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Research Article

Commuting graph of an aperiodic Brandt Semigroup

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Abstract: The commuting graph of a finite non-commutative semigroup S, denoted by $\Delta(S)$, is the simple graph whose vertices are the non-central elements of S and two distinct vertices x, y are adjacent if $xy = yx$. In this paper, we study the commuting graph of an important class of inverse semigroups viz. Brandt semigroup B_n. In this connection, we obtain the automorphism group $\mathrm{Aut}(\Delta(B_n))$ and the endomorphism monoid $\mathrm{End}(\Delta(B_n))$ of $\Delta(B_n)$. We show that $\mathrm{Aut}(\Delta(B_n)) \cong S_n \times \mathbb{Z}_2$, where S_n is the symmetric group of degree n and $\mathbb{Z}_2$ is the additive group of integers modulo 2. Further, for $n \geq 4$, we prove that $\mathrm{End}(\Delta(B_n)) = \mathrm{Aut}(\Delta(B_n))$. Moreover, we provide the vertex connectivity and edge connectivity of $\Delta(B_n)$. This paper provides a partial answer to a question posed in [3] and so we ascertained a class of inverse semigroups whose commuting graph is Hamiltonian.

Keywords: commuting graph, Brandt semigroups, automorphism group of a graph.

AMS Subject classification: 05C25

1. Introduction

The investigation of algebraic graphs is one of the popular topic in algebraic graph theory. Various graphs associated to groups, rings, and semigroups have been studied extensively by several researchers (see [1, 10, 12, 13, 24, 29, 30]). Such study provides the interplay between the property of algebraic structure and the graph theoretic property of its associated graph. The commuting graph of a finite non-abelian group

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G is a simple graph (undirected graph with no loops or repeated edges) whose vertices are the non-central elements of G and two distinct vertices x, y are adjacent if $xy = yx$. Commuting graphs of various groups have been studied by several authors (cf. [4, 5, 8, 20]). Moreover, [33–35] use combinatorial parameters of certain commuting graphs to establish long standing conjectures in the theory of division algebras. The concept of commuting graph can be defined analogously for semigroups. Let S be a finite non-commutative semigroup with centre $Z(S) = \{a \in S : ab = ba \text{ for all } b \in S\}$. The commuting graph of S, denoted by $\Delta(S)$, is the simple graph whose vertex set is $S - Z(S)$ and two distinct vertices a, b are adjacent if $ab = ba$. In 2011, Araújo et al. [3] initiated the study of commuting graph on finite semigroups and calculated the diameter of commuting graphs of various ideals of full transformation semigroup. Also, for every natural number $n \geq 2$, a finite semigroup whose commuting graph has diameter n has been constructed in [3]. Further, various graph theoretic properties (viz. clique number and diameter) of $\Delta(\mathcal{I}(X))$, where $\mathcal{I}(X)$ is the symmetric inverse semigroup of partial injective transformations on a finite set X, have been studied in [2]. In order to provide answers to few of the problems posed in [3], T. Bauer et al. [6] have established a semigroup whose knit degree is 3. For a wider class of semigroups, it was shown in [6], that the diameter of their commuting graphs is effectively bounded by the rank of the semigroups. Further, the construction of monomial semigroups with a bounded number of generators, whose commuting graphs have an arbitrary clique number have been provided in [6]. Motivated with the work in [3] and the questions posed in its Section 6, in this paper, we study various graph invariants of the commuting graph associated with an important class of inverse semigroups. This work leads to answer partially to some of the problems posed in [3]. Moreover, the results obtained in this paper may be useful into the study of commuting graphs on completely 0-simple inverse semigroups.

Let G be a finite group. For a natural number n, we write $[n] = \{1, 2, \ldots, n\}$. Recall that the *Brandt semigroup*, denoted by $B_n(G)$, has underlying set $([n] \times G \times [n]) \cup \{0\}$ and the binary operation '$\cdot$' on $B_n(G)$ is defined as

$$(i, a, j) \cdot (k, b, l) = \begin{cases} (i, ab, l) & \text{if } j = k; \\ 0 & \text{if } j \neq k \end{cases}$$

and, for all $\alpha \in B_n(G)$, $\alpha \cdot 0 = 0 \cdot \alpha = 0$. Note that 0 is the (two sided) zero element in $B_n(G)$.

Theorem 1 ([17, Theorem 5.1.8]). *A finite semigroup S is both completely 0-simple and an inverse semigroup if and only if S is isomorphic to the semigroup $B_n(G)$ for some group G.*

Since all completely 0-simple inverse semigroups are exhausted by Brandt semigroups, their consideration seems interesting and useful in various aspects. Brandt semigroups have been studied extensively by various authors, see [21, 31, 32] and the references therein. When G is the trivial group, the Brandt semigroup $B_n(\{e\})$ is denoted by B_n. Thus, the semigroup B_n can be described as the set $([n] \times [n]) \cup \{0\}$, where 0 is the zero element and the product $(i, j) \cdot (k, l) = (i, l)$, if $j = k$ and 0, otherwise. Since Green's $\mathcal{H}$-class of B_n is trivial, it is also known as aperiodic Brandt semigroup. As a Rees matrix semigroup [17], B_n is isomorphic to the Rees matrix semigroup $M^0(\{1, \ldots, n\}, 1, \{1, \ldots, n\}, I_n)$, where I_n is the $n \times n$ identity matrix. Brandt semigroup B_n play an important role in inverse semigroup theory and arises in number of different ways, see [11, 22] and the references therein. Endomorphism seminear-rings on B_n have been classified by Gilbert and Samman [14]. Further, various aspects of affine near-semirings generated by affine maps on B_n have been studied in [25]. The combinatorial study of B_n have been related with theory of matroids and simplicial complexes in [27]. Various ranks of B_n have been obtained in [18, 19, 28], where some of the ranks of B_n were obtained by using graph theoretic properties of some graph associated on B_n. Cayley graphs associated with Brandt semigroups have been studied in [15, 23]. Recently, various graph invariants of the commuting graph of Brandt semigroup have been studied in [26].

In this paper, we have further investigated the commuting graph of B_n as follows. In Section 2, we provide necessary background material and notations used throughout the paper. In Section 3, the automorphism group as well as endomorphism monoid of $\Delta(B_n)$ is described. In Section 4, we investigate the vertex connectivity and edge connectivity of $\Delta(B_n)$.

2. Preliminaries

In this section, we recall necessary definitions, results and notations of graph theory from [36]. A graph $\mathcal{G}$ is a pair $\mathcal{G} = (V, E)$, where $V = V(\mathcal{G})$ and $E = E(\mathcal{G})$ are the set of vertices and edges of $\mathcal{G}$, respectively. We say that two different vertices a, b are *adjacent*, denoted by $a \sim b$, if there is an edge between a and b. We are considering simple graphs, i.e. undirected graphs with no loops or repeated edges. If a and b are not adjacent, then we write $a \nsim b$. The *neighbourhood* $N(x)$ of a vertex x is the set all vertices adjacent to x in $\mathcal{G}$. Additionally, we denote $N[x] = N(x) \cup \{x\}$. A subgraph of a graph $\mathcal{G}$ is a graph $\mathcal{G}'$ such that $V(\mathcal{G}') \subseteq V(\mathcal{G})$ and $E(\mathcal{G}') \subseteq E(\mathcal{G})$. A *walk* λ in $\mathcal{G}$ from the vertex u to the vertex w is a sequence of vertices $u = v_1, v_2, \cdots, v_m = w$ $(m > 1)$ such that $v_i \sim v_{i+1}$ for every $i \in \{1, 2, \ldots, m-1\}$. If no edge is repeated in λ, then it is called a *trail* in $\mathcal{G}$. A trail whose initial and end vertices are identical is called a *closed trail*. A walk is said to be a *path* if no vertex is repeated. The length of a path is the number of edges it contains. If $U \subseteq V(\mathcal{G})$, then the subgraph of $\mathcal{G}$ induced by U is the graph $\mathcal{G}'$ with vertex set U, and with two vertices adjacent in $\mathcal{G}'$ if and only if they are adjacent in $\mathcal{G}$. A graph $\mathcal{G}$ is said to be *connected* if there is a path between every pair of vertex. A graph $\mathcal{G}$ is said to be *complete* if any two distinct vertices are adjacent. A path that begins and ends on the same vertex is called a *cycle*. A cycle in a graph $\mathcal{G}$ that includes every vertex of $\mathcal{G}$ is called a *Hamiltonian cycle* of $\mathcal{G}$. If $\mathcal{G}$ contains a Hamiltonian cycle, then $\mathcal{G}$ is called a *Hamiltonian graph*.

Also, recall that the *degree* of a vertex v is the number of edges incident to v and it is denoted as $\deg(v)$. A *clique* of a graph $\mathcal{G}$ is a complete subgraph of $\mathcal{G}$ and the number of vertices in a clique of maximum size is called the *clique number* of $\mathcal{G}$ and it is denoted by $\omega(\mathcal{G})$. A *vertex (edge) cut-set* in a connected graph $\mathcal{G}$ is a set of vertices (edges) whose deletion increases the number of connected components of $\mathcal{G}$. The *vertex connectivity* (*edge connectivity*) of a connected graph $\mathcal{G}$ is the minimum size of a vertex (edge) cut-set and it is denoted by $\kappa(\mathcal{G})$ ($\kappa'(\mathcal{G})$). For $k \geq 1$, graph $\mathcal{G}$ is k-*connected* if $\kappa(\mathcal{G}) \geq k$. It is well known that $\kappa(\mathcal{G}) \leq \kappa'(\mathcal{G}) \leq \delta(\mathcal{G})$.
The *commuting graph* of a finite semigroup S, denoted by $\Delta(S)$, is the simple graph whose vertices are the non-central elements of S and two distinct vertices x, y are adjacent if $xy = yx$. The following fundamental results are useful in the sequel.

Lemma 1 ([26, Lemma 3.1]). *In the graph $\Delta(B_n)$, we have the following:*

(i) $N[(i,i)] = \{(j,k) \,:\, j,k \in [n],\ j,k \neq i\} \cup \{(i,i)\}$.

(ii) $N[(i,j)] = \{(i,l) \,:\, l \in [n], l \neq i,j\} \cup \{(l,j) \,:\, l \in [n], l \neq i,j\} \cup \{(k,l) \,:\, k,l \in [n],\ k \neq i,j \text{ and } l \neq i,j\} \cup \{(i,j)\}$, *where* $i \neq j$.

Remark 1. Two distinct vertices (i,j) and (k,i) are not adjacent in $\Delta(B_n)$.

Corollary 1. *In the commuting graph $\Delta(B_n)$, the degree of idempotent vertices is $(n-1)^2$ and the degree of non-idempotent vertices is $n(n-2)$.*

Theorem 2 ([26, Lemma 3.2]). *For $n \geq 3$, the commuting graph $\Delta(B_n)$ Hamiltonian.*

Notation: We denote $\mathcal{K}$ as the set of all cliques of $\Delta(B_n)$ having no idempotent element and $\mathcal{E}$ as the set of non-zero idempotents of B_n.

