(\alpha, \beta \geq 1)$ with G has a unique subgroup of each of distinct prime orders p, q;*

(2) *$\mathscr{I}^c(G)$ is bipartite;*

(3) *$\mathscr{I}^c(G)$ is C_3-free.*

Proof. Suppose $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where p_is are pairwise distinct primes and $\alpha_i \geq 1$. If $k \geq 3$, then G has at least three subgroups of distinct prime orders and so they form C_3 as a subgraph of $\mathscr{I}^c(G)$. Now we assume that $k = 1$. If G has a unique subgroup of order p_1, then $G \cong \mathbb{Z}_{p_1^{\alpha_1}}(\alpha_1 \geq 1)$ or $Q_{2^{\alpha_1}}(\alpha_1 \geq 3)$ and so by Theorem 1, $\mathscr{I}^c(G)$ is bipartite. Otherwise, G has three subgroups of order p_1, so they form C_3 as a subgraph of $\mathscr{I}^c(G)$. Next we assume that $k = 2$. Then $\mathscr{I}^c(G)$ is bipartite if and only if G has a unique subgroup of each of orders p_1 and p_2. For otherwise, the subgroups of prime orders forms C_3 as a proper subgraph of $\mathscr{I}^c(G)$. In this case, $\mathscr{I}^c(G)$ is bipartite with bipartition X and Y, where X is the set of all proper subgroups of G which contains the subgroup of order p_1 and Y is the set of all proper subgroups of G which contains the subgroup of order p_2 . So the proof follows. □

Corollary 1. *Let G be a finite group. Then the following are equivalent.*

(1) *$G \cong \mathbb{Z}_{pq}$, where p and q are distinct prime numbers;*

(2) *$\mathscr{I}^c(G)$ is a tree;*

(3) *$\mathscr{I}^c(G)$ is complete bipartite;*

(4) *$\mathscr{I}^c(G)$ is a star graph.*

Proof. We use Theorem 2 to prove this result, since the three type of graphs mentioned in this result are bipartite. If $G \cong \mathbb{Z}_{p^\alpha}(\alpha \geq 1)$, $Q_{2^\alpha}(\alpha \geq 3)$, then by Theorem 1, $\mathscr{I}^c(G)$ is neither a tree nor complete bipartite. Now we assume that $|G| = p^\alpha q^\beta(\alpha, \beta \geq 1)$ with G has a unique subgroup of each of distinct prime orders p and q. Suppose G has a subgroup of order pq, then this subgroup is an isolated vertex in $\mathscr{I}^c(G)$ and so $\mathscr{I}^c(G)$ is disconnected. Consequently, $\mathscr{I}^c(G)$ is neither a tree nor complete bipartite. Finally, suppose $G \cong \mathbb{Z}_{pq}$, then $\mathscr{I}^c(\mathbb{Z}_{pq}) \cong K_2$, which is a tree and a star graph. □

3. Planarity of $\mathscr{I}^c(G)$

In this section, we characterize all finite groups whose complement of intersection graph of subgroups is planar. The well-known Kuratowski's theorem [8, Theorem

11.13] states that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.
The main result of this section is the following.

Theorem 3. *Let G be a finite group. Then $\mathscr{I}^c(G)$ is planar if and only if G is isomorphic to one the following groups.*

(1) $\mathbb{Z}_{p^\alpha}$ $(\alpha \geq 1)$, $\mathbb{Z}_{p^\alpha q^\beta}$ $(\alpha+\beta \leq 5)$, $\mathbb{Z}_{p^\alpha q^\beta r^\gamma}$ $(\alpha+\beta+\gamma \leq 5)$, $\mathbb{Z}_{pqrs}$, $\mathbb{Z}_{2^\alpha} \times \mathbb{Z}_2 (\alpha \geq 2)$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $Q_{2^\alpha}(\alpha \geq 3)$, S_3, $Q_{2^\alpha} \times \mathbb{Z}_p(\alpha \geq 3)$, $Q_{2^\alpha} \times \mathbb{Z}_{p^2}(\alpha \geq 3)$, *where p, q, r, s are distinct primes;*

(2) $\langle a, b \mid a^q = b^{p^2} = 1, bab^{-1} = a^i, ord_q(i) = p^2 \rangle \cong \mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}$, *where p, q are distinct primes with $p < q$ and $p^2 \mid (q-1)$.*

(3) $|G| = p^\alpha q$ *or* $p^\alpha q^2$ $(\alpha \geq 3)$ *with G has a unique Sylow q-subgroup; Sylow p-subgroup is not unique and each of them is isomorphic to $\mathbb{Z}_{p^\alpha}$ or Q_{2^α} and they intersect with each other non-trivially, where p, q are distinct primes.*

To prove this main result, we start with the following.

