Communications in Combinatorics and Optimization
Vol. 9, No. 4 (2024), pp. 813-825
https://doi.org/10.22049/cco.2023.28762.1707

CCO
Commun. Comb. Optim.

Short Note

Some observations on sombor coindex of graphs

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Received: 18 June 2023; Accepted: 5 September 2023
Published Online: 10 September 2023

Abstract: Let $G = (V, E)$, $V = \{v_1, v_2, \dots, v_n\}$, be a simple graph of order n and size m, without isolated vertices. The Sombor coindex of a graph G is defined as $\overline{SO}(G) = \sum_{i \nsim j} \sqrt{d_i^2 + d_j^2}$, where $d_i = d(v_i)$ is a degree of vertex v_i, $i = 1, 2, \dots, n$. In this paper we investigate a relationship between Sombor coindex and a number of other topological coindices.

Keywords: Sombor index, Sombor coindex

AMS Subject classification: 05C07, 05C35

1. Introduction

Let $G = (V, E)$ be a simple graph without isolated vertices with vertex set $V = \{v_1, v_2, \dots, v_n\}$, edge set $E = \{e_1, e_2, \dots, e_m\}$, and with vertex–degree sequence $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$, $d_i = d(v_i)$. If vertices v_i and v_j are adjacent in G, we write $i \sim j$, otherwise we write $i \nsim j$. The complement of G is a graph $\overline{G} = (V, \overline{E})$ which has the same vertex set V and two vertices v_i and v_j are adjacent in $\overline{G}$ if and only if they are not adjacent in G and vice versa. The number of edges in $\overline{G}$ is equal to

$$\overline{m} = \frac{n(n-1)}{2} - m. \tag{1}$$

A topological index (graph invariant) of a graph is a numerical quantity which is invariant under automorphisms of the graph. Topological indices are an important tool used to relate molecular structure with physicochemical characteristics of chemical

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compounds, especially those relevant for their pharmacological, medical, toxicological, and similar properties (see, for example, [11, 26]).
Topological indices can be defined in terms of vertex degrees, such as Zagreb indices [15]), in terms of edge degrees, such as Platt index [23], vertex distance in graph, such as Wiener index [29], or on some graph spectra, such as Kemeny's constant [17].
Many vertex–degree–based topological indices can be represented in the form

$$TI(G) = \sum_{i\sim j} F(d_i, d_j), \tag{2}$$

where $F(x, y)$ is a real non–negative symmetric function, $F(x, y) = F(y, x)$, defined on a cartesian product $D \times D$, where $D = \{d_1, d_2, \ldots, d_n\}$. Here we list some particular cases obtained from (2) by appropriate choice of function $F(x, y)$ that are of interest for the present paper.

- For $F(x, y) = x + y$ we obtain the first Zagreb index, $M_1(G)$, defined in [16] (see also [7]) as

$$M_1(G) = \sum_{i\sim j}(d_i + d_j) = \sum_{i=1}^{n} d_i^2.$$

  Let $e = \{v_i, v_j\} \in E$ be an arbitrary edge in graph G. The degree of edge e, $d(e)$, is $d(e) = d_i + d_j - 2$. Hence we have that

$$M_1(G) = \sum_{i\sim j}(d_i + d_j) = \sum_{i=1}^{m}(d(e_i) + 2)\,.$$

  Therefore $M_1(G)$ can be considered as an edge–degree–based topological index, as well (see [18]).

- For $F(x, y) = xy$ we get the second Zagreb index, $M_2(G)$, introduced in [15] as

$$M_2(G) = \sum_{i\sim j} d_i d_j.$$

- For $F(x, y) = x^2 + y^2$ we obtain the forgotten topological index, $F(G)$, defined in [10] as

$$F(G) = \sum_{i\sim j}(d_i^2 + d_j^2) = \sum_{i=1}^{n} d_i^3.$$

  Since

$$F(G) = \sum_{i\sim j}(d_i^2 + d_j^2) = \sum_{i\sim j}(d_i + d_j)^2 - 2\sum_{i\sim j} d_i d_j = \sum_{i=1}^{m}(d(e_i) + 2)^2 - 2\sum_{i\sim j} d_i d_j,$$

  this topological index can also be considered as an edge–degree–based topological index.