Lemma 2 ([26, Lemma 3.4]). *For $K \in \mathcal{K}$, we have* $|K| \leq \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even;} \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd.} \end{cases}$

Corollary 2 ([26, Corollary 3.5]). *For $n \geq 4$, there exists $K \in \mathcal{K}$ such that*

$$|K| = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even;} \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 3 ([26, Lemma 3.6]). *For $n \in \{2,3,4\}$, the set $\mathcal{E}$ forms a clique of maximum size. Moreover, in this case $\omega(\Delta(B_n)) = n$.*

Theorem 3 ([26, Theorem 3.7]). *For $n > 4$, the clique number of $\Delta(B_n)$ is given below:*

$$\omega(\Delta(B_n)) = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even;} \\ \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 4. *For $n > 4$, let K be a clique of maximum size in $\Delta(B_n)$. Then all elements of K are non-idempotent.*

Proof. Suppose K is a clique of maximum size such that K contains m idempotents viz. $(i_1, i_1), (i_2, i_2), \ldots, (i_m, i_m)$. Without loss of generality, we assume that $\{i_1, i_2, \ldots, i_m\} = \{n - m + 1, n - m + 2, \ldots, n\}$. For $1 \leq r \leq m$, K contains (i_r, i_r) and no element of the form (x, i_r) or (i_r, x) $(x \in [n]), x \neq i_r$ is in K. Thus $K \setminus \{(i_1, i_1), \ldots, (i_m, i_m)\}$ is a clique in $\Delta(B_{n-m})$ which does not contain any idempotent. Clearly, $|K \setminus \{(i_1, i_1), \ldots, (i_m, i_m)\}| = \omega(\Delta(B_{n-m}))$. Then by Corollary 2

$$|K \setminus \{(i_1, i_1), \ldots, (i_m, i_m)\}| = \begin{cases} \frac{(n-m)^2}{4} & \text{if } n-m \text{ is even;} \\ \\ \frac{(n-m)^2-1}{4} & \text{if } n-m \text{ is odd.} \end{cases}$$

Thus,

$$|K| = \begin{cases} \frac{(n-m)^2}{4} + m & \text{if } n-m \text{ is even;} \\ \frac{(n-m)^2-1}{4} + m & \text{if } n-m \text{ is odd.} \end{cases}$$

Since $n > 4$ and for $m > 0$, one can observe that

$$|K| < \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even;} \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd;} \end{cases}$$

a contradiction of the fact that K is a clique of maximum size (see Theorem 3). □

Lemma 5. *For $n > 4$ and $(i, j) \notin \mathcal{E}$, there exists a clique K of maximum size such that $(i, j) \in K$.*

Proof. Consider a partition A and B of a set $[n]$ such that $i \in A$, $j \in B$ and $|A| = \frac{n}{2}$ when n is even, otherwise $|A| = \frac{n-1}{2}$. In view of Lemma 1 and Theorem 3, note that $A \times B$ forms a clique of maximum size that contain the vertex (i, j). □

Lemma 6. *For $n = 4$, let K be any clique in $\Delta(B_n)$ of size 4. Then K is either $\mathcal{E}$ or $K = A \times B$, where A and B are disjoint subset of $\{1, 2, 3, 4\}$ of size two.*

Proof. First note that $K = \{(1,1),(2,2),(3,3),(4,4)\}$ is a clique in $\Delta(B_4)$. Suppose K' is a clique of maximum size. If K' does not contain an idempotent, then by Lemma 2, $|K'| = 4$. Thus, K is also a clique of maximum size. On the other hand, we may now assume that K contains an idempotent. Without loss of generality, let $(4,4) \in K'$. Then $K' \setminus \{(4,4)\}$ is a clique of maximum size in $\Delta(B_3)$. Since $\{(1,1),(2,2),(3,3)\}$ is the only clique in $\Delta(B_3)$ of maximum size. Thus, $K' \setminus \{(4,4)\} = \{(1,1),(2,2),(3,3)\}$. Consequently, $K' = \{(1,1),(2,2),(3,3),(4,4)\} = K$. Hence, we have the result. □

3. Algebraic properties of $\Delta(B_n)$

In order to study algebraic aspects of the graph $\Delta(B_n)$, in this section we obtain automorphism group (see Theorem 4) and endomorphism monoid (see Theorem 6) of $\Delta(B_n)$.

3.1. Automorphism group of $\Delta(B_n)$

An *automorphism* of a graph $\mathcal{G}$ is a permutation f on $V(\mathcal{G})$ with the property that, for any vertices u and v, we have $uf \sim vf$ if and only if $u \sim v$. The set $\mathrm{Aut}(\mathcal{G})$ of all graph automorphisms of a graph $\mathcal{G}$ forms a group with respect to composition of mappings. The symmetric group of degree n is denoted by S_n. For $n = 1$, the group $\mathrm{Aut}(\Delta(B_n))$ is trivial. For the remaining subsection, we assume $n \geq 2$.

Lemma 7. *Let $x \in V(\Delta(B_n))$ and $f \in \mathrm{Aut}(\Delta(B_n))$. Then x is an idempotent if and only if xf is an idempotent.*

Proof. Since f is an automorphism, we have $\deg(x) = \deg(xf)$. By Corollary 1, the result holds. □

Lemma 8. *For $f \in \mathrm{Aut}(\Delta(B_n))$ and $i, j, k, k' \in [n]$ such that $(i,i)f = (k,k)$ and $(j,j)f = (k',k')$, we have either $(i,j)f = (k,k')$ or $(i,j)f = (k',k)$.*

Proof. For $i \neq j$, suppose that $(i,j)f = (x,y)$. Clearly, $(i,j) \nsim (i,i)$ so that $(x,y) = (i,j)f \nsim (i,i)f = (k,k)$. Since $(x,y) \nsim (k,k)$, we get either $x = k$ or $y = k$. Similarly, for $(i,j) \nsim (j,j)$, we have either $x = k'$ or $y = k'$. Thus, by Lemma 7, we have $(x,y) = (k,k')$ or $(x,y) = (k',k)$. □

Lemma 9. *For $\sigma \in S_n$, let $\phi_\sigma : V(\Delta(B_n)) \to V(\Delta(B_n))$ defined by $(i,j)\phi_\sigma = (i\sigma, j\sigma)$. Then $\phi_\sigma \in \mathrm{Aut}(\Delta(B_n))$.*

Proof. It is easy to verify that ϕ_σ is a permutation on $V(\Delta(B_n))$. Now we show that ϕ_σ preserves adjacency. Let $(i,j),(x,y) \in V(\Delta(B_n))$ such that $(i,j) \sim (x,y)$.

Now,

$$\begin{aligned}(i,j)\sim(x,y) &\iff x\neq j \text{ and } y\neq i\\ &\iff \text{ for } \sigma\in S_n, \text{ we have } x\sigma\neq j\sigma \text{ and } y\sigma\neq i\sigma\\ &\iff (i\sigma,j\sigma)\sim(x\sigma,y\sigma)\\ &\iff (i,j)\phi_\sigma\sim(x,y)\phi_\sigma.\end{aligned}$$

Hence, $\phi_\sigma \in \mathrm{Aut}(\Delta(B_n))$. □

Lemma 10. *Let* $\alpha : V(\Delta(B_n)) \to V(\Delta(B_n))$ *be a mapping defined by* $(i,j)\alpha = (j,i)$. *Then* $\alpha \in \mathrm{Aut}(\Delta(B_n))$.

Proof. It is straightforward to verify that α is a one-one and onto map on $V(\Delta(B_n))$. Note that

$$\begin{aligned}(i,j)\sim(x,y) &\iff x\neq j \text{ and } y\neq i\\ &\iff (j,i)\sim(y,x)\\ &\iff (i,j)\alpha\sim(x,y)\alpha.\end{aligned}$$

Hence, $\alpha \in \mathrm{Aut}(\Delta(B_n))$. □

Remark 2. For ϕ_σ and α, defined in Lemma 9 and 10, we have $\phi_\sigma \circ \alpha = \alpha \circ \phi_\sigma$.

Proposition 1. *For each* $f \in \mathrm{Aut}(\Delta(B_n))$, *we have either* $f = \phi_\sigma$ *or* $f = \phi_\sigma \circ \alpha$ *for some* $\sigma \in S_n$.

Proof. Since $f \in \mathrm{Aut}(\Delta(B_n))$, by Lemma 7, note that there exists a permutation $\sigma : [n] \to [n]$ such that $i\sigma = j \iff (i,i)f = (j,j)$, determined by f. Thus, we have $(i,i)f = (i\sigma, i\sigma)$ for all $i \in [n]$. Let $j \neq i$. Then by Lemma 8, we get either $(i,j)f = (i\sigma, j\sigma)$ or $(i,j)f = (j\sigma, i\sigma)$. First, let $(i,j)f = (i\sigma, j\sigma)$. Then for every vertical and horizontal neighbour (x', y') of (i,j), we have $(x', y')f = (x'\sigma, y'\sigma)$ because $(i\sigma, j\sigma)$ and $(y'\sigma, x'\sigma)$ are not adjacent. Since every other vertex is connected to (i,j) by vertical and horizontal neighbors, we have $(x,y)f = (x\sigma, y\sigma)$ for all $(x,y) \in V(\Delta(B_n))$. Consequently, $f = \phi_\sigma$. If $(i,j)f = (j\sigma, i\sigma)$, then $(i,j)(f\circ\alpha) = (i\sigma, j\sigma)$. Thus, $(x,y)(f\circ\alpha) = (x\sigma, y\sigma)$ for all $(x,y) \in V(\Delta(B_n))$. Therefore, $(x,y)f = (y\sigma, x\sigma)$, and hence $f = \phi_\sigma \circ \alpha$. □

Theorem 4. *For* $n \geq 2$, *we have* $\mathrm{Aut}(\Delta(B_n)) \cong S_n \times \mathbb{Z}_2$. *Moreover,* $|\mathrm{Aut}(\Delta(B_n))| = 2(n!)$.

Proof. In view of Lemmas 1, 9 and 10, note that the underlying set of the automorphism group of $\Delta(B_n)$ is

$$\mathrm{Aut}(\Delta(B_n)) = \{\phi_\sigma \ : \ \sigma \in S_n\} \cup \{\phi_\sigma \circ \alpha \ : \ \sigma \in S_n\},$$

where S_n is a symmetric group of degree n. Note that the groups $\mathrm{Aut}(\Delta(B_n))$ and $S_n \times \mathbb{Z}_2$ are isomorphic under the assignment $\phi_\sigma \mapsto (\sigma, \bar{0})$ and $\phi_\sigma \circ \alpha \mapsto (\sigma, \bar{1})$. Since, all the elements in $\mathrm{Aut}(\Delta(B_n))$ are distinct, we have $|\mathrm{Aut}(\Delta(B_n))| = 2(n!)$. □

3.2. Endomorphism monoid of $\Delta(B_n)$

A mapping f from a graph $\mathcal{G}$ to $\mathcal{G}'$ is said to be a *homomorphism* if $x \sim y$, then $xf \sim yf$ for all $x, y \in V(\mathcal{G})$. If $\mathcal{G}' = \mathcal{G}$, then we say f is an *endomorphism*. Note that the set $\mathrm{End}(\mathcal{G})$ of all endomorphisms on $\mathcal{G}$ forms a monoid with respect to the composition of mappings. First we obtain the endomorphism monoid of $\Delta(B_n)$ for $n \in \{2, 3\}$. The following remark is useful in the sequel.

Remark 3. Let $f \in \mathrm{End}(\mathcal{G})$ and K be a clique of maximum size in $\mathcal{G}$. Then Kf is again a clique of maximum size.