Proposition 1. *If G is a finite group whose order has at least five distinct prime factors, then $\mathscr{I}^c(G)$ contains K_5.*

Proof. By [18, Proposition 3.1], $\mathscr{I}^c(G)$ contains K_n, $n \geq 5$, so the proof follows. ☐

Proposition 2. *Let G be a group of order $p_1^{\alpha_1} p_1^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4}$, where p_is are pairwise distinct prime numbers and $\alpha_i \geq 1$ for $i = 1, 2, 3, 4$. Then $\mathscr{I}^c(G)$ is planar if and only if $G \cong \mathbb{Z}_{p_1 p_2 p_3 p_4}$.*

Proof. Suppose G is nilpotent. Then G is the direct product of Sylow p_i-subgroups for $i = 1, 2, 3, 4$. If $\alpha_i \geq 2$ for some i; with out loss of generality, we assume that $\alpha_1 \geq 2$. Then $\mathscr{I}^c(G)$ contains $K_{3,3}$ as a subgraph with bipartition (X, Y), where X is a set of three subgroups of G whose orders are $p_1^{\alpha_1}$, p_1, p_2, respectively and Y is a set of three subgroups of G whose orders are p_3, p_4, p_3p_4, respectively. If $\alpha_i = 1$ for every $i = 1, 2, 3, 4$, then $G \cong \mathbb{Z}_{p_1 p_2 p_3 p_4}$ and so $\mathscr{I}^c(G)$ is planar as shown in Figure 1, where H_i, $i = 1, 2, \ldots, 14$ are the subgroups of G of order p_1, p_2, p_3, p_4, p_1p_2, p_1p_3, p_1p_4, p_2p_3, p_2p_4, p_3p_4, $p_1p_2p_3$, $p_1p_2p_4$, $p_1p_3p_4$, $p_2p_3p_4$ respectively.
Next, suppose G is non-nilpotent. With out loss of generality, we may assume that Sylow p_1-subgroup of G is not unique. Then G has at least three Sylow p_1-subgroups. In this case, $\mathscr{I}^c(G)$ contains $K_{3,3}$ as a subgraph with bipartition (X, Y), where X is a set of three Sylow p_1-subgroups of G and Y is a set of three subgroups of G whose orders are $p_2^{\alpha_2}$, $p_3^{\alpha_3}$, $p_4^{\alpha_4}$, respectively. ☐

Proposition 3. *Let G be a group of order $p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$, where p_is are pairwise distinct prime numbers and $\alpha_i \geq 1$ for $i = 1, 2, 3$. Then $\mathscr{I}^c(G)$ is planar if and only if $G \cong \mathbb{Z}_{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}}$ with $\alpha_1 + \alpha_2 + \alpha_3 \leq 4$.*

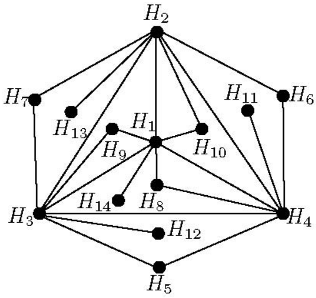

Figure 1. $\mathscr{I}^c(\mathbb{Z}_{p_1p_2p_3p_4})$

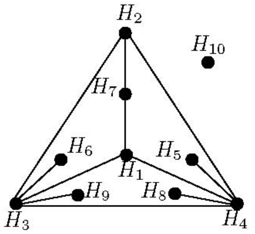

Figure 2. $\mathscr{I}^c(\mathbb{Z}_{p_1^2p_2p_3})$

Proof. Suppose G is nilpotent. Then G is the direct product of Sylow p_i-subgroups for $i = 1, 2, 3$. Then we have three cases to consider.