- For $F(x,y) = \sqrt{x^2+y^2}$, the Sombor index introduced in [12] is obtained

$$SO(G) = \sum_{i\sim j} \sqrt{d_i^2 + d_j^2}\,.$$

- For $F(x,y) = \frac{xy}{x+y}$ we get the inverse sum indeg index, $ISI(G)$, defined in [28] as

$$ISI(G) = \sum_{i\sim j} \frac{d_i d_j}{d_i + d_j}.$$

- For $F(x,y) = \frac{x^2+y^2}{xy}$ we get symmetric division deg index, $SDD(G)$, defined in [27] as

$$SDD(G) = \sum_{i\sim j} \frac{d_i^2 + d_j^2}{d_i d_j}.$$

- For $F(x,y) = \frac{2}{x+y}$ and $F(x,y) = \frac{1}{x^2} + \frac{1}{y^2}$ harmonic index, $H(G)$, and inverse degree index, $ID(G)$, are obtained respectively (see [8])

$$H(G) = \sum_{i\sim j} \frac{2}{d_i + d_j} \quad \text{and} \quad ID(G) = \sum_{i\sim j} \left(\frac{1}{d_i^2} + \frac{1}{d_j^2} \right) = \sum_{i=1}^{n} \frac{1}{d_i}.$$

- For $F(x,y) = |x-y|$ the Albertson index, $Alb(G)$, used as an irregularity measure of a graph is obtained. It is defined in [1] as

$$Alb(G) = \sum_{i\sim j} |d_i - d_j|.$$

- For $F(x,y) = \frac{1}{xy}$ we obtain the modified second Zagreb index, $M_2^*(G)$, defined in [20] as

$$M_2^*(G) = \sum_{i\sim j} \frac{1}{d_i d_j}.$$

  This index is also known as the general Randić index, $R_{-1}(G)$, (see, for example, [3, 4, 25]), or reciprocal second Zagreb index, $RM_2(G)$, (see [14]).

A concept of coindices of graphs was introduced in [6]. In this case the sum runs over the edges of the complement of G. In a view of (2), the corresponding coindex of G can be defined as

$$\overline{TI}(G) = \sum_{i\nsim j} F(d_i, d_j). \tag{3}$$

Here we are concerned with the Sombor coindex defined as

$$\overline{SO}(G) = \sum_{i\nsim j} \sqrt{d_i^2 + d_j^2}\,,$$

and investigate its relationship with some other topological coindices obtained based on (2).

2. Preliminaries

In this section we recall some results from the literature that are of interest for our paper. In [22] the following results were proven.

Lemma 1. [22, Theorem 7] *Let G be a graph on n vertices and m edges. Then*

$$\overline{SO}(G) \leq \sqrt{\overline{m}\Delta\overline{M}_1(G)}. \tag{4}$$

Lemma 2. [22, Theorem 8] *Let G be a graph on n vertices and m edges. Then*

$$\overline{SO}(G) \leq \sqrt{\left(\frac{\Delta}{\delta}+\frac{\delta}{\Delta}\right)\overline{m}\overline{M}_2(G)}. \tag{5}$$

Lemma 3. [22, Theorem 2] *Let G be a graph on n vertices and m edges. Then*

$$\overline{SO}(G) \geq \frac{\delta n}{\sqrt{2}}(n-1-\Delta). \tag{6}$$

Equality holds if G is a regular graph.

Remark 1. In [19] it was proven that

$$SO(G) \leq \sqrt{mF(G)}\,.$$

On the other hand, in [22], it was stated, without proof, that

$$\overline{SO}(G) \leq \sqrt{\overline{m}\overline{F}(G)}. \tag{7}$$

Since

$$\overline{F}(G) \leq \Delta\overline{M}_1(G),$$

it follows that (4) is a corollary of (7).
For every i and j, $i,j=1,2,\ldots,n$, the following inequalities are valid

$$(d_i-\delta)(d_i-\Delta)\leq 0 \quad \text{and} \quad (d_j-\delta)(d_j-\Delta)\leq 0\,,$$

that is

$$d_i^2+\delta\Delta \leq (\Delta+\delta)d_i \quad \text{and} \quad d_j+\delta\Delta \leq (\Delta+\delta)d_j\,.$$

The sum of the above inequalities yields

$$d_i^2+d_j^2 \leq (\Delta+\delta)(d_i+d_j)-2\Delta\delta\,. \tag{8}$$

After summation of (8) over all pairs of non–adjacent vertices v_i and v_j in G, we obtain

$$\sum_{i\nsim j}(d_i^2+d_j^2) \leq (\Delta+\delta)\sum_{i\nsim j}(d_i+d_j)-2\Delta\delta\sum_{i\nsim j}1\,,$$

that is

$$\overline{F}(G) \leq (\Delta+\delta)\overline{M}_1(G) - 2\overline{m}\Delta\delta,$$

From the above and (7) we obtain

$$\overline{SO}(G) \leq \sqrt{\overline{m}\left((\Delta+\delta)\overline{M}_1(G) - 2\overline{m}\Delta\delta\right)},$$

which is stronger than (4).