Lemma 11. $\mathrm{End}(\Delta(B_2)) = \{f : V(\Delta(B_2)) \to V(\Delta(B_2)) \ : \ \mathcal{E}f = \mathcal{E}\}$, where $\mathcal{E} = \{(1,1), (2,2)\}$.

Proof. For $x, y \in V(\Delta(B_2))$, note that $x \sim y$ if and only if x, y belongs to $\mathcal{E}$. Hence, we have the result. □

For $\sigma \in S_3$, we define the mappings f^σ and g^σ on $V(\Delta(B_3))$ by

- $(i,i) \xmapsto{f^\sigma} (i\sigma, i\sigma), (1,2) \xmapsto{f^\sigma} (1\sigma, 1\sigma), (1,3) \xmapsto{f^\sigma} (3\sigma, 3\sigma), (2,3) \xmapsto{f^\sigma} (2\sigma, 2\sigma), (2,1) \xmapsto{f^\sigma} (1\sigma, 1\sigma), (3,1) \xmapsto{f^\sigma} (3\sigma, 3\sigma),\ (3,2) \xmapsto{f^\sigma} (2\sigma, 2\sigma)$, and
- $(i,i) \xmapsto{g^\sigma} (i\sigma, i\sigma), (1,2) \xmapsto{g^\sigma} (2\sigma, 2\sigma), (3,2) \xmapsto{g^\sigma} (3\sigma, 3\sigma), (3,1) \xmapsto{g^\sigma} (1\sigma, 1\sigma), (2,1) \xmapsto{g^\sigma} (2\sigma, 2\sigma), (2,3) \xmapsto{g^\sigma} (3\sigma, 3\sigma),\ (1,3) \xmapsto{g^\sigma} (1\sigma, 1\sigma)$, respectively.

It is routine to verify that $f^\sigma, g^\sigma \in \mathrm{End}(\Delta(B_3))$.

Lemma 12. $\mathrm{End}(\Delta(B_3)) = \mathrm{Aut}(\Delta(B_3)) \cup \{f^\sigma \ : \ \sigma \in S_3\} \cup \{g^\sigma \ : \ \sigma \in S_3\}$, where f^σ and g^σ are the endomorphisms on $V(\Delta(B_3))$ as defined above.

Proof. Let $\psi \in \mathrm{End}(\Delta(B_3))$. By Figure 1, note that $\{(1,1), (2,2), (3,3)\}$ is the only clique of maximum size in $\Delta(B_3)$. Since the image of a clique of maximum size under an endomorphism is again a clique of maximum size, we get $(i,i)\psi$ is an idempotent element for all $i \in \{1,2,3\}$. Also note that restriction of ψ to $\mathcal{E} = \{(1,1), (2,2), (3,3)\}$ is a bijective map from $\mathcal{E}$ to $\mathcal{E}$. If $(i,i)\psi = (j,j)$ for some $j \in \{1,2,3\}$, then define $\sigma : \{1,2,3\} \to \{1,2,3\}$ by $i\sigma = j$. Consequently, $\sigma \in S_3$. Suppose $(i,j)\psi$ is an idempotent element for some distinct $i, j \in \{1,2,3\}$. Without loss of generality,

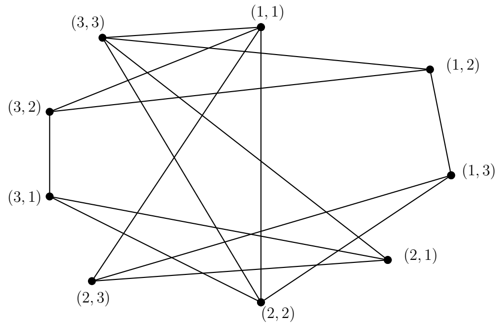

Figure 1. **The commuting graph of B_3**

let $i = 1$ and $j = 2$. Since $(1, 2) \sim (3, 3)$ we have $(1, 2)\psi \sim (3, 3)\psi = (3\sigma, 3\sigma)$. Consequently, $(1, 2)\psi \in \{(1\sigma, 1\sigma), (2\sigma, 2\sigma)\}$. If $(1, 2)\psi = (1\sigma, 1\sigma)$, then $\psi = f^\sigma$. Otherwise, $\psi = g^\sigma$. On the other hand, if $(i, j)\psi$ is a non-idempotent for all $i \neq j$. Let $(i, j)\psi = (x, y)$, where $x \neq y$. For $k \neq i, j$, we have $(x, y) = (i, j)\psi \sim (k, k)\psi$. Thus, $(i, j)\psi$ is either $(i\sigma, j\sigma)$ or $(j\sigma, i\sigma)$. By the similar argument used in Proposition 1, we have $\psi \in \mathrm{Aut}(\Delta(B_3))$. □

Now, we obtain $\mathrm{End}(\Delta(B_n))$ for $n \geq 4$. We begin with few definitions and necessary results. If $\mathcal{G}'$ is a subgraph of $\mathcal{G}$, then a homomorphism $f : \mathcal{G} \to \mathcal{G}'$ such that $xf = x$ for all $x \in \mathcal{G}'$ is called a *retraction* of $\mathcal{G}$ onto $\mathcal{G}'$ and $\mathcal{G}'$ is said to be a *retract* of $\mathcal{G}$. A subgraph $\mathcal{G}'$ of $\mathcal{G}$ is said to be a *core* of $\mathcal{G}$ if and only if it admits no proper retracts (cf. [16]). Let $X \subset A$, $Y \subseteq B$ and f be any mapping from the set A to B such that $Xf \subseteq Y$. We write the *restriction map* of f from X to Y as $f_{X\times Y}$ i.e $f_{X\times Y} : X \to Y$ such that $xf_{X\times Y} = xf$.

Proposition 2 ([9, Proposition 2.4]). *A graph $\mathcal{G}$ is a core if and only if* $\mathrm{End}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})$.

Lemma 13. *Let f be a retraction of $\Delta(B_4)$. Then a non-idempotent element maps to a non-idempotent element of B_4 under f.*

Proof. Let, if possible there exists a non-idempotent element (i, j) of B_4 such that $(i, j)f$ is an idempotent element. In order to get a contradiction, first we show that $(a, b)f \in \mathcal{E} = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ for all $a \neq b \in \{1, 2, 3, 4\}$. Without loss of generality, we may assume that $i = 1$ and $j = 2$. In view of Remark 6 , any clique K in $\Delta(B_4)$ of maximum size is either $K = \mathcal{E}$ or $K = A \times B$, where A and B are disjoint subsets of $\{1, 2, 3, 4\}$ of size two. Therefore, $\Delta(B_4)$ has two cliques of maximum size which contains $(1, 2)$ viz. $K_1 = \{1, 3\} \times \{2, 4\}$ and $K_2 = \{1, 4\} \times \{2, 3\}$. Note that for disjoint subsets A and B of $\{1, 2, 3, 4\}$, the clique $A \times B$ does not contain an idempotent element. Since $(1, 2)f$ is an idempotent element and by Remark 3, we have $K_1f = K_2f = \mathcal{E}$. By using the other elements of $(K_1f \cup K_2f) \setminus \{(1, 2)f\}$, in a similar manner, one can observe that the image of remaining non-idempotent elements belongs to $\mathcal{E}$. Thus, $(a,b)f \in \mathcal{E}$ for all $a \neq b \in [n]$. Now, we show that for any two distinct $x, y \in \{1,2,3,4\}$, $(x,y)f$ is either (x,x) or (y,y). Since image of non-idempotent element is an idempotent so that $(x,y)f = (p,p)$ for some $p \in \{1,2,3,4\}$. Note that $p \in \{x,y\}$. Otherwise, $(p,p) \sim (x,y)$ implies $(p,p) = (p,p)f \sim (x,y)f = (p,p)$; which is not possible. Now suppose $(1,2)f = (1,1)$. Since $(1,2) \sim (1,k)$ for $k \neq 1,2$, we get $(1,1) = (1,2)f \sim (1,k)f$. Consequently, $(1,k)f = (k,k)$. Similarly, we get $(2,k)f = (2,2)$. Therefore, $(2,3)f = (2,4)f = (2,2)$. We get a contradiction as $(2,4) \sim (2,3)$. Similarly, we get a contradiction when $(1,2)f = (2,2)$. Hence, the result hold. □

Lemma 14. *For $n \geq 5$, let $f \in \mathrm{End}(\Delta(B_n))$. Then a non-idempotent element maps to a non-idempotent element of B_n under f.*

Proof. Let (i,j) be a non-idempotent element of B_n. By Remark 5, there exists a clique K of maximum size which contains (i,j). In view of Remarks 4 and 3, all the elements of Kf are non-idempotent. Thus, $(i,j)f$ is a non-idempotent element. □

Proposition 3. *For $n \geq 4$, let $\mathcal{G}'$ be a retract of $\Delta(B_n)$ such that $(i,i) \in \mathcal{G}'$ for all $i \in [n]$. Then $\mathcal{G}' = \Delta(B_n)$.*

Proof. Since $\mathcal{G}'$ is a retract of $\Delta(B_n)$, there exists a homomorphism $f : \Delta(B_n) \to \mathcal{G}'$ such that $xf = x$ for all $x \in V(\mathcal{G}')$. Let (i,j) be a non-idempotent element of B_n. Then $(i,j)f$ is a non-idempotent element of B_n (cf. Lemmas 13 and 14). Let $(i,j)f = (x,y)$, where $x \neq y$. For $k \in [n] \setminus \{i,j\}$, we have $(i,j) \sim (k,k)$. Since $(k,k) \in \mathcal{G}'$, we get $(x,y) \in N[(k,k)]$. By Lemma 1(i), $x, y \neq k$. Consequently, $(x,y) \in \{(i,j),(j,i)\}$. Thus, either $(i,j)f = (i,j)$ or (j,i). Now to prove $\mathcal{G}' = \Delta(B_n)$, we show that f is an identity map. Since $(i,i) \in \mathcal{G}'$, it is sufficient to prove that for any $i, j \in [n]$ such that $i \neq j$, we have $(i,j)f = (i,j)$. Let if possible, $(i,j)f = (j,i)$ for some $i \neq j$. Then $(j,i)f = (j,i)$. For $p \in [n] \setminus \{i,j\}$, note that $(j,p)f = (j,p)$ because if $(j,p)f = (p,j)$, then $(j,p) \sim (j,i)$ implies $(j,p)f = (p,j) \nsim (j,i) = (j,i)f$; a contradiction. Further, note that $(i,p)f \notin \{(i,p),(p,i)\}$ which is not possible. For instance, if $(i,p)f = (i,p)$ then $(i,p) \sim (i,j)$ gives $(i,p)f \sim (i,j)f$. Consequently, we get $(i,p) \sim (j,i)$; a contradiction. On the other hand, if $(i,p)f = (p,i)f$ then $(i,p) \sim (j,p)$ gives $(i,p)f = (p,i) \nsim (j,p) = (j,p)f$; a contradiction. Hence, f is an identity map so that $\mathcal{G}' = \Delta(B_n)$. □

To obtain the $\mathrm{End}(\Delta(B_n))$, following lemmas will be useful.