Case 1. If $\alpha_1 + \alpha_2 + \alpha_3 \leq 4$, then G is abelian. If G is cyclic, then $\mathscr{I}^c(G)$ is planar as shown in Figure 2, where H_i, $i = 1, 2, \ldots, 10$ are subgroups of G of order p_1, p_1^2, p_2, p_3, p_1p_2, p_1p_3, p_2p_3, $p_1^2p_3$, $p_1^2p_3$, $p_1p_2p_3$ respectively. If G is non-cyclic, then G has at least five subgroups of prime orders and so they form K_5 as a subgraph of $\mathscr{I}^c(G)$.

Case 2. If $\alpha_1 \geq 3$, $\alpha_2 = \alpha_3 = 1$, then $\mathscr{I}^c(G)$ contains $K_{3,3}$ as a proper subgraph with bipartition (X, Y), where X is a set of three subgroups of G whose orders are p_1, p_1^2, p_1^3, respectively and Y is a set of three subgroups of G whose orders are p_2, p_3, p_2p_3, respectively.

Case 3. If $\alpha_1 \geq 2$, $\alpha_2 \geq 2$, $\alpha_3 = 1$, then G has subgroups H_i, $i = 1, 2, \ldots, 7$ of order p_1, p_1^2, p_2, p_2^2, p_3, p_2p_3, p_1p_3, respectively. It follows that $\mathscr{I}^c(G)$ contains a subdivision of K_5 as shown in Figure 3.

Next, suppose G is non-nilpotent. With out loss of generality, we may assume that Sylow p_1-subgroup of G is not unique. Then G has at least three Sylow p_1- subgroups. It follows that $\mathscr{I}^c(G)$ contains $K_{3,3}$ as a subgraph with bipartition (X, Y), where X is a set of three Sylow p_1-subgroups of G and Y is a set of three subgroups of G whose orders are $p_2, p_3, p_2^ip_3^j$, respectively. □

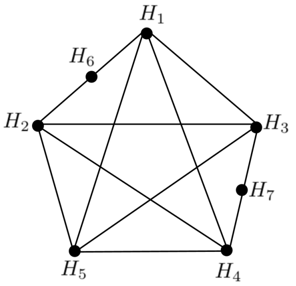

Figure 3. **A subdivision of K_5 in $\mathscr{I}^c(G)$**

Proposition 4. *Let G be an abelian group of order either $p_1^{\alpha_1}$ or $p_1^{\alpha_1}p_2^{\alpha_2}$, where p_1, p_2 are distinct prime numbers and α_1, $\alpha_2 \geq 1$. Then $\mathscr{I}^c(G)$ is planar if and only if G is isomorphic to either $\mathbb{Z}_{p_1^{\alpha_1}}$, $\mathbb{Z}_{p_1^{\alpha_1}p_2^{\alpha_2}}$ $(\alpha_1 + \alpha_2 \leq 5)$, $\mathbb{Z}_{2^{\alpha_1}} \times \mathbb{Z}_2$, $\alpha_1 \geq 1$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

Proof. Proof is divided in to two cases.

Case 1. Suppose G is cyclic. If $|G| = p_1^{\alpha_1}$, then by Theorem 1, $\mathscr{I}^c(G)$ is planar. If $|G| = p_1^{\alpha_1}p_2^{\alpha_2}$, then the subgroups of G are H_i, K_j, N_{ij}, where $|H_i| = p_1^i$, $|K_j| = p_2^j$, $|N_{ij}| = p_1^i p_2^j$ for $i = 1, 2, \ldots, \alpha_1$, $j = 1, 2, \ldots, \alpha_2$. It can be seen that $\mathscr{I}^c(G) \cong K_{\alpha_1,\alpha_2} \cup K_{\alpha_1\alpha_2 - 1}$. Therefore, $\mathscr{I}^c(G)$ is planar only when $\alpha_1 + \alpha_2 \leq 5$.

Case 2. Suppose G is non-cyclic.

Subcase 2a. If $G \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1}$, then $\mathscr{I}^c(G) \cong K_{p_1+1}$ and so it is planar only when $p = 2$, 3.

Subcase 2b. If $G \cong \mathbb{Z}_{p_1^2} \times \mathbb{Z}_{p_1}$, then from the subgroup lattice of G, it can be seen that

$$\mathscr{I}^c(G) \cong K_1 \cup (K_{p_1} + \overline{K}_{p_1+1}) \tag{1}$$

and so it is planar only when $p = 2$.

Subcase 2c. If $G \cong \mathbb{Z}_{p_1p_2} \times \mathbb{Z}_{p_1}$, then G has exactly $p_1 + 2$ subgroups of prime order and they have trivial intersection with each other. The remaining subgroups of G have non-trivial intersection with each other. It follows that $\mathscr{I}^c(G)$ is planar only when $p_1 = 2$.