In Lemma 2 the condition that G is without isolated vertices is missing. Otherwise, $\delta = 0$, and the right-hand side of (5) is not defined.

In [24] the following inequality for real number sequences was proven.

Lemma 4. [24] *Let $x = (x_i)$ and $a = (a_i)$, $i = 1, 2, \ldots, n$, be positive real number sequences. Then for any $r \geq 0$ holds*

$$\sum_{i=1}^{n} \frac{x_i^{r+1}}{a_i^r} \geq \frac{\left(\sum_{i=1}^{n} x_i\right)^{r+1}}{\left(\sum_{i=1}^{n} a_i\right)^{r}}. \tag{9}$$

Equality holds if and only if $r = 0$, or $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

Remark 2. The inequality (9) is known as Radon's inequality and it is given in its original form. However, it is not difficult to observe that it is valid for any real r such that $r \leq -1$ or $r \geq 0$, and when $-1 \leq r \leq 0$ the opposite inequality holds. The equality is also attained when $r = -1$.

3. Main results

In the next theorem we establish a relationship between $\overline{SO}(G)$ and $\overline{M}_1(G)$ and $\overline{ISI}(G)$.

Theorem 1. *Let G be a simple graph of order n and size m without isolated vertices. Then we have*

$$\overline{SO}(G) \leq \sqrt{\overline{M}_1(G)(\overline{M}_1(G) - 2\overline{ISI}(G))}. \tag{10}$$

Equality holds if and only if $\frac{d_i}{d_j} = const$ for any pair of non–adjacent vertices v_i and v_j $(i < j)$ in G.

Proof. For any i and j, $1 \leq i \leq n$, $1 \leq j \leq n$, we have

$$d_i + d_j - \frac{2d_id_j}{d_i+d_j} = \frac{d_i^2+d_j^2}{d_i+d_j}.$$

After summing the above equality over all pairs of non–adjacent vertices v_i and v_j of G, we obtain

$$\overline{M}_1(G) - 2\overline{ISI}(G) = \sum_{i \nsim j} \frac{d_i^2 + d_j^2}{d_i + d_j}. \tag{11}$$

On the other hand, for $r = 1$, $x_i := \sqrt{d_i^2 + d_j^2}$, $a_i := d_i + d_j$, with summing performed over all non–adjacent vertices v_i and v_j in graph G, the inequality (9) becomes

$$\sum_{i \nsim j} \frac{d_i^2 + d_j^2}{d_i + d_j} \geq \frac{\left(\sum_{i \nsim j} \sqrt{d_i^2 + d_j^2} \right)^2}{\sum_{i \nsim j} (d_i + d_j)} = \frac{\overline{SO}(G)^2}{\overline{M}_1(G)}. \tag{12}$$

Now, according to (11) and (12) we obtain

$$\frac{\overline{SO}(G)^2}{\overline{M}_1(G)} \leq \overline{M}_1(G) - 2\overline{ISI}(G),$$

from which we arrive at (10).

Equality in (12) holds if and only if $\frac{\sqrt{d_i^2+d_j^2}}{d_i+d_j} = const$ for any pair of non–adjacent vertices v_i and v_j in G. This means that $\sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}}$ is constant for any pair of non–adjacent vertices v_i and v_j in G. A short analysis shows that equality in (10) holds if and only if $\frac{d_i}{d_j} \in \{c, 1/c\}$, $c = const$, for any pair of non–adjacent vertices v_i and v_j $(i < j)$ in G. $\square$

Remark 3. Since

$$\overline{M}_1(G) - 2\overline{ISI}(G) \leq \overline{m}\Delta,$$

the inequality (10) is stronger than (4).

In the next theorem we establish a relationship between $\overline{SO}(G)$ and $\overline{M}_2(G)$ and $\overline{SDD}(G)$.