Lemma 15. *For $n \geq 4$, let f be a retraction of $\Delta(B_n)$ onto $\mathcal{G}'$. Then there exists a clique K of maximum size in $\mathcal{G}'$ such that $K = A \times B$ where A and B forms a partition of $[n]$. Moreover,*

(i) *if n is even then $|A| = |B| = \frac{n}{2}$, or*

(ii) *if n is odd then either $|A| = \frac{n-1}{2}, |B| = \frac{n+1}{2}$ or $|A| = \frac{n+1}{2}, |B| = \frac{n-1}{2}$.*

Proof. Let f be a retraction on $\Delta(B_n)$. For $n \geq 4$, in view of Corollary 2, Lemma 3 and Theorem 3, $\Delta(B_n)$ contains a clique K' of maximum size such that all the elements of K' are non-idempotent. By Remark 3 and Lemmas 13, 14, $K'f$ is a clique of maximum size and all of its elements are non-idempotents. Now consider $K'f = K$, by the proof of Lemma 2, we get $K = A \times B$ where A and B forms a partition of $[n]$ together with (i) or (ii). □

In the following lemma, we provide the possible images of non-idempotent elements of B_n under a retraction.

Lemma 16. *Let f be a retraction of $\Delta(B_n)$ onto $\mathcal{G}'$, where $n \geq 4$. There exists a partition $\{A, B\}$ of $[n]$ such that for any for $p \neq q$, we have $(p,q)f \in \{(t,p) : t \in A\} \cup \{(q,t) : t \in B\} \cup \{(p,q)\}$. Moreover,*

(i) *if $p \in A$, then $(p,q)f \neq (t,p)$ for any $t \in A$.*

(ii) *if $q \in B$, then $(p,q)f \neq (q,t)$ for any $t \in B$.*

Proof. In view of Lemma 15, there exists a clique $K = A \times B$ of maximum size in $\mathcal{G}'$ for some partition $\{A, B\}$ of $[n]$. Suppose $(p,q)f = (x,y)$. Then, by Lemmas 13 and 14, we have $x \neq y$. If $(p,q)f = (p,q)$ then there is nothing to prove. Now let $(p,q)f = (x,y)$ where $(x,y) \neq (p,q)$. If $x, y \notin \{p,q\}$, then $(p,q) \sim (x,y)$ gives $(p,q)f = (x,y)f = (x,y)$; a contradiction. Then either $x \in \{p,q\}$ or $y \in \{p,q\}$. If $x = p$, then clearly $y \notin \{p,q\}$. Consequently, $(p,q) \sim (x,y)$ provides again a contradiction. Therefore, $x \neq p$. Similarly, one can show that $y \neq q$. It follows that $(p,q)f = (x,y)$ where either $x = q$ or $y = p$. Now observe that if $y = p$, then $x \in A$. If possible, let $x \in B$. Then for $\alpha \in A \setminus \{q\}$, $(\alpha, x)f = (\alpha, x)$ as $(\alpha, x) \in A \times B \subseteq \mathcal{G}'$. Since $x \neq p$ as $x \neq y$, we get $(p,q) \sim (\alpha, x)$ so that $(p,q)f = (x,p) \sim (\alpha,x) = (\alpha,x)f$; a contradiction of Remark 1. In a similar manner it is not difficult to observe if $x = q$, then $y \in B$.
To prove addition part of the lemma, suppose $p \in A$ and $(p,q)f = (t,p)$ for some $t \in A$. For $r \in B$ such that $r \neq q$, we have $(p,q) \sim (p,r)$ and $(p,r)f = (p,r)$ as $(p,r) \in K \subseteq \mathcal{G}'$. Consequently, we get $(p,q)f = (t,p) \sim (p,r) = (p,r)f$; a contradiction of Remark 1. Thus, $(p,q)f \neq (t,p)$. Using similar argument, observe that for $q \in B$, $(p,q)f \neq (q,t)$ for any $t \in B$. Thus, the result hold. □

Theorem 5. *For $n = 4$, we have* $\mathrm{End}(\Delta(B_n)) = \mathrm{Aut}(\Delta(B_n))$.

Proof. In view of Proposition 2, we show that $\Delta(B_n)$ is a core. For that it is sufficient to show $\Delta(B_n)$ admits no proper retract (cf. [16]). On contrary, suppose $\Delta(B_n)$ admits a proper retract $\mathcal{G}'$. Then there exists a homomorphism $f : \Delta(B_n) \to \mathcal{G}'$ such that $xf = x$ for all $x \in \mathcal{G}'$. Since the set $\mathcal{E} = \{(1,1),(2,2),(3,3),(4,4)\}$ forms a clique of maximum size as $\omega(\Delta(B_4)) = 4$ (cf. Lemma 3) so that $\mathcal{E}f$ is a clique of size 4 (see Remark 3). By Remark 6, we have either $\mathcal{E}f = \mathcal{E}$ or $\mathcal{E}f = A \times B$ where $A, B \subseteq \{1,2,3,4\}$ with $|A| = |B| = 2$. If $\mathcal{E}f = \mathcal{E}$, then by Proposition 3, $\mathcal{G}' = \Delta(B_n)$; a contradiction. Thus, $\mathcal{E}f = A \times B$. Let $(1,1)f = (i,j)$ where $i \neq j$. Then $(i,j)f = (i,j)$ as $(i,j) \in \mathcal{G}'$. Note that either $i = 1$ or $j = 1$. If both $i, j \neq 1$, then $(i,j) \sim (1,1)$. Consequently, $(1,1)f \sim (i,j)f$ which is not possible as $(i,j)f = (1,1)f = (i,j)$. Without Loss of generality, we assume that $i = 1$ and $j = 2$. Similarly, $(2,2)f \in \{(2,k),(k,2)\}$ for some $k \neq 1,2$. Since $(2,2)f \sim (1,2) = (1,1)f$ as $(1,1) \sim (2,2)$. If $(2,2)f = (2,k)$, then $(2,k) \sim (1,2)$; a contradiction of Remark 1 so $(2,2)f = (k,2)$ for some $k \neq 1,2$. Without loss of generality, we suppose $k = 3$. In the same way, we get $(3,3)f = (3,4)$ and $(4,4)f = (1,4)$. Therefore, we have $A = \{1,3\}$ and $B = \{2,4\}$. In view of Lemma 16, $(2,4)f \in \{(1,2),(3,2),(2,4)\}$. Since $(1,1) \sim (2,4)$ so that $(1,1)f = (1,2) \sim (2,4)f$ gives $(2,4)f = (3,2)$. Similarly, we get $(2,3)f = (3,4)$. Again by Lemma 16, we have $(1,3)f \in \{(3,2),(3,4),(1,3)\}$. For $(1,3) \sim (2,3)$ and $(1,3) \sim (2,4)$ we obtained $(1,3)f \sim (3,4)$ and $(1,3)f \sim (3,2)$. Consequently, we get a contradiction of Remark 1. □

Theorem 6. *For $n \geq 5$, we have* $\mathrm{End}(\Delta(B_n)) = \mathrm{Aut}(\Delta(B_n))$.

Proof. In order to prove the result, we show that $\Delta(B_n)$ is a core (see Proposition 2). For that it is sufficient to show $\Delta(B_n)$ admits no proper retract (cf. [16]). On contrary, suppose $\Delta(B_n)$ admits a proper retract $\mathcal{G}'$. Then there exists an onto homomorphism $f : \Delta(B_n) \to \mathcal{G}'$ such that $xf = x$ for all $x \in \mathcal{G}'$. In view of Lemma 15, there exists a clique $K = A \times B$ where A and B forms a partition of $[n]$. Without loss of generality, we may assume that $A = \{1,2,\ldots,t\}$ and $B = \{t+1, t+2, \ldots, n\}$ where $t \in \{\frac{n}{2}, \frac{n-1}{2}, \frac{n+1}{2}\}$. Consider the set

$$X = \{i \in A \setminus \{1\} : (1,i)f = (1,i)\} \cup \{1 : (2,1)f = (2,1)\}.$$

The following claims will be useful in the sequel.

Claim 1. (i) *For $i \in X$ and $r \in A \setminus \{i\}$, we have $(r,i)f = (r,i)$.*

(ii) *For $i \in A \setminus X$ and $r \in A \setminus \{i\}$, we have $(r,i)f = (i,s)$ for some $s \in B$.*

Proof of Claim (i) First, suppose that $1 \notin X$. Then $i \neq 1$ and so $(1,i)f = (1,i)$. Thus, the result holds for $r = 1$. We may now suppose that $r \in A \setminus \{1,i\}$. Then by Lemma 16, we have either $(r,i)f = (r,i)$ or $(r,i)f = (i,s)$ where $s \in B$. If $(r,i)f = (i,s)$ for some $s \in B$, then $(i,s) = (r,i)f \sim (1,i)f = (1,i)$; a contradiction to the Remark 1. Thus, $(r,i)f = (r,i)$ for all $r \in A \setminus \{i\}$. Now we assume that $1 \in X$. For $i \neq 1$, similar to the above, we obtain $(r,i)f = (r,i)$ for all $r \in A \setminus \{i\}$. Further, we consider $i = 1$. Note that $(2,1)f = (2,1)$. Therefore, the result holds for $r = 2$. Let $r \in A \setminus \{1,2\}$. In view of Lemma 16, we have either $(r,1)f = (r,1)$ or $(r,1)f = (1,s)$ where $s \in B$. Suppose $(r,1)f = (1,s)$ for some $s \in B$. Since $(r,1) \sim (2,1)$, we get

$(1,s) = (r,1)f \sim (2,1)f = (2,1)$ which is not possible. Hence, $(r,i)f = (r,i)$ for all $r \in A \setminus \{i\}$.

(ii) First, suppose that $1 \notin X$ and let $i \in A \setminus (X \cup \{1\})$. In view of Lemma 16, we have either $(1,i)f = (1,i)$ or $(1,i)f = (i,s)$ for some $s \in B$. Note that $(1,i)f \neq (1,i)$ as $i \in A \setminus X$. It follows that $(1,i)f = (i,s)$ for some $s \in B$. Thus, the result holds for $r = 1$. If $r \in A \setminus \{1,i\}$, then we have either $(r,i)f = (r,i)$ or $(r,i)f = (i,s')$ where $s' \in B$ (cf. Lemma 16). Suppose $(r,i)f = (r,i)$. Since $(r,i) \sim (1,i)$, it implies that $(r,i)f \sim (1,i)f$ and so $(r,i) \sim (i,s)$; a contradiction of Remark 1. Thus, $(r,i)f = (i,s')$ for some $s' \in B$. Now consider $i = 1 \in A \setminus X$ and let $r \in A \setminus \{1\}$. Then $(2,1)f \neq (2,1)$. By Lemma 16, we get $(2,1)f = (1,s)$ for some $s \in B$. Therefore, the result holds for $r = 2$. For $r \in A \setminus \{1,2\}$, we get eiher $(r,1)f = (r,1)$ or $(r,1)f = (1,z)$ where $z \in B$ (cf. Lemma 16). If $(r,1)f = (r,1)$, then $(r,1) = (r,1)f \sim (2,1)f = (1,s)$ which is not possible. Thus, $(r,1)f = (1,s)$ for some $s \in B$. Further, we assume that $1 \in X$. Note that $i \neq 1$ and so $(1,i)f \neq (1,i)$. Again by Lemma 16, $(1,i)f = (i,t)$ for some $t \in B$. Thus, the result holds for $r = 1$. Let $r \in A \setminus \{1,i\}$. Then either $(r,i)f = (r,i)$ or $(r,i)f = (i,s')$ where $s' \in B$ (cf. Lemma 16). If $(r,i)f = (r,i)$, then $(r,i) = (r,i)f \sim (1,i)f = (i,t)$; a contradiction to Remark 1. This completes the proof our claim.