Subcase 2d. If $G \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_1}$, then from the subgroup lattice of G, we have

$$\mathscr{I}^c(G) \cong K_{\alpha_1-2} \cup (K_{p_1} + \overline{K}_{(\alpha_1-2)p_1+1}) \tag{2}$$

It follows that $\mathscr{I}^c(G)$ is planar only when $p = 2$.

Subcase 2e. If $G \cong \mathbb{Z}_{p_1^2} \times \mathbb{Z}_{p_1^2} := \langle a, b \mid a^{p_1^2} = b^{p_1^2} = 1, ab = ba \rangle$. Then $K_{3,3}$ is a subgraph of $\mathscr{I}^c(G)$ with bipartition $\{\langle a \rangle, \langle a, b^2 \rangle, \langle a^2 \rangle\}$ and $\{\langle a, b \rangle, \langle a^2, b \rangle, \langle a^3, b \rangle\}$.

Subcase 2f. If $G \cong \mathbb{Z}_{p_1^2p_2} \times \mathbb{Z}_{p_1}$ or $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1}$, then G has at least five subgroups of prime order and so $\mathscr{I}^c(G)$ contains K_5.

Subcase 2g. If $G \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}}, k \geq 2$, then G has one of $\mathbb{Z}_{p_1^2} \times \mathbb{Z}_{p_1^2}$, $\mathbb{Z}_{p_1^2 p_2} \times \mathbb{Z}_{p_1}$ or $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1}$ as a subgroup. Then by Subcases 2e and 2f, $\mathscr{I}^c(G)$ contains either $K_{3,3}$ or K_5. □

Proposition 5. *Let G be a non-abelian group of order p^α, where p is a prime number and $\alpha \geq 3$. Then $\mathscr{I}^c(G)$ is planar if and only if $G \cong Q_{2^\alpha}$ or M_{2^α}.*

Proof. Suppose $\alpha = 3$. Up to isomorphism, there are four groups of order p^3, including the group Q_8. Except Q_8, the remaining three groups have at least five subgroups of prime order and so they form K_5 as a subgraph of $\mathscr{I}^c(G)$.
Suppose $p > 2$. Then G has a non-cyclic subgroup H of order $p^{\alpha-1}$. So by the above argument and Proposition 4, $\mathscr{I}^c(H)$ is non-planar.
Suppose $p = 2$ and $\alpha \geq 4$. If $G \not\cong Q_{2^\alpha}$, $Q_{2^{\alpha-1}} \times \mathbb{Z}_2$ and M_{2^α}, then either G contains a non-cyclic subgroup, say H of order $2^{\alpha-1}$ or it contains at least five subgroups of order 2. So by above argument and Proposition 4, $\mathscr{I}^c(H)$ is non-planar or $\mathscr{I}^c(G)$ contains K_5. Next we investigate the remaining possibilities.
If $G \cong Q_{2^\alpha}$, then by Proposition 1, $\mathscr{I}^c(Q_{2^\alpha})$ is planar.
If $G \cong \mathbb{Q}_{2^{\alpha-1}} \times \mathbb{Z}_2$, then we split the set of all non-trivial proper subgroups of G in to five mutually disjoint subsets: The first subset consist of subgroups $\langle a^{2^{\alpha-3}} c\rangle$ and $\langle c\rangle$, where c denotes the generator of $\mathbb{Z}_2$. The second subset consists of the subgroups $\langle ac\rangle$, $\langle a^2c\rangle, \ldots, \langle a^{2^{\alpha-4}}c\rangle$, $\langle bc\rangle$, $\langle abc\rangle$, $\langle a^2bc\rangle, \ldots, \langle a^{2^{\alpha-3}-3}c\rangle$. The third subset consists of the subgroup $\langle a^{2^{\alpha-3}}\rangle$. The fourth subset consists of all the subgroups of $Q_{2^{\alpha-1}}$ except $\{e\}$. The fifth subset consists of the remaining subgroups of G. It can be seen that any two subgroups in the union of these subsets, except the first subset intersect non-trivially. Each subgroup in the first subset intersect trivially with the subgroups in the second, third and fourth subsets; at the same time it intersect non-trivially with the subgroups in the fifth subset. Also the two subgroups in the first subset intersect trivially. From the above description, it is easy to see that the structure of $\mathscr{I}^c(G)$ as shown in Figure 4 and so it is planar.
If $G \cong M_{2^\alpha}$ then its subgroup lattice is isomorphic to the subgroup lattice of $\mathbb{Z}_{2^\alpha} \times \mathbb{Z}_2$. By Theorem 4, $\mathscr{I}^c(G)$ is planar and

$$\mathscr{I}^c(M_{2^\alpha}) \cong K_{\alpha-2} \cup (K_2 + \overline{K}_{2\alpha-4}). \tag{3}$$

