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Research Article

On zero-divisor graph of the ring $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$

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Abstract: In this article, we discussed the zero-divisor graph of a commutative ring with identity $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$ where $u^3 = 0$ and p is an odd prime. We find the clique number, chromatic number, vertex connectivity, edge connectivity, diameter and girth of a zero-divisor graph associated with the ring. We find some of topological indices and the main parameters of the code derived from the incidence matrix of the zero-divisor graph $\Gamma(R)$. Also, we find the eigenvalues, energy and spectral radius of both adjacency and Laplacian matrices of $\Gamma(R)$.

Keywords: zero-divisor graph, Laplacian matrix, spectral radius.

AMS Subject classification: 05C09, 05C40, 05C50

1. Introduction

The zero-divisor graph has attracted a lot of attention in the last few years. In 1988, Beck [6] introduced the zero-divisor graph. He included the additive identity of a ring R in the definition and was mainly interested in the coloring of commutative rings. Let Γ be a simple graph whose vertices are the set of zero-divisors of the ring R, and two distinct vertices are adjacent if the product is zero. Later it was modified by Anderson and Livingston [1]. They redefined the definition as a simple graph that only considers the non-zero zero-divisors of a commutative ring R.

Let R be a commutative ring with identity and $Z(R)$ be the set of zero-divisors of R. The zero-divisor graph $\Gamma(R)$ of a ring R is an undirected graph whose vertices are the non-zero zero-divisors of R with two distinct vertices x and y are adjacent if and only if $xy = 0$. In this article, we consider the zero-divisor graph $\Gamma(R)$ as a graph with vertex set $Z^*(R)$ the set of non-zero zero-divisors of the ring R. Many researchers are doing research in this area [9, 11, 13, 14].

Let $\Gamma = (V, E)$ be a simple undirected graph with vertex set V, edge set E. The incidence matrix of a graph Γ is a $|V| \times |E|$ matrix $Q(\Gamma)$ whose rows are labelled by the vertices and columns by the edges and entries $q_{ij} = 1$ if the vertex labelled by row i is incident with the edge labelled by column j and $q_{ij} = 0$ otherwise.

The adjacency matrix $A(\Gamma)$ of the graph Γ, is the $|V| \times |V|$ matrix defined as follows. The rows and the columns of $A(\Gamma)$ are indexed by V. If $i \neq j$ then the (i, j)-entry of $A(\Gamma)$ is 0 for vertices i and j which are nonadjacent, and the (i, j)-entry is 1 for i and j which are adjacent. The (i, i)-entry of $A(\Gamma)$ is 0 for $i = 1, \ldots, |V|$. For any graph Γ, the energy of the graph is defined as

$$\varepsilon(\Gamma) = \sum_{i=1}^{|V|} |\lambda_i|,$$

where $\lambda_1, \ldots, \lambda_{|V|}$ are the eigenvalues of $A(\Gamma)$ of Γ.

The Laplacian matrix $L(\Gamma)$ of Γ is the $|V| \times |V|$ matrix defined as follows. The rows and columns of $L(\Gamma)$ are indexed by V. If $i \neq j$ then the (i, j)-entry of $L(\Gamma)$ is 0 if vertex i and j are not adjacent, and it is -1 if i and j are adjacent. The (i, i)-entry of $L(\Gamma)$ is d_i, the degree of the vertex i, $i = 1, 2, \ldots, |V|$. Let $D(\Gamma)$ be the diagonal matrix of vertex degrees. If $A(\Gamma)$ is the adjacency matrix of Γ, then note that $L(\Gamma) = D(\Gamma) - A(\Gamma)$. Let $\mu_1, \mu_2, \ldots, \mu_{|V|}$ are eigenvalues of $L(\Gamma)$. Then the Laplacian energy $LE(\Gamma)$ is given by

$$LE(\Gamma) = \sum_{i=1}^{|V|} \left| \mu_i - \frac{2|E|}{|V|} \right|.$$

Lemma 1. *[5] Let $\Gamma = (V, E)$ be a graph, and let $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{|V|}$ be the eigenvalues of its Laplacian matrix $L(\Gamma)$. Then, $\mu_2 > 0$ if and only if Γ is connected.*

The Wiener index of a connected graph Γ is defined as the sum of distances between each pair of vertices, i.e.,

$$W(\Gamma) = \sum_{\substack{a,b \in V \\ a \neq b}} d(a, b),$$

where $d(a, b)$ is the length of shortest path joining a and b.

The degree of $v \in V$, denoted by d_v, is the number of vertices adjacent to v.

The Randić index (also known under the name connectivity index) is a much investigated degree-based topological index. It was invented in 1976 by Milan Randić [12] and is defined as

$$R(\Gamma) = \sum_{(a,b) \in E} \frac{1}{\sqrt{d_a d_b}}$$

with summation going over all pairs of adjacent vertices of the graph.

The Zagreb indices were introduced more than 50 years ago by Gutman and Trinajestić [8]. For a graph Γ, the first Zagreb index $M_1(\Gamma)$ and the second Zagreb index $M_2(\Gamma)$ are, respectively, defined as follows:

$$M_1(\Gamma) = \sum_{a \in V} d_a^2$$

$$M_2(\Gamma) = \sum_{(a,b) \in E} d_a d_b.$$

An edge-cut of a connected graph Γ is the set $S \subseteq E$ such that $\Gamma - S = (V, E - S)$ is disconnected. The edge-connectivity $\lambda(\Gamma)$ is the minimum cardinality of an edge-cut. The minimum k for which there exists a k-vertex cut is called the vertex connectivity or simply the connectivity of Γ it is denoted by $\kappa(\Gamma)$.

For any connected graph Γ, we have $\lambda(\Gamma) \leq \delta(\Gamma)$ where $\delta(\Gamma)$ is minimum degree of the graph Γ.