In view of the set X, we have the following cases.

Case 1: Suppose $|X| > |A \setminus X|$. Then $|X| \geq 2$ as $n \geq 5$. In order to get a contradiction of the fact that $\mathcal{G}'$ is a proper retract of $\Delta(B_n)$, we prove that f is an identity map in this case. First we show that each non-idempotent element of $\Delta(B_n)$ maps to itself under f through the following claim.

Note: If $n > 5$, then $|A| \geq 3$. For $n = 5$, we have either $|A| = 2$, $|B| = 3$ or $|A| = 3$, $|B| = 2$. If $|A| = 2$ and $|B| = 3$, then $X = A = \{1,2\}$. This case we will discuss separately in the following claim (vi). Therefore, in part (ii) to (v), we assume that $|A| \geq 3$.

Claim 2. (i) *For $p \in A, q \in B$, we have $(p,q)f = (p,q)$.*

(ii) *If $p \neq q$ such that $(p,q)f = (a,p)$ for some $a \in A$, then $a \in A \setminus X$.*

(iii) *For $p \in B, q \in A$, we have $(p,q)f = (p,q)$.*

(iv) *For distinct $p,q \in B$, we have $(p,q)f = (p,q)$.*

(v) *For distinct $p,q \in A$, we have $(p,q)f = (p,q)$.*

(vi) *For $n = 5$, $|A| = 2$, $|B| = 3$ and $p \neq q$, we have $(p,q)f = (p,q)$.*

Proof of Claim: (i) Since $K = A \times B$ is contained in $\mathcal{G}'$ so that $(p,q)f = (p,q)$ for all $p \in A, q \in B$.

(ii) On contrary, we assume that $a \in X$. Clearly, $a \neq p$ (cf. Lemma 14). If $p \in A$, then by Claim 1(i), we get $(p,a)f = (p,a)$. Note that $q \neq a$, otherwise $(p,q)f = (p,q) = (q,p)$ implies $p = q$; a contradiction. Consequently, $(p,q) \sim (p,a)$ gives

$(p,a)f = (p,a) \sim (a,p) = (p,q)f$; a contradiction of Remark 1. Thus, $p \in B$. For $r \in A \setminus \{a,q\}$, by Claim 1(i), we have $(r,a)f = (r,a)$. Since $(p,q) \sim (r,a)$ as $a \neq p$ and $r \neq q$ so that $(p,q)f = (a,p) \sim (r,a) = (r,a)f$ which is not possible. Thus, $a \notin X$.

(iii) Let $p \in B$ and $q \in A$. First suppose that $q \in X$. Then by Lemma 16, $(p,q)f \in \{(s,p) : s \in A\} \cup \{(q,s) : s \in B\} \cup \{(p,q)\}$. For $r \in A \setminus \{q\}$, we have $(r,q)f = (r,q)$ (cf. Claim 1(i)). Note that $(p,q)f \neq (q,s)$ for any $s \in B$. For instance, if $(p,q)f = (q,s)$ for some $s \in B$, then $(p,q)f = (q,s) \sim (r,q) = (r,q)f$ as $(p,q) \sim (r,q)$, where $r \in A \setminus \{q\}$; a contradiction of Remark 1. It follows that $(p,q)f \in \{(s,p) : s \in A\} \cup \{(p,q)\}$. Suppose $(p,q)f = (s,p)$ for some $s \in A$. Note that $s \in A \setminus X$ (see part (ii)). Now we claim that for any $j \in X \setminus \{q\}$, we have $(p,j)f = (s',p)$ for some $s' \in A \setminus X$. In view of Lemma 16, $(p,j)f \in \{(s',p) : s \in A\} \cup \{(j,s') : s' \in B\} \cup \{(p,j)\}$. Note that $(p,j)f \neq (p,j)$ because $(p,q) \sim (p,j)$ but $(p,q)f = (s,p) \nsim (p,j)$ (cf. Remark 1). In a similar manner of $(p,q)f \neq (q,s)$ for any $s \in B$, one can show that $(p,j)f \neq (j,s')$ for any $s' \in B$. It follows $(p,j)f = (s',p)$ for some $s' \in A$. By part (ii), we get $(p,j)f = (s',p)$ for some $s' \in A \setminus X$. Clearly, the subgraph induced by the vertices of the form (p,j) where $j \in X$ forms a clique. Consequently, for any $i \in X \setminus \{j\}$, we get $(p,i)f = (s,p)$ and $(p,j)f = (s',p)$ are distinct for some $s, s' \in A \setminus X$. Therefore, we have $|X| \leq |A \setminus X|$; a contradiction. Thus, $(p,q)f = (p,q)$ for all $p \in B$ and $q \in X$. Now we assume $q \in A \setminus X$. In view of Lemma 16, $(p,q)f \in \{(\alpha,p) : \alpha \in A\} \cup \{(q,\beta) : \beta \in B\} \cup \{(p,q)\}$. Suppose $(p,q)f = (\alpha,p)$ for some $\alpha \in A$. In fact $\alpha \in A \setminus X$ (see part (ii)). Choose $i \in X$ as $|X| > |A \setminus X|$, from above we get $(p,i)f = (p,i)$ as $p \in B$. Since $(p,q) \sim (p,i)$ so that $(p,q)f = (\alpha,p) \sim (p,i) = (p,i)f$ which is not possible. Therefore, we have $(p,q)f = (q,\beta)$ for some $\beta \in B$ if $(p,q)f \neq (p,q)$. Again for $i \in X$ and from the above we get $(\beta,i)f = (\beta,i)$. Since $(p,q) \sim (\beta,i)$ as $p,\beta \in B$ and $q,i \in A$ gives $(p,q)f = (q,\beta) \sim (\beta,i) = (\beta,i)f$; a contradiction of Remark 1. Thus, $(p,q)f = (p,q)$ $\forall p \in B$ and $q \in A \setminus X$ and hence the result hold.

(iv) Let $p \neq q \in B$. In view of Lemma 16, $(p,q)f \in \{(s,p) : s \in A\} \cup \{(p,q)\}$. Suppose $(p,q)f = (s,p)$ for some $s \in A$. Since $(p,s) \sim (p,q)$ so that $(p,s)f = (p,s) \sim (s,p) = (p,q)f$; a contradiction of Remark 1. Thus, $(p,q)f = (p,q)$ for all distinct $p,q \in B$.

(v) By Claim 1(i), we have $(p,q)f = (p,q)$ when $q \in X$. It is sufficient to prove the result for $q \in A \setminus X$. In view of Lemma 16, $(p,q)f \in \{(q,s) : s \in B\} \cup \{p,q)\}$. Suppose $(p,q)f = (q,s)$ for some $s \in B$. Then by (iv) part, we have $(s,x)f = (s,x)$ where $x \in B \setminus \{s\}$. For $p,q \in A$ and $s,x \in B$, we get $(p,q) \sim (s,x)$ gives $(p,q)f = (q,s) \sim (s,x) = (s,x)f$; a contradiction of Remark 1. Thus, $(p,q)f = (p,q)$ for all distinct $p,q \in A$.

(vi) Suppose $n = 5$, $|A| = 2$, $|B| = 3$ and $p \neq q$. Then $X = A$ so $(p,q)f = (p,q)$ for all $p,q \in A$ (see Claim 1(i)). If $p,q \in B$, then by Lemma 16, $(p,q)f \in \{(s,p) : s \in A\} \cup \{p,q)\}$. Suppose $(p,q)f = (s,p)$ for some $s \in A$. Then there exists $s' \in A$ as $|A| = 2$. Consequently, $(s',s)f = (s',s)$ and $(p,q) \sim (s',s)$ gives $(p,q)f = (s,p) \sim (s',s) = (s',s)f$ which is not possible. Thus, $(p,q)f = (p,q)$ for all $p,q \in B$. Now we suppose that $p \in B$ and $q \in A$. In view of Lemma 16, we have $(p,q)f \in \{(r,p) : r \in A\} \cup \{(q,r') : r' \in B\} \cup \{(p,q)\}$. Suppose $(p,q)f = (r,p)$ for some $r \in A = X$. For

$\beta \in B \setminus \{p\}$, we get $(p,q) \sim (p,\beta)$ and $(p,\beta)f = (p,\beta)$ provides $(s,p) \sim (p,\beta)$ which is not possible. Therefore, $(p,q)f \in \{(q,r') : r' \in B\} \cup \{(p,q)\}$. Let $(p,q)f = (q,r')$ for some $r' \in B$. Since $|B| = 3$ so that there exists $z \in B \setminus \{p,r'\}$. As a consequence, we have $(r',z) \sim (p,q)$ and $(r',z)f = (r',z)$ implies $(r',z)f = (r',z) \sim (q,r') = (p,q)f$; a contradiction. For $p \in A$ and $q \in B$, note that $(p,q) \in A \times B \subseteq \mathcal{G}'$ and $\mathcal{G}'$ is a retraction. Consequently, $(p,q)f = f(p,q)$ because all the elements of $\mathcal{G}'$ are fixed by f. Thus, $(p,q)f = (p,q)$ for all $p \neq q \in [n]$.

Thus, by Claim 2, we have $(p,q)f = (p,q)$ for all $p \neq q$. Now we show that $(p,p)f = (p,p)$ for all $p \in [n]$. On contrary assume that $(p,p)f = (x,y)$ for some $(x,y) \neq (p,p) \in B_n$. Then $(x,y)f = (x,y)$ as f is a retraction on $\Delta(B_n)$. Note that $x \neq y$. Otherwise, $(p,p) \sim (x,y)$ but $(p,p)f = (x,y)f = (x,y)$; a contradiction. Also, observe that $p \in \{x,y\}$. Otherwise, being an adjacent elements (x,y) and (p,p) have same images; again a contradiction. Without loss of generality assume that $x = p$. For $z \in [n] \setminus \{y,p\}$, we get $(p,p) \sim (y,z)$ so that $(p,p)f = (p,y) \sim (y,z) = (y,z)f$; a contradiction of Remark 1. Thus, f is an identity map. Consequently, $\mathcal{G}' = \Delta(B_n)$; a contradiction. Thus, **Case 1** is not possible.

Case 2: Suppose $|X| \leq |A \setminus X|$. Then $X \neq A$. Now, we have the following subcases depend on n. In each subcase, we prove that $A = X$ which is a contradiction.

Subcase 1: n is even. The following claim will be useful in the sequel.