This completes the proof. □

Proposition 6. *Let G be the non-abelian group of order pq, where p and q are distinct primes with $p < q$ and p divides $q - 1$. Then $\mathscr{I}^c(G)$ is planar if and only if $G \cong S_3$.*

Proof. Since G has $q + 1$ subgroups of prime order and these are the only proper subgroups of G, it follows that $\mathscr{I}^c(G) \cong K_{q+1}$. Therefore, $\mathscr{I}^c(G)$ is planar if and only if $q = 3$. □

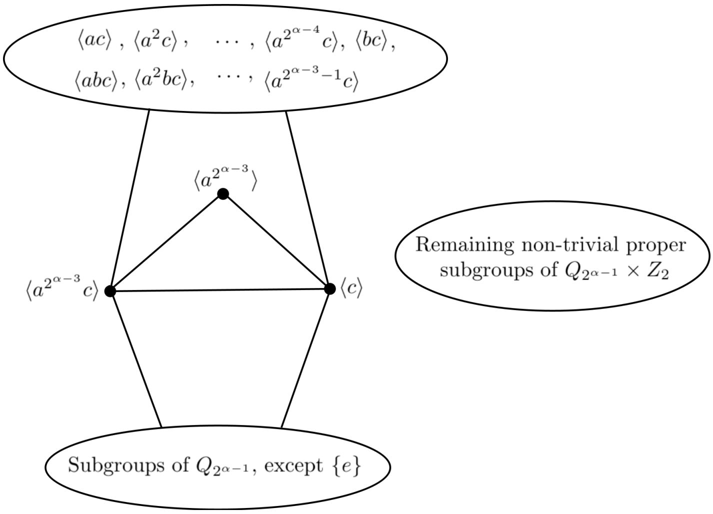

Figure 4. **The structure of $\mathscr{I}^c(Q_{2\alpha-1}\times\mathbb{Z}_2)$**

Proposition 7. *Let G be a non-abelian group of order p^2q, where p, q are distinct primes. Then $\mathscr{I}^c(G)$ is planar if and only if $G\cong\langle a,b\mid a^q=b^{p^2}=1, bab^{-1}=a^i, ord_q(i)=p^2\rangle\cong\mathbb{Z}_q\rtimes\mathbb{Z}_{p^2}$, where p,q are distinct primes with $p<q$ and $p^2\mid(q-1)$.*

Proof. According to [4], there are eight groups of order p^2q. It can be seen that all these groups, except $\mathbb{Z}_q\rtimes\mathbb{Z}_{p^2}$ have at least five subgroups of prime order and so they form K_5 as a subgraph of $\mathscr{I}^c(G)$. If $G\cong\mathbb{Z}_q\rtimes\mathbb{Z}_{p^2}=\langle a,b\mid a^q=b^{p^2}=1, bab^{-1}=a^i, ord_q(i)=p^2\rangle$, where p,q are distinct primes with $p<q$ and $p^2\mid(q-1)$. Then $\langle a\rangle$, $\langle b^p\rangle$, $\langle ab^p\rangle$ and $\langle a^ib\rangle$, where $i=1,2,\ldots,q$ are the only nontrivial proper subgroups of G. Here $\langle b^p\rangle$ is contained in these subgroups, except $\langle a\rangle$. Also $\langle a\rangle$ is a subgroup of $\langle ab^p\rangle$. It follows that

$$\mathscr{I}^c(\mathbb{Z}_q\rtimes\mathbb{Z}_{p^2})\cong K_1\cup K_{1,q+1}. \tag{4}$$