Theorem 2. *Let G be a simple graph of order $n \geq 2$ and size m without isolated vertices. Then*

$$\overline{SO}(G) \leq \sqrt{\overline{M}_2(G)\overline{SDD}(G)}. \tag{13}$$

Equality holds if and only if for any pair of vertices v_j and v_k in G which are not adjacent to v_i holds $d_j = d_k$.

Proof. For $r = 1$, $x_i := \sqrt{d_i^2 + d_j^2}$, $a_i := d_i d_j$, with summing performed over all non–adjacent vertices v_i and v_j in graph G, the inequality (9) transforms into

$$\sum_{i \nsim j} \frac{d_i^2 + d_j^2}{d_i d_j} \geq \frac{\left(\sum_{i \nsim j} \sqrt{d_i^2 + d_j^2} \right)^2}{\sum_{i \nsim j} d_i d_j}, \tag{14}$$

that is

$$\overline{SDD}(G) \geq \frac{\overline{SO}(G)^2}{\overline{M}_2(G)},$$

from which we get (13).

Equality in (14) holds if and only if $\frac{\sqrt{d_i^2+d_j^2}}{d_i d_j} = const$ for any pair of non–adjacent vertices v_i and v_j in G, i.e. if and only if $\frac{1}{d_i^2} + \frac{1}{d_j^2}$ is constant for any pair of non–adjacent vertices v_i and v_j in G. Let v_j and v_k be two vertices non–adjacent to vertex v_i in G. Then we have $\frac{1}{d_i^2} + \frac{1}{d_j^2} = \frac{1}{d_i^2} + \frac{1}{d_k^2}$, that is $d_j = d_k$. Therefore equality in (13) holds if for any pair of vertices v_j and v_k in G which are not adjacent to v_i holds $d_j = d_k$. □

Remark 4. Some new bounds on $\overline{SDD}(G)$ can be found in [21].

Remark 5. Since

$$\overline{SDD}(G) \leq \overline{m}\left(\frac{\Delta}{\delta} + \frac{\delta}{\Delta}\right),$$

the inequality (13) is stronger than (5).

In the next theorem we determine a relationship between $\overline{SO}(G)$ and $\overline{M}_1(G)$ and $\overline{H}(G)$, when minimum degree, δ, and maximum degree, Δ, are known.

Theorem 3. *Let G be a simple graph of order $n \geq 2$ without isolated vertices, minimum vertex degree, δ, and maximum vertex degree, Δ. Then we have*

$$\overline{SO}(G) \leq \sqrt{\overline{M}_1(G)(\overline{m}(\Delta+\delta) - \Delta\delta\overline{H}(G))}. \tag{15}$$

Equality holds if and only if for any two non–adjacent vertices v_i and v_j in G holds $d_i = d_j$, $d_i \in \{\Delta, \delta\}$ or $d_i \neq d_j$, $d_i, d_j \in \{\Delta, \delta\}$.

Proof. After dividing the the inequality (8) by $d_i + d_j$ and summing over all pairs of non–adjacent vertices v_i and v_j in G we obtain

$$\sum_{i \nsim j} \frac{d_i^2 + d_j^2}{d_i + d_j} \leq (\Delta+\delta)\sum_{i \nsim j} 1 - 2\Delta\delta \sum_{i \nsim j} \frac{1}{d_i + d_j}, \tag{16}$$

i.e.

$$\sum_{i \nsim j} \frac{d_i^2 + d_j^2}{d_i + d_j} \leq (\Delta+\delta)\overline{m} - \Delta\delta\overline{H}(G).$$

From the above and (12) we arrive at (15).

Equality in (16), and therefore in (15), is attained if and only if for any two non–adjacent vertices v_i and v_j in G holds $d_i = d_j$, $d_i \in \{\Delta, \delta\}$ or $d_i \neq d_j$, $d_i, d_j \in \{\Delta, \delta\}$. □

In the next theorem we establish a relationship between $\overline{SO}(G)$ and $\overline{M}_2(G)$, $\overline{M}_2^*(G)$, and $ID(G)$, when minimum and maximum vertex degrees are known.

Theorem 4. *Let G be a simple graph of order $n \geq 2$ without isolated vertices, minimum vertex degree δ and maximum vertex degree Δ. Then*

$$\overline{SO}(G) \leq \sqrt{\overline{M}_2(G)((\Delta+\delta)((n-1)ID(G)-n)-2\Delta\delta\overline{M}_2^*(G))}. \tag{17}$$

Equality is attained if and only if for any two non–adjacent vertices v_i and v_j in G holds $d_i = d_j = \Delta$, or $d_i = d_j = \delta$, or $d_i \neq d_j$, $d_i, d_j \in \{\Delta, \delta\}$.