Claim 3. (i) *Let $i \in A \setminus X$. Then there exists a unique $s_i \in B$ such that the restriction map $f_{A_i \times B_{s_i}}$ of f is a bijection from $A_i = \{(r,i) : r \in A \setminus \{i\}\}$ onto $B_{s_i} = \{(i,s) : s \in B \setminus \{s_i\}\}$.*

(ii) *In view of part (i), for $Y = \{s_i \in B : i \in A \setminus X\}$, we have $Y = B$. Moreover, for $i \neq j \in A \setminus X$, we have $s_i \neq s_j$.*

(iii) *If $x \neq y \in B$, then $(x,y)f = (x,y)$.*

(iv) *If $i \neq j \in A$, then $(i,j)f = (i,j)$.*

Proof of Claim: (i) Let $i \in A \setminus X$. Then for $r \in A \setminus \{i\}$, we have $(r,i)f = (i,s)$ for some $s \in B$ (see Claim 1(ii)). Consequently, $A_i f \subset \{(i,s) : s \in B\}$. Since f is one-one on A_i because A_i forms a clique, we get $|A_i f| = |A_i| = |A| - 1 = |B| - 1$ as n is even. Thus, there exists $s_i \in B$ such that $A_i f = B_{s_i}$, where $B_{s_i} = \{(i,s) : s \in B \setminus \{s_i\}\}$. Hence, $f_{A_i \times B_{s_i}}$ is a one-one map from A_i onto B_{s_i}.

(ii) Clearly $Y \subseteq B$. We show that $Y \subset B$ is not possible. On contrary, if $Y \subset B$ so there exists $s \in B \setminus Y$. Let $x \in B \setminus \{s\}$. By Lemma 16, $(s,x)f \in \{(\alpha,s) : \alpha \in A\} \cup \{(s,x)\}$. We provide a contradiction for both the possibilities of $(s,x)f$. Suppose $(s,x)f = (\alpha,s)$ for some $\alpha \in A$. By Claim 2(ii), in fact we have $(s,x)f = (\alpha,s)$ for some $\alpha \in A \setminus X$. Then by part (i) there exists $s_\alpha \in B$ such that the map $f_{A_\alpha \times B_{s_\alpha}}$ is a bijection. As $s_\alpha \in Y$, $s \neq s_\alpha$ so that $(\alpha,s) \in B_{s_\alpha}$. Consequently, there exists $r_\alpha \in A \setminus \{\alpha\}$ such that $(r_\alpha,\alpha)f = (\alpha,s)$. Now since $r_\alpha, \alpha \in A$ and $s,x \in B$ we get $(r_\alpha,\alpha) \sim (s,x)$ as A and B forms a partition of $[n]$ so that $(r_\alpha,\alpha)f \sim (s,x)f$.

But $(r_\alpha, \alpha)f = (s, x)f = (\alpha, s)$ which is not possible. It follows that $(s, x)f = (s, x)$. For $i \in A \setminus X$, there exists $s_i \in Y$ such that the map $f_{A_i \times B_{s_i}}$ is a bijection. Since $s \neq s_i$ as $s \notin Y$ gives $(i, s) \in B_{s_i}$. As a result, there exists $r \in A \setminus \{i\}$ such that $(r, i)f = (i, s)$. For $r, i \in A$ and $s, x \in B$, we get $(s, x) \sim (r, i)$; again a contradiction as $(s, x)f = (s, x) \sim (i, s) = (r_i, i)f$. Hence, $Y = B$.

(iii) Let $x, y \in B$. Then by Lemma 16, $(x, y)f \in \{(\alpha, x) : \alpha \in A\} \cup \{(x, y)\}$. Suppose $(x, y)f = (\alpha, x)$ for some $\alpha \in A$. In fact $\alpha \in A \setminus X$ (see Claim 2(ii)). For $x \in B = Y$, there exists $i_x \in A \setminus X$ such that $f_{A_{i_x} \times B_x}$ is a bijection. If $\alpha \in A \setminus (X \cup \{i_x\})$, then by part (i) there exists $s_\alpha \in B \setminus \{x\}$ such that the restriction map $f_{A_\alpha \times B_{s_\alpha}}$ is a bijective map and $(\alpha, x) \in B_{s_\alpha}$. Consequently, we get $(r, \alpha)f = (\alpha, x)$ for some $r \in A \setminus \{\alpha\}$. But $(x, y) \sim (r, \alpha)$ as $x, y \in B$ and $r, \alpha \in A$ gives $(x, y)f \neq (r, \alpha)f$. However, we have $(x, y)f = (r, \alpha)f$; a contradiction. It follows that $\alpha = i_x$. In view of Lemma 16, for $y' \in B \setminus \{x, y\}$, note that $(x, y')f \in \{(\alpha', x) : \alpha' \in A\} \cup \{(x, y')\}$. Now observe that $(x, y')f \neq (x, y')$. If $(x, y')f = (x, y')$, then $(x, y) \sim (x, y')$ provides $(\alpha, x) \sim (x, y')$; a contradiction of Remark 1. Thus, $(x, y')f = (\alpha', x)$ for some $\alpha' \in A \setminus X$. Further note that $\alpha' \neq \alpha$. Otherwise, $(x, y) \sim (x, y')$ gives $(x, y)f \sim (x, y')$ but $(x, y)f = (x, y')f = (\alpha, x)$ which is not possible. Consequently, $\alpha' \neq i_x$. By the similar argument used for $\alpha \neq i_x$, we get $(r', \alpha')f = (\alpha', x)$ for some $r' \in A \setminus \{\alpha'\}$. Since $(r', \alpha') \sim (x, y')$ we get $(r', \alpha')f \sim (x, y')f$ but $(r', \alpha')f = (x, y')f = (\alpha', x)$ is not possible. Hence, $(x, y)f = (x, y)$ for all $x \neq y \in B$.

(iv) Suppose $i \neq j \in A$. Then by Lemma 16, $(i, j)f \in \{(j, \beta) : \beta \in B\} \cup \{(i, j)\}$. If $(i, j)f = (j, \beta)$ for some $\beta \in B$ then for $x \in B \setminus \{\beta\}$ note that $(i, j) \sim (\beta, x)$ but $(i, j)f = (j, \beta) \nsim (\beta, x) = (\beta, x)f$ (cf. part (iii)). Thus, $(i, j)f = (i, j)$.

By Claim 3(iv), we get $A = X$. Therefore, Case 2 is not possible when n is even.

Subcase 2: n is odd. By Lemma 15, we have either $|A| = \frac{n+1}{2}$, $|B| = \frac{n-1}{2}$ or $|A| = \frac{n-1}{2}$, $|B| = \frac{n+1}{2}$ (see proof of Lemma 2). First we prove the following claim.

Claim 4. (i) *If x and y are distinct elements of B, then $(x, y)f = (x, y)$.*

(ii) *If $x \in B$ and $i \in A$, then $(x, i)f = (x, i)$.*

Proof of Claim: (i) First, we suppose that $|A| = \frac{n+1}{2}$ and $|B| = \frac{n-1}{2}$. Let $x \neq y \in B$. Then by Lemma 16, we get either $(x, y)f = (i, x)$ for some $i \in A$ or $(x, y)f = (x, y)$. Let if possible, $(x, y)f = (i, x)$ for some $i \in A$. In fact $i \in A \setminus X$ (cf. Claim 2(ii)). Also, for $r \in A \setminus \{i\}$ and $i \in A \setminus X$, by Claim 1(ii), we get $(r, i)f = (i, s)$ for some $s \in B$. As a result, $A_i f \subseteq B_i$ where $A_i = \{(r, i) : r \in A \setminus \{i\}\}$ and $B_i = \{(i, s) : s \in B\}$. Since A_i forms a clique, we have f is one-one on A_i. Moreover, $|A_i f| = |A_i| = |A| - 1 = |B| = |B_i|$. Therefore, we get a bijection $f_{A_i \times B_i}$ from A_i onto B_i. Then there exists $r \in A \setminus \{i\}$ such that $(r, i)f = (i, x)$. Note that $(x, y) \sim (r, i)$ but $(x, y)f = (r, i)f = (i, x)$ which is not possible. Thus, $(x, y)f = (x, y)$ for all $x \neq y \in B$.

On the other hand, we may assume that $|A| = \frac{n-1}{2}$ and $|B| = \frac{n+1}{2}$. Then $|B| \geq 3$. First, we claim that there exist $x, y \neq B$ such that $(x, y)f = (x, y)$. On contrary, we assume that $(x, y)f \neq (x, y)$ for all $x \neq y$ in B. Let $x \neq y \in B$. By Lemma 16 and Claim 2(ii), we have $(x, y)f = (\alpha, x)$ for some $\alpha \in A \setminus X$. Similarly, for any $y' \in B \setminus \{x, y\}$, we have $(x, y')f = (\alpha', x)$ for some $\alpha' \in A \setminus X$. It follows that $B_x f \subseteq A_x$ where $B_x = \{(x, z) : z \in B \setminus \{x\}\}$ and $A_x = \{(i, x) : i \in A \setminus X\}$. Since the set B_x forms a clique so that f is one-one on B_x provide $|B_x f| = |B_x| = |B| - 1 = |A| = |A_x| = |A \setminus X|$. Consequently, we get $f_{B_x \times A_x}$ is a bijection and $X = \varnothing$. For $r \in A \setminus \{\alpha\}$, we have $(r, \alpha)f = (\alpha, \beta)$ for some $\beta \in B$ (cf. Claim 1(ii)). If $\beta = x$, then $(x, y)f = (r, \alpha)f = (\alpha, x)$ but $(x, y) \sim (r, \alpha)$ which is not possible. For $\beta \neq x$, by using the similar argument used for x, there exist the subsets B_β and A_β such that the restriction map $f_{B_\beta \times A_\beta}$ is a bijective map. As a consequence $(\alpha, \beta) \in A_\beta$ so that there exists $(\beta, s) \in B_\beta$ such that $(\beta, s)f = (\alpha, \beta)$. As $r, \alpha \in A$ and $\beta, s \in B$, $(r, \alpha) \sim (\beta, s)$ gives $(r, \alpha)f \sim (\beta, s)f$ but $(r, \alpha)f = (\beta, s)f = (\alpha, \beta)$ which is not possible. Thus, there exist $p \neq q \in B$ such that $(p, q)f = (p, q)$.
For any $w \in B \setminus \{p, q\}$, we have either $(p, w)f = (p, w)$ or $(p, w)f = (i, p)$ for some $i \in A$. Since $(p, q) \sim (p, w)$ so that $(p, q)f = (p, q) \sim (p, w)f$ implies $(p, w)f \neq (i, p)$ for any $i \in A$. Therefore, $(p, w)f = (p, w)$. Consider the subsets $A' = A \cup \{p\}$ and $B' = B \setminus \{p\}$ of $[n]$. Note that A' and B' are the disjoint subsets of $[n]$ with $|A'| = \frac{n+1}{2}$ and $|B'| = \frac{n-1}{2}$ so $A' \times B'$ forms a clique of maximum size in $\mathcal{G}'$. If $|X| > |A' \setminus X|$, then in Claim 2(iv), replace A and B with A' and B' respectively, we get $(a, b)f = (a, b)$ for all $a, b \in B'$. For $|X| \leq |A' \setminus X|$, by using the similar concept used above we have $(a, b)f = (a, b)$ for all $a, b \in B'$. Since $(p, w)f = (p, w)$ for all $w \in B \setminus \{p\}$ so that $(a, b)f = (a, b)$ for all distinct $a, b \in B$ and $b \neq p$. If possible, let $(a, p)f \neq (a, p)$, then by Lemma 16, $(a, p)f = (l, a)$ for some $l \in A$. Choose $\beta \in B \setminus \{a, p\}$ so $(a, \beta) \sim (a, p)$ and $(a, \beta)f = (a, \beta)$ as $a, \beta \in B'$ we obtained $(a, \beta)f = (a, \beta) \sim (l, a) = (a, p)$; a contradiction of remark 1. Hence, $(a, b)f = (a, b)$ for all distinct $a, b \in B$.