Therefore, $\mathscr{I}^c(\mathbb{Z}_q\rtimes\mathbb{Z}_{p^2})$ is planar. □

Proposition 8. *Let G be a non-abelian group of order $p^\alpha q$, where p, q are distinct primes and $\alpha\geq 3$. Then $\mathscr{I}^c(G)$ is planar if and only if $G\cong Q_{2^\alpha}\times\mathbb{Z}_q$ or G has a unique Sylow q-subgroup; Sylow p-subgroup is not unique and each of them is isomorphic to $\mathbb{Z}_{p^\alpha}$ or Q_{2^α} and they intersect with each other non-trivially.*

Proof. Let P and Q be a Sylow p-subgroup and a Sylow q-subgroup of G, respectively. Suppose $\mathscr{I}^c(P)$ is non-planar, then $\mathscr{I}^c(G)$ is so. Therefore, it is enough to consider the cases when $\mathscr{I}^c(P)$ is planar. By Propositions 4 and 5, $P\cong\mathbb{Z}_{p^\alpha}$, $\mathbb{Z}_{2^\alpha}\rtimes\mathbb{Z}_2$, M_{p^α}, $Q_{2^{\alpha-1}}\times\mathbb{Z}_2$ or Q_{2^α}.

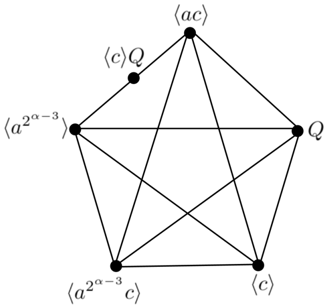

Figure 5. **A subdivision of K_5 in $\mathscr{I}^c(G)$**

Case 1. Suppose Sylow q-subgroup of G is not unique. Then $\mathscr{I}^c(G)$ has $K_{3,3}$ as a subgraph with bipartition (X, Y), where X is a set of three subgroups of G whose orders are p, p^2, p^3, respectively and Y is a set of three subgroups of G whose orders are q, q, q, respectively.

Case 2. Suppose Sylow q-subgroup of G is unique.
If $P \cong Q_{2^{\alpha-1}} \times \mathbb{Z}_2$, then $\langle c\rangle$, $\langle ac\rangle$, $\langle a^{2^{\alpha-3}}c\rangle$, $\langle a^{2^{\alpha-3}}\rangle$ are subgroups of P as mentioned in the proof of Proposition 5. These four subgroups together with Q, $\langle c\rangle Q$ forms a subdivision of K_5 in $\mathscr{I}^c(G)$, which is shown in Figure 5. So $\mathscr{I}^c(G)$ is non-planar. Next we investigate the remaining possibilities.

Subcase 2a. Suppose Sylow p-subgroup is not unique and these subgroups intersect with each other trivially. Then G has subgroups H_i, $i = 1, 2, \ldots, 6$ of order p^α, p^α, p^α, q, p, p^2, respectively such that H_6 is a subgroup of H_1; but not a subgroup of H_2 and H_3. These subgroups form a subdivision of K_5 in $\mathscr{I}^c(G)$, which is isomorphic to the graph shown in Figure 5.

Subcase 2b. Suppose Sylow p-subgroup is not unique and these subgroups intersect with each other non-trivially. If $P \cong \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_2$ or M_{2^α}, then by (2), $\mathscr{I}^c(G)$ has $K_{2,3}$ as a subgraph. Since $\mathbb{Z}_q$ is adjacent with all the vertices of $\mathscr{I}^c(P)$, it follows that $\mathscr{I}^c(G)$ has $K_{3,3}$ as a subgraph. If $P \cong \mathbb{Z}_{p^\alpha}$ or Q_{2^α}, then G has exactly two subgroups, one having order 2 and the other having order q; subgroups of $\mathbb{Z}_{p^\alpha}$ are adjacent with $\mathbb{Z}_q$ and the remaining subgroups of G intersect non-trivially. It follows that $\mathscr{I}^c(G)$ is planar.