Proof. Dividing (8) by $d_i d_j$ and summing over all pairs of non–adjacent vertices v_i and v_j in G yields

$$\sum_{i \nsim j} \frac{d_i^2 + d_j^2}{d_i d_j} \leq (\Delta+\delta) \sum_{i \nsim j} \left(\frac{1}{d_i} + \frac{1}{d_j}\right) - 2\Delta\delta \sum_{i \nsim j} \frac{1}{d_i d_j},$$

i.e.

$$\sum_{i \nsim j} \frac{d_i^2 + d_j^2}{d_i d_j} \leq (\Delta+\delta) \sum_{i=1}^{n} \frac{n-1-d_i}{d_i} - 2\Delta\delta \sum_{i \nsim j} \frac{1}{d_i d_j},$$

$$\sum_{i \nsim j} \frac{d_i^2 + d_j^2}{d_i d_j} \leq (\Delta+\delta)((n-1)ID(G)-n) - 2\Delta\delta\overline{M}_2^*(G).$$

From the above and (14) we arrive at (17). □

In the next theorem we determine a relationship between Sobor coindex, first Zagreb coindex and Albertson coindex.

Theorem 5. *Let G be a simple graph graph of order $n \geq 2$ without isolated vertices. Then*

$$\frac{\sqrt{2}}{2}\sqrt{\overline{M}_1(G)^2 + \overline{Alb}(G)^2} \leq \overline{SO}(G) \leq \frac{\sqrt{2}}{2}\left(\overline{M}_1(G) + \overline{Alb}(G)\right). \tag{18}$$

Equality on the right-hand side holds if and only if $d_i = d_j$ for any pair of non–adjacent vertices v_i and v_j in G. Equality on the left-hand side holds if and only if $\frac{d_i}{d_j}$ is constant for any pair of non–adjacent vertices v_i and v_j in G.

Proof. For any two nonnegative real numbers a and b hold

$$\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}, \tag{19}$$

with equality if and only if $a = 0$ or $b = 0$.
For $a := \frac{1}{2}(d_i + d_j)^2$, $b := \frac{1}{2}(d_i - d_j)^2$, the inequality (19) becomes

$$\sqrt{d_i^2 + d_j^2} \leq \frac{\sqrt{2}}{2}(d_i + d_j + |d_i - d_j|). \tag{20}$$

After summing the above inequality over all non–adjacent vertices v_i and v_j in graph G, we get the right-hand side of inequality (18).
Since G does not contain isolated vertices, equality on (20), and therefore on the right-hand side of (18), holds if and only if $d_i = d_j$ for any pair of non–adjacent vertices v_i and v_j in G.
Next we prove the left–hand side of (18). The following equalities are valid

$$\overline{SO}(G) - \sum_{i \nsim j} \frac{2d_i d_j}{\sqrt{d_i^2 + d_j^2}} = \sum_{i \nsim j} \frac{(d_i - d_j)^2}{\sqrt{d_i^2 + d_j^2}}$$

and

$$\overline{SO}(G) + \sum_{i \nsim j} \frac{2d_i d_j}{\sqrt{d_i^2 + d_j^2}} = \sum_{i \nsim j} \frac{(d_i + d_j)^2}{\sqrt{d_i^2 + d_j^2}}.$$

After summing these two equalities we get

$$2\overline{SO}(G) = \sum_{i \nsim j} \frac{(d_i + d_j)^2}{\sqrt{d_i^2 + d_j^2}} + \sum_{i \nsim j} \frac{(d_i - d_j)^2}{\sqrt{d_i^2 + d_j^2}}. \tag{21}$$

For $r = 1$, $x_i := d_i + d_j$, $a_i := \sqrt{d_i^2 + d_j^2}$, with summing performed over all non–adjacent vertices v_i and v_j in graph G, the inequality (9) becomes

$$\sum_{i \nsim j} \frac{(d_i + d_j)^2}{\sqrt{d_i^2 + d_j^2}} \geq \frac{\left(\sum_{i \nsim j} (d_i + d_j) \right)^2}{\sum_{i \nsim j} \sqrt{d_i^2 + d_j^2}} = \frac{\overline{M}_1(G)^2}{\overline{SO}(G)}. \tag{22}$$