(ii) Let $x \in B$ and $i \in A$. Then by Lemma 16, we have $(x, i)f \in \{(\alpha, x) : \alpha \in A\} \cup \{(i, \beta) : \beta \in B\} \cup \{(x, i)\}$. Note $(x, i)f \neq (\alpha, x)$ for any $\alpha \in A$. For instance if $(x, i)f = (\alpha, x)$ for some $\alpha \in A$, then $(x, y) \sim (x, i)$ where $y \in B \setminus \{x\}$ gives $(x, y)f \sim (x, i)f$. By part (i), we get $(x, y)f = (x, y)$ so $(x, y) \sim (\alpha, x)$; a contradiction of Remark 1. On the other hand now we get a contradiction for $(x, i)f = (i, \beta)$ for some $\beta \in B$. If $\beta = x$ then for $\gamma \in B \setminus \{x\}$, we have $(x, \gamma)f = (x, \gamma)$ (by part (i)). Since $(x, i) \sim (x, \gamma)$ but $(x, i)f = (i, x) \nsim (x, \gamma) = (x, \gamma)f$ which is not possible so $\beta \neq x$. For $n \geq 5$, we have $|B| \geq 2$. If $|B| = 2$, then $|A| = 3$. There exist $j, k \in A \setminus \{i\}$. Consequently, $(j, i)f = (i, y)$ and $(k, i)f = (i, z)$ for some $y, z \in B$ (cf. Lemma 16). Because if $(j, i)f = (j, i)$ then $(x, i) \sim (j, i)$ gives $(x, i)f = (i, \beta) \sim (j, i) = (j, i)f$; a contradiction of Remark 1. Similarly, $(k, i)f = (k, i)$ is not possible. Note that $\{(x, i), (j, i), (k, i)\}$ forms a clique of size 3 so that $\{(x, i)f, (j, i)f, (k, i)f\} = \{(i, \beta), (i, y), (i, z)\}$. Consequently, β, y, z are the elements of B. Thus, $|B| \geq 3$; a contradiction of $|B| = 2$. It follows that $|B| \geq 3$. For $z \in B \setminus \{x, \beta\}$ we have $(x, i) \sim (\beta, z)$. By part (i), $(\beta, z)f = (\beta, z)$. Consequently, $(x, i)f = (i, \beta) \sim (\beta, z) = (\beta, z)f$ which is not possible. Hence, $(x, i)f = (x, i)$.

Now if $x \in A$, then $i \in A \setminus X$. For $x \in B$, by Claim 4(ii), we have $(x, i)f = (x, i)$. Since $(1, i) \sim (x, i)$ so that $(1, i)f = (i, s) \sim (x, i) = (x, i)f$; a contradiction of Remark 1. Thus, $X \subset A$ is not possible. Consequently, $X = A$; a contradiction of **Case 2**. In view of **Case 1** and **Case 2** such X is not possible. Thus, $\Delta(B_n)$ admits no proper retract. Hence, $\Delta(B_n)$ is a core. □

4. Graph invariants of $\Delta(B_n)$

In this section, we obtained the vertex connectivity and edge connectivity of $\Delta(B_n)$.

Theorem 7. *For $n \geq 3$, the vertex connectivity of $\Delta(B_n)$ is $n(n-2)$.*

Proof. By Theorem 4.1.9 of [36] and Corollary 1, we have $\kappa(\Delta(B_n)) \leq n(n-2)$. By Menger's theorem (cf. [7, Theorem 3.2]), to prove another inequality, it is sufficient to show that there exist at least $n(n-2)$ internally disjoint paths between arbitrary pair of vertices. Let (a, b) and (c, d) be arbitrary pair of vertices in $V(\Delta(B_n))$. Now consider

$$A = \{(b, x) : \ x \in [n]\} \cup \{(x, a) : \ x \in [n]\}$$

and

$$B = \{(d, x) : \ x \in [n]\} \cup \{(x, c) : \ x \in [n]\}.$$

Note that $|A| = |B| = 2n - 1$ and each element of A and B is not adjacent with (a, b) and (c, d), respectively (see Remark 1). If $T = A \cup B \cup \{(a, b), (c, d)\}$, then note that every element of $T' = V(\Delta(B_n)) \setminus T$, commutes with (a, b) and (c, d). Thus, for each element (x, y) of T', we have a path $(a, b) \sim (x, y) \sim (c, d)$. Consequently, there are at least $|T'|$ many internally disjoint paths between (a, b) and (c, d). We show that there exist $n(n-2)$ internally disjoint paths between (a, b) and (c, d) in the following cases.

Case 1: Both (a, b) and (c, d) are distinct idempotents. Clearly $a = b, c = d$ and $a \neq c$. Then, we have $A \cap B = \{(a, c), (c, a)\}$ so that $|T'| = n^2 - 4n + 4$. As a consequence, we get $n^2 - 4n + 4$ internally disjoint paths between (a, b) and (c, d). Furthermore, for $x \in [n] \setminus \{a, c\}$, we have $(a, a) \sim (c, x) \sim (a, x) \sim (c, c)$ and $(a, a) \sim (x, c) \sim (x, a) \sim (c, c)$ internally disjoint paths between (a, b) and (c, d) which are $2n - 4$ in total. Thus, there are at least $n^2 - 2n$ internally disjoint paths between (a, b) and (c, d).

Case 2: Either (a, b) or (c, d) is idempotent. Without loss of generality, let $c = d$. Further, we have the following subcases.

Subcase 2.1: $c \neq a, b$. Then $A \cap B = \{(b, c), (c, a)\}$ so that $|T'| = n^2 - 4n + 3$. Consequently, we get $n^2 - 4n + 3$ internally disjoint paths between (a, b) and (c, d). In addition to that, for $x \in [n] \setminus \{a, b, c\}$, we have

$$(a, b) \sim (c, x) \sim (b, x) \sim (c, c),$$

$$(a,b) \sim (x,c) \sim (x,a) \sim (c,c)$$

internally disjoint paths between (a,b) and (c,d) which are $2n-6$ in total. Further, we have three more paths between (a,b) and (c,d) as follows:

$$(a,b) \sim (c,b) \sim (a,a) \sim (c,c),$$

$$(a,b) \sim (a,c) \sim (b,b) \sim (c,c),$$

$$(a,b) \sim (c,c).$$

Thus, there are at least n^2-2n internally disjoint paths between (a,b) and (c,d).
Subcase 2.2: $c=a$ or $c=b$. First suppose $c=a$. Then, we have $A \cap B = \{(x,a) : x \in [n]\}$ so that $|T'| = n^2-3n+2$. Therefore, $\Delta(B_n)$ contains n^2-3n+2 internally disjoint paths between (a,b) and (c,d). Additionally, for $x \in [n] \setminus \{a,b\}$, we have $n-1$ internally disjoint paths $(a,b) \sim (a,x) \sim (b,x) \sim (a,a)$ between (a,b) and (c,d). Thus, there are at least n^2-2n internally disjoint paths between (a,b) and (c,d). Similarly, for $c=b$, at least n^2-2n internally disjoint paths between (a,b) and (c,d) can be obtained.
Case 3: Both (a,b) and (c,d) are non-idempotent elements. Clearly, $a \neq b$ and $c \neq d$. Further, we have the following subcases.
Subcase 3.1: a,b,c,d all are distinct. Then, we have $A \cap B = \{(b,c),(d,a)\}$ so that $|T'| = n^2-4n+2$. Thus, there are n^2-4n+2 internally disjoint paths between (a,b) and (c,d). In addition to that, for $x \in [n] \setminus \{a,b,c,d\}$, we have $(a,b) \sim (x,c) \sim (x,a) \sim (c,d)$ and $(a,b) \sim (d,x) \sim (b,x) \sim (c,d)$ internally disjoint paths between (a,b) and (c,d) which are $2n-8$ in total. Moreover, we have six additional paths between (a,b) and (c,d) as follows:

$$(a,b) \sim (a,c) \sim (b,b) \sim (c,d),$$

$$(a,b) \sim (c,c) \sim (b,d) \sim (c,d),$$

$$(a,b) \sim (d,c) \sim (a,a) \sim (c,d),$$

$$(a,b) \sim (d,d) \sim (b,a) \sim (c,d),$$

$$(a,b) \sim (d,b) \sim (c,a) \sim (c,d),$$

$$(a,b) \sim (c,d).$$

Thus, there are at least n^2-2n internally disjoint paths between (a,b) and (c,d).

Subcase 3.2: $c \in \{a,b\}$. If $c=a$, then $A \cap B = \{(x,a) : x \in [n]\}$ so that $|T'| = n^2-3n$. Therefore, $\Delta(B_n)$ contains n^2-3n internally disjoint paths between (a,b) and (c,d). Additionally, for $x \in [n] \setminus \{a,b,d\}$, we have $(a,b) \sim (d,x) \sim (b,x) \sim (a,d)$ internally disjoint paths between (a, b) and (c, d) which are $n - 3$ in total. Besides these paths, we have three paths between (a, b) and (c, d) as follows:

$$(a, b) \sim (d, b) \sim (a, a) \sim (b, d) \sim (a, d),$$

$$(a, b) \sim (d, d) \sim (b, b) \sim (a, d),$$

$$(a, b) \sim (a, d).$$

Thus, there are at least $n^2 - 2n$ internally disjoint paths between (a, b) and (c, d). On the other hand $c = b$. Now we have the two possibilities (i) $d = a$ (ii) a, b, d are distinct. If $d = a$, then $A \cap B = \{(b, b), (a, a)\}$ so that $|T'| = n^2 - 4n + 4$. Consequently, we get $n^2 - 4n + 4$ internally disjoint paths between (a, b) and (c, d). In addition to that, for $x \in [n] \setminus \{a, b\}$, we have $(a, b) \sim (x, b) \sim (x, a) \sim (b, a)$ and $(a, b) \sim (a, x) \sim (b, x) \sim (b, a)$ internally disjoint paths between (a, b) and (c, d) which are $2n - 4$ in total. Thus, we get at least $n^2 - 2n$ internally disjoint paths between (a, b) and (c, d). For distinct a, b and d, we get $A \cap B = \{(d, a), (b, b)\}$ so that $|T'| = n^2 - 4n + 4$. Consequently, we get $n^2 - 4n + 4$ internally disjoint paths between (a, b) and (c, d). Additionally, for $x \in [n] \setminus \{a, b, d\}$, we have $2n - 6$ internally disjoint paths

$$(a, b) \sim (x, b) \sim (x, a) \sim (b, d),$$

$$(a, b) \sim (d, x) \sim (b, x) \sim (b, d)$$

between (a, b) and (c, d). Besides these paths, we have two more paths $(a, b) \sim (d, b) \sim (a, a) \sim (b, d)$ and $(a, b) \sim (d, d) \sim (b, a) \sim (b, d)$. Thus, there are at least $n^2 - 2n$ internally disjoint paths between (a, b) and (c, d).