Subcase 2c. Suppose Sylow p-subgroup is unique. Then G is the direct product of its Sylow p-subgroup and Sylow q-subgroup. If $P \cong \mathbb{Z}_{p^\alpha}$, $\mathbb{Z}_{2^\alpha} \times \mathbb{Z}_2$, then G is abelian, which is not possible. So it forces that $P \cong M_{2^4}$ or Q_{2^α}. If $G \cong M_{2^4} \times \mathbb{Z}_q$, then by (2) and by the above argument, $\mathscr{I}^c(G)$ has $K_{3,3}$ as a subgraph. If $G \cong Q_{2^\alpha} \times \mathbb{Z}_q$, then G has exactly two subgroups, one of which has order 2 other has order q; and so the remaining subgroups of G intersect non-trivially. It follows that $\mathscr{I}^c(G)$ is planar. □

Proposition 9. *Let G be a non-abelian group of order p^2q^2, where p, q are distinct prime numbers. Then $\mathscr{I}^c(G)$ is non-planar.*

Proof. According to [11], there are four subgroups of order p^2q^2 when $(p,q) \neq (3,2)$ and nine groups of order 36 when $(p,q) = (3,2)$. Using the subgroups information of these groups given in [6, pages 40-43], it can be directly seen that the complement of intersection graph of subgroups of these groups have K_5 as a subgraph, except the following two groups, which have to be considered separately.
The first group is $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{q^2} := \langle a, b, c \mid a^p = b^p = c^{q^2} = 1, ab = ba, cac^{-1} = a^i b^j, cbc^{-1} = a^k b^l \rangle$, where $\left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right)$ has order q^2 in $GL_2(p)$ and the second group is $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_q \times \mathbb{Z}_q)$ where $(p,q) \neq (3,2)$. Notice that in each of these groups $p \geq 5$ and it has $\mathbb{Z}_p \times \mathbb{Z}_p$ as its subgroup, so by Proposition 4, $\mathscr{I}^c(\mathbb{Z}_p \times \mathbb{Z}_p)$ is non-planar. Consequently, the complement of the intersection graph of subgroups of these two groups are non-planar. ☐

Proposition 10. *Let G be a non-abelian group of order $p^\alpha q^2$, where p, q are distinct prime numbers, $\alpha \geq 3$. Then $\mathscr{I}^c(G)$ is planar if and only if $G \cong Q_{2^\alpha} \times \mathbb{Z}_{q^2}$ or G has a unique Sylow q-subgroup; Sylow p-subgroup is not unique and each of them is isomorphic to $\mathbb{Z}_{p^\alpha}$ or Q_{2^α} and they intersect with each other non-trivially.*

Proof. Let P and Q be a Sylow p-subgroup and a Sylow q-subgroup of G, respectively. Suppose either $\mathscr{I}^c(P)$ or $\mathscr{I}^c(Q)$ is non-planar, then $\mathscr{I}^c(G)$ is so. Therefore, it is enough to consider the cases when both $\mathscr{I}^c(P)$ and $\mathscr{I}^c(Q)$ are planar. By Propositions 4 and 5, $P \cong \mathbb{Z}_{p^\alpha}$, $\mathbb{Z}_{2^\alpha} \times \mathbb{Z}_2$, M_{2^α}, $Q_{2^{\alpha-1}} \times \mathbb{Z}_2$ or Q_{2^α} and $Q \cong \mathbb{Z}_{q^2}$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Case 1. Suppose Q is not unique. Then $\mathscr{I}^c(G)$ contains $K_{3,3}$ as a subgraph with bipartition (X, Y), where X is a set of three subgroups of G whose orders are p^3, p^2, p, respectively and Y is a set of three subgroups of G whose orders are q, q, q^2, respectively.

Case 2. Suppose Q is unique.
If $P \cong Q_{2^{\alpha-1}} \times \mathbb{Z}_2$, then $\langle c \rangle$, $\langle ac \rangle$, $\langle a^{2^{\alpha-3}} c \rangle$, $\langle a^{2^{\alpha-3}} \rangle$ are subgroups of P as mentioned in the proof of Proposition 5. These four subgroups together with Q, $\langle c \rangle Q$ forms a subdivision of K_5 in $\mathscr{I}^c(G)$, which is shown in Figure 5. So $\mathscr{I}^c(G)$ is non-planar.
Next we investigate the remaining possibilities.