Similarly, for $r = 1$, $x_i := |d_i - d_j|$, $a_i := \sqrt{d_i^2 + d_j^2}$, with summing performed over all non–adjacent vertices v_i and v_j in graph G, the inequality (9) becomes

$$\sum_{i \nsim j} \frac{(d_i - d_j)^2}{\sqrt{d_i^2 + d_j^2}} \geq \frac{\left(\sum_{i \nsim j} |d_i - d_j| \right)^2}{\sum_{i \nsim j} \sqrt{d_i^2 + d_j^2}} = \frac{\overline{Alb}(G)^2}{\overline{SO}(G)}. \tag{23}$$

Now, according to (21), (22) and (23) we get

$$2\overline{SO}(G) \geq \frac{\overline{M}_1(G)^2}{\overline{SO}(G)} + \frac{\overline{Alb}(G)^2}{\overline{SO}(G)},$$

from which the left-hand side of (18) is obtained.
Equality in (22) holds if and only if $\frac{d_i + d_j}{\sqrt{d_i^2 + d_j^2}} = const$ for any pair of non–adjacent vertices v_i and v_j in G. Equality in (23) holds if and only if $\frac{|d_i - d_j|}{\sqrt{d_i^2 + d_j^2}} = const$ for any pair of non–adjacent vertices v_i and v_j in G. This means that equality on the left-hand side of (18) holds if and only if $\frac{d_i}{d_j}$ is constant for any pair of non–adjacent vertices v_i and v_j in G. □

We have the following corollary of Theorem 5.

Corollary 1. *Let G be a simple graph without isolated vertices. Then*

$$\overline{SO}(G) \geq \frac{1}{2}\left(\overline{M}_1(G) + \overline{Alb}(G)\right). \tag{24}$$

Equality holds if and only if $G \cong K_n$.

Proof. For any two nonnegative real numbers a and b the following inequality is valid

$$\sqrt{a+b} \geq \frac{\sqrt{2}}{2}\left(\sqrt{a} + \sqrt{b}\right), \tag{25}$$

with equality if and only if $a = b$.
For $a := \overline{M}_1(G)^2$, $b := \overline{Alb}(G)^2$, the inequality (25) transforms into

$$\sqrt{\overline{M}_1(G)^2 + \overline{Alb}(G)^2} \geq \frac{\sqrt{2}}{2}\left(\overline{M}_1(G) + \overline{Alb}(G)\right).$$

From the above and the left-hand side of (18) we obtain (24). □

In the next theorem we provide a relationship between Sombor index and coindex.

Theorem 6. *Let G be a simple graph of order $n \geq 2$ and size m without isolated vertices. Then*

$$SO(G) + \overline{SO}(G) \geq \sqrt{2}m(n-1). \tag{26}$$

Equality holds if and only if G is a regular graph.

Proof. For $a := d_i^2$, $b := d_j^2$, where v_i and v_j are two arbitrary vertices in G, the inequality (25) becomes

$$\sqrt{d_i^2 + d_j^2} \geq \frac{\sqrt{2}}{2}\left(d_i + d_j\right). \tag{27}$$

After summing the above inequality over all pairs of non–adjacent vertices v_i and v_j in G, we obtain

$$\overline{SO}(G) \geq \frac{\sqrt{2}}{2}\overline{M}_1(G). \tag{28}$$

Next, after summation of (27) over all pairs of adjacent vertices v_i and v_j in G, we get

$$SO(G) \geq \frac{\sqrt{2}}{2}M_1(G). \tag{29}$$

Now, from (28) and (29) we have

$$SO(G) + \overline{SO}(G) \geq \frac{\sqrt{2}}{2}\left(M_1(G) + \overline{M}_1(G)\right). \tag{30}$$

In [5] (see also [2]) the following identity was proven

$$M_1(G) + \overline{M}_1(G) = 2m(n-1).$$

From the above and (30) we arrive at (26).
Equality in (28) holds if and only if $d_i = d_j$ for any pair of non–adjacent vertices v_i and v_j in G. Equality in (29) holds if and only if $d_i = d_j$ for any pair of adjacent vertices v_i and v_j in G. This means that equality in (26) holds if and only if G is a regular graph. □

Remark 6. The inequality (29) was proven in [19] (see also [9, 13]).

Remark 7. The inequality (28) is stronger than (6).

Acknowledgement: Research partly supported by the Serbian Ministry of Science, Technological Development and Innovations, grant No. 451-03-47/2023-01/ 200102.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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