Subcase 3.3: $d \in \{a, b\}$. If $d = a$, then $A \cap B = \{(b, c), (a, a)\}$ so that $|T'| = n^2 - 4n + 4$. Consequently, we get $n^2 - 4n + 4$ internally disjoint paths between (a, b) and (c, d). In addition to that, for $x \in [n] \setminus \{a, b, c\}$, we have $(a, b) \sim (a, x) \sim (b, x) \sim (c, a)$ and $(a, b) \sim (x, c) \sim (x, a) \sim (c, a)$ internally disjoint paths between (a, b) and (c, d) which are $2n - 6$ in total. Moreover, we have two paths $(a, b) \sim (a, c) \sim (b, b) \sim (c, a)$ and $(a, b) \sim (c, c) \sim (b, a) \sim (c, a)$ between (a, b) and (c, d). Thus, there are at least $n^2 - 2n$ internally disjoint paths between (a, b) and (c, d). On the other hand, let $d = b$. Then $A \cap B = \{(b, x) : x \in [n]\}$ so that $|T'| = n^2 - 3n$. As a consequence, we get $n^2 - 3n$ internally disjoint paths between (a, b) and (c, d). Furthermore, for $x \in [n] \setminus \{a, b, c\}$, we have $n - 3$ internally disjoint paths $(a, b) \sim (x, c) \sim (x, a) \sim (c, b)$ between (a, b) and (c, d). Besides these paths, we have three more paths between (a, b) and (c, d) as follows:

$$(a, b) \sim (c, c) \sim (a, a) \sim (c, b),$$

$$(a, b) \sim (a, c) \sim (b, b) \sim (c, a) \sim (c, b),$$

$$(a, b) \sim (c, b).$$

Thus, there are at least $n^2 - 2n$ internally disjoint paths between (a, b) and (c, d). □

In view of Lemma 1 and since $\kappa(\mathcal{G}) \leq \kappa'(\mathcal{G}) \leq \delta(\mathcal{G})$, we have the following corollary.

Corollary 3. *For $n \geq 3$, the edge connectivity of $\Delta(B_n)$ is $n(n-2)$.*

Open Problem: The work in this paper can be carried out for other class of semigroups viz. the semigroup of all partial maps on a finite set and its various subsemigroups. In view of Theorem 1; to investigate the commuting graph of finite 0-simple inverse semigroup, it is sufficient to investigate $\Delta(B_n(G))$. In this connection, the results obtained in this paper might be useful. For example, using the result of $\Delta(B_n)$, in particular Theorem 2(iii), we prove the following theorem which gives a partial answer to the problem posed in [3, Section 6].

Theorem 8. *For $n \geq 3$, $\Delta(B_n(G))$ is Hamiltonian.*

Proof. Let $G = \{a_1, a_2, \ldots, a_m\}$. We show that there exists a Hamiltonian cycle in $\Delta(B_n(G))$. First note that if $(i,j) \sim (k,l)$ in $\Delta(B_n)$, then $(i,a,j) \sim (k,b,l)$ in $\Delta(B_n(G))$ for all $a, b \in G$. Let $G_{a_1} = \{(i, a_1, j) : i, j \in [n]\}$. Since $\Delta(B_n)$ is Hamiltonian (see Theorem 2), we assume that there exists a Hamiltonian cycle C . Corresponding to the cycle C, choose a Hamiltonian path P whose first vertex is (i,j) and the end vertex is (k,l). For the path P, there exists a Hamiltonian path in the subgraph induced by G_{a_1} whose first vertex is (i, a_1, j) and the end vertex is (k, a_1, l). Since $(i,j) \sim (k,l)$ in $\Delta(B_n)$, we have $(k, a_1, l) \sim (i, a_2, j)$. By the similar way, we get a Hamiltonian path in the subgraph induced by G_{a_2} whose first vertex is (i, a_2, j) and the end vertex is (k, a_2, l). On Continuing this process, we get a Hamiltonian path in $\Delta(B_n(G))$ with first vertex is (i, a_1, j) and the end vertex is (k, a_m, l). For $(i,j) \sim (k,l)$, we get $(i, a_1, j) \sim (k, a_m, l)$. Thus, $\Delta(B_n(G))$ is Hamiltonian. □

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References

[1] A. Alilou and J. Amjadi, *The sum-annihilating essential ideal graph of a commutative ring*, Commun. Comb. Optim. **1** (2016), no. 2, 117–135. https://doi.org/10.22049/cco.2016.13555.

[2] J. Araújo, W. Bentz, and K. Janusz, *The commuting graph of the symmetric inverse semigroup*, Israel J. Math. **207** (2015), no. 1, 103–149.
https://doi.org/10.1007/s11856-015-1173-9.

[3] J. Araújo, M. Kinyon, and J. Konieczny, *Minimal paths in the commuting graphs of semigroups*, European J. Combin. **32** (2011), no. 2, 178–197.
https://doi.org/10.1016/j.ejc.2010.09.004.

[4] C. Bates, D. Bundy, S. Perkins, and P. Rowley, *Commuting involution graphs for finite Coxeter groups*, J. Group Theory **6** (2003), no. 4, 461–476.
https://doi.org/10.1515/jgth.2003.032.

[5] ———, *Commuting involution graphs for symmetric groups*, J. Algebra **266** (2003), no. 1, 133–153.
https://doi.org/10.1016/S0021-8693(03)00302-8.

[6] T. Bauer and B. Greenfeld, *Commuting graphs of boundedly generated semigroups*, European J. Combin. **56** (2016), 40–45.
https://doi.org/10.1016/j.ejc.2016.02.009.

[7] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Elsevier Publishing, New York, 1976.

[8] D. Bundy, *The connectivity of commuting graphs*, J. Combin. Theory Ser. A **113** (2006), no. 6, 995–1007.
https://doi.org/10.1016/j.jcta.2005.09.003.

[9] P.J. Cameron, *Graph Homomorphisms*, Combinatorics Study Group Notes (2006), Manuscript.

[10] I. Chakrabarty and J.V. Kureethara, *A survey on the intersection graphs of ideals of rings*, Commun. Comb. Optim. **7** (2022), no. 2, 121–167.
https://doi.org/10.22049/cco.2021.26990.1176.

[11] M. Ciric and S. Bogdanovic, *The five-element Brandt semigroup as a forbidden divisor*, Semigroup Forum **61** (2000), no. 3, 363–372.
https://doi.org/10.1007/PL00006035.

[12] S. Dalal, *Graphs associated with groups and semigroups*, Ph.D. thesis, BITS Pilani, Pilani, 2021.

[13] S. Dalal and J. Kumar, *Chromatic number of the cyclic graph of infinite semigroup*, Graphs Combin. **36** (2020), no. 1, 109–113.
https://doi.org/10.1007/s00373-019-02120-4.

[14] N.D. Gilbert and M. Samman, *Endomorphism seminear-rings of Brandt semigroups*, Comm. Algebra **38** (2010), no. 11, 4028–4041.
https://doi.org/10.1080/00927870903286892.

[15] Y. Hao, X. Gao, and Y. Luo, *On the Cayley graphs of Brandt semigroups*, Comm. Algebra **39** (2011), no. 8, 2874–2883.
https://doi.org/10.1080/00927872.2011.568028.

[16] P. Hell and J. Nešetřil, *The core of a graph*, Discrete Math. **109** (1992), no. 1-3, 117–126.
https://doi.org/10.1016/0012-365X(92)90282-K.

[17] J.M. Howie, *Fundamentals of Semigroup Theory*, Oxford University Press, Oxford, 1995.

[18] J.M. Howie and M.I.M. Ribeiro, *Rank properties in finite semigroups*, Comm. Algebra **27** (1999), no. 11, 5333–5347.
https://doi.org/10.1080/00927879908826758.

[19] ———, *Rank properties in finite semigroups II: The small rank and the large rank*, Southeast Asian Bull. Math. **24** (2000), no. 2, 231–237.
https://doi.org/10.1007/s10012-000-0231-2.

[20] A. Iranmanesh and A. Jafarzadeh, *On the commuting graph associated with the symmetric and alternating groups*, J. Algebra .Appl. **7** (2008), no. 1, 129–146.
https://doi.org/10.1142/S0219498808002710.

[21] M. Jackson and M. Volkov, *Undecidable problems for completely 0-simple semigroups*, J. Pure Appl. Algebra **213** (2009), no. 10, 1961–1978.
https://doi.org/10.1016/j.jpaa.2009.02.011.

[22] K. Kátai-Urbán and C. Szabó, *Free spectrum of the variety generated by the five element combinatorial Brandt semigroup*, **73** (2006), no. 2, 253–260.
https://doi.org/10.1007/s00233-006-0615-4.

[23] B. Khosravi and B. Khosravi, *A characterization of Cayley graphs of Brandt semigroups*, Bull. Malays. Math. Sci. Soc. **35** (2012), no. 2, 399–410.

[24] A. Kumar, L. Selvaganesh, P.J. Cameron, and T.T. Chelvam, *Recent developments on the power graph of finite groups–A survey*, AKCE Int. J. Graphs Comb. **18** (2021), no. 2, 65–94.
https://doi.org/10.1080/09728600.2021.1953359.

[25] J. Kumar, *Affine near-semirings over Brandt semigroups*, Ph.D. thesis, IIT Guwahati, Guwahati, 2014.

[26] J. Kumar, S. Dalal, and P. Pandey, *On the structure of the commuting graph of Brandt semigroups*, International Conference on Semigroups and Applications, Springer Proceedings in Mathematics & Statistics, 2019, pp. 95–105.

[27] S. Margolis, J. Rhodes, and P.V. Silva, *On the subsemigroup complex of an aperiodic Brandt semigroup*, **97** (2018), no. 1, 7–31.
https://doi.org/10.1007/s00233-018-9927-4.

[28] J.D. Mitchell, *Turán's graph theorem and maximum independent sets in Brandt semigroups*, Semigroups and languages, World Sci. Publ., River Edge, NJ, 2004, pp. 151–162.

[29] F. Movahedi, *The energy and edge energy of some Cayley graphs on the abelian group* $\mathbb{Z}_n^4$, Commun. Comb. Optim. **9** (2024), no. 1, 119–130.
https://doi.org/10.22049/cco.2023.28642.1647.

[30] R.P. Panda, S. Dalal, and J. Kumar, *On the enhanced power graph of a finite group*, Comm. Algebra **49** (2021), no. 4, 1697–1716.
https://doi.org/10.1080/00927872.2020.1847289.

[31] M.M. Sadar, *Pseudo-amenability of Brandt semigroup algebras*, Comment. Math. Univ. Carolin. **50** (2009), no. 3, 413–419.
http://eudml.org/doc/33324.

[32] M.M. Sadr, *Morita equivalence of Brandt semigroup algebras*, Int. J. Math. Math. Sci. **2012**, Article ID: 280636.
https://doi.org/10.1155/2012/280636.

[33] Y. Segev, *On finite homomorphic images of the multiplicative group of a division algebra*, Annals Math. **149** (1999), no. 1, 219–251.
https://doi.org/10.2307/121024.

[34] ———, *The commuting graph of minimal nonsolvable groups*, Geometriae Dedicata **88** (2001), no. 1, 55–66.
http://doi.org/10.1023/A:1013180005982.

[35] Y. Segev and G.M. Seitz, *Anisotropic groups of type A_n and the commuting graph of finite simple groups*, Pacific J. Math. **202** (2002), no. 1, 125–225.
http://doi.org/10.2140/pjm.2002.202.125.

[36] D.B. West, *Introduction to Graph Theory*, Prentice hall Upper Saddle River, 2001.