Subcase 2a. Suppose P is unique. Then $G \cong P \times Q$ and so by the above argument, G is isomorphic to one of $Q_{2^\alpha} \times \mathbb{Z}_{q^2}$, $Q_{2^\alpha} \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $M_{2^\alpha} \times \mathbb{Z}_{q^2}$ or $M_{2^\alpha} \times \mathbb{Z}_3 \times \mathbb{Z}_3$. If $G \cong Q_{2^\alpha} \times \mathbb{Z}_3 \times \mathbb{Z}_3$, then G has five subgroups of prime order and so they form K_5 as a subgraph of $\mathscr{I}^c(G)$. If $G \cong Q_{2^\alpha} \times \mathbb{Z}_{q^2}$, then G contains unique subgroups H_1 and H_2 of order 2 and 3, respectively. Here H_1 and H_2 are adjacent with all the subgroups of $\mathbb{Z}_{p^2}$ and Q_{2^α}, respectively. The remaining proper subgroups of G contains H_1 and H_2. It follows that $\mathscr{I}^c(G)$ is planar. If $G \cong M_{2^\alpha} \times \mathbb{Z}_{q^2}$, then by (2), $\mathscr{I}^c(M_{2^\alpha})$ has $K_{2,3}$ as a subgraph. Notice that $\mathbb{Z}_{p^2}$ has a trivial intersection with the subgroups corresponding to the vertices of $K_{2,3}$ and so $\mathscr{I}^c(G)$ has $K_{3,3}$ as a subgraph. If $G \cong M_{2^\alpha} \times \mathbb{Z}_3 \times \mathbb{Z}_3$, then by a similar argument as above, it can be seen that $\mathscr{I}^c(G)$ has $K_{3,3}$ as a subgraph.

Subcase 2b. Suppose P is not unique and Sylow p-subgroups of G mutually intersect non-trivially.
Suppose $Q \cong \mathbb{Z}_{q^2}$. If $P \cong \mathbb{Z}_{p^\alpha}$ or Q_{2^α}, then G contains unique subgroups of each of orders p and q and these subgroups are contained in the remaining proper subgroups of G. Therefore, $\mathscr{I}^c(G)$ is planar. If $P \cong \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_2$ or M_{2^α}, then by (2), $\mathscr{I}^c(M_{2^\alpha})$ has $K_{2,3}$ as a subgraph. Here $\mathbb{Z}_{p^2}$ is adjacent with all the vertices of $K_{2,3}$ and so $\mathscr{I}^c(G)$ contains $K_{3,3}$ as a subgraph.
Suppose $Q \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $\mathscr{I}^c(G)$ has $K_{3,3}$ as a proper subgraph with bipartition (X, Y), where X is a set of three subgroups of G each having order 2 and Y is a set of three subgroups of G whose orders are p, p^2, p^3, respectively. Suppose $Q \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then G has four subgroups each having order 3 and has a subgroup of order p. These five subgroups form K_5 as a subgraph of $\mathscr{I}^c(G)$.

Subcase 2c. Suppose P is not unique and Sylow p-subgroups of G mutually intersect trivially.
If $Q \cong \mathbb{Z}_{q^2}$, then G contains distinct subgroups of order p, p, p^2, p^3, q, q^2. They form a subdivision of K_5 in $\mathscr{I}^c(G)$, which is isomorphic to the graph shown in Figure 5.
For the remaining cases, we can apply the same argument as in Subcase 2b and obtain that $\mathscr{I}^c(G)$ contains either K_5 or $K_{3,3}$. ☐

Proposition 11. *Let G be a non-abelian group of order $p^\alpha q^\beta$, where p, q are distinct prime numbers, $\alpha, \beta \geq 3$, $\alpha + \beta \geq 6$. Then $\mathscr{I}^c(G)$ is non-planar.*

Proof. Here $\mathscr{I}^c(G)$ has $K_{3,3}$ as a subgraph with bipartition (X, Y), where X is a set of three subgroups of G whose orders are p^α, p, p^2, respectively and Y is a set of three subgroups of G whose orders are q^α, q, q^2, respectively. ☐

Proof of Theorem 3: Combining all the results that have been established thus far in this section, we arrive at the desired outcome. ☐

Notice that $\mathbb{Z}_q \rtimes_t \mathbb{Z}_{p^\alpha} := \langle a, b \mid a^q = b^{p^\alpha} = 1, bab^{-1} = a^i, ord_q(i) = p^t \rangle$, where $p^t \mid (q-1)$ and $\mathbb{Z}_{q^2} \rtimes_t \mathbb{Z}_{p^\alpha} := \langle a, b \mid a^{q^2} = b^{p^\alpha} = 1, bab^{-1} = a^i, ord_{q^2}(i) = p^t \rangle$, where $p^t \mid (q^2 - 1)$ shows the existence of groups of order $p^\alpha q$ and $p^\alpha q^2$, respectively satisfying the condition (2) of Theorem 3. However, classifying these groups seems to be a challenging issue that requires more research.

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