The chromatic number of a graph Γ is the minimum number of colors needed to color the vertices of Γ so that adjacent vertices of Γ receive distinct colors and is denoted by $\chi(\Gamma)$. A clique of a graph Γ is a complete subgraph of Γ. The clique number $\omega(\Gamma)$ of a graph Γ is the number of vertices in a maximum clique of Γ. Note that for any graph Γ, $\omega(\Gamma) \leq \chi(\Gamma)$. The girth of an undirected graph is the length of a shortest cycle contained in the graph.

Beck [6] conjectured that if R is a finite chromatic ring, then $\omega(\Gamma(R)) = \chi(\Gamma(R))$ where $\omega(\Gamma(R)), \chi(\Gamma(R))$ are the clique number and the chromatic number of $\Gamma(R)$, respectively. He also verified that the conjecture is true for several examples of rings. Anderson and Naseer, in [1], disproved the above conjecture with a counterexample. $\omega(\Gamma(R))$ and $\chi(\Gamma(R))$ of the zero-divisor graph associated to the ring $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$ are same. For basic graph theory, one can refer [4, 5].

Let $\mathbb{F}_q$ be a finite field with q elements. Let $x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$, then the Hamming weight $w_H(x)$ of x is defined by the number of non-zero coordinates in x. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{F}_q^n$, the Hamming distance $d_H(x, y)$ between x and y is defined by the number of coordinates in which they differ.

A q-ary code of length n is a non-empty subset C of $\mathbb{F}_q^n$. If C is a subspace of $\mathbb{F}_q^n$, then C is called a q-ary linear code of length n. An element of C is called a *codeword*. The minimum Hamming distance of a code C is defined by

$$d_H(C) = \min\{d_H(c_1, c_2) \mid c_1 \neq c_2, c_1, c_2 \in C\}.$$

The minimum weight $w_H(C)$ of a code C is the smallest among all weights of the non-zero codewords of C. For q-ary linear code, we have $d_H(C) = w_H(C)$. For basic coding theory, we refer [10].

A linear code of length n, dimension k and minimum distance d is denoted by $[n, k, d]_q$. The code generated by the rows of the incidence matrix $Q(\Gamma)$ of the graph Γ is denoted by $C_p(\Gamma)$ over the finite field $\mathbb{F}_p$.

Theorem 1. *[7]*

1. *Let $\Gamma = (V, E)$ be a connected graph and let G be a $|V| \times |E|$ incidence matrix for Γ. Then, the main parameters of the code $C_2(G)$ is $[|E|, |V| - 1, \lambda(\Gamma)]_2$.*

2. *Let $\Gamma = (V, E)$ be a connected bipartite graph and let G be a $|V| \times |E|$ incidence matrix for Γ. Then the incidence matrix generates $[|E|, |V| - 1, \lambda(\Gamma)]_p$ code for odd prime p.*

Codes from the row span of incidence matrix or adjacency matrix of various graphs are studied in [2, 3, 7, 15, 16].
Let p be an odd prime. The ring $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$ is defined as a characteristic p ring subject to restrictions $u^3 = 0$. The ring isomorphism $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p \cong \frac{\mathbb{F}_p[x]}{\langle x^3 \rangle}$ is obvious to see. An element $a + ub + u^2c \in R$ is unit if and only if $a \neq 0$.

Throughout this article, we denote the ring $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$ by R. In this article, we discussed the zero-divisor graph of a commutative ring with identity $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$ where $u^3 = 0$ and we find the clique number, chromatic number, vertex connectivity, edge connectivity, diameter, and girth of the graph $\Gamma(R)$, in Section 2. In Section 3, we find some of topological indices of $\Gamma(R)$. In Section 4, we find the main parameters of the code derived from incidence matrix of the zero-divisor graph $\Gamma(R)$. Finally, We find the eigenvalues, energy and spectral radius of both adjacency and Laplacian matrices in Section 5.

2. Zero-divisor graph $\Gamma(R)$ of the ring R

In this section, we discuss the zero-divisor graph $\Gamma(R)$ of the ring R and we find the clique number, chromatic number, vertex connectivity, edge connectivity, diameter, and girth of the graph $\Gamma(R)$.

Let $A_u = \{xu \mid x \in \mathbb{F}_p^*\}$, $A_{u^2} = \{xu^2 \mid x \in \mathbb{F}_p^*\}$ and $A_{u+u^2} = \{xu + yu^2 \mid x, y \in \mathbb{F}_p^*\}$. Then $|A_u| = (p-1)$, $|A_{u^2}| = (p-1)$ and $|A_{u+u^2}| = (p-1)^2$. Therefore, $Z^*(R) = A_u \cup A_{u^2} \cup A_{u+u^2}$ and $|Z^*(R)| = |A_u| + |A_{u^2}| + |A_{u+u^2}| = (p-1) + (p-1) + (p-1)^2 = p^2 - 1$. As $u^3 = 0$, every vertices of A_u is adjacent with every vertices of A_{u^2}, every vertices

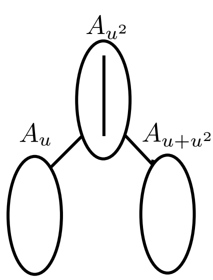

Figure 1. **Zero-divisor graph of** $R = \mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$

of A_{u^2} is adjacent with every vertices of A_{u+u^2} and any two distinct vertices of A_{u^2} are adjacent. From the diagram, the graph $\Gamma(R)$ is connected with p^2-1 vertices and $(p-1)^2+(p-1)^3+\frac{(p-1)(p-2)}{2}=\frac{1}{2}(2p^3-3p^2-p+2)$ edges.

Example 1. For $p=3$, $R=\mathbb{F}_3+u\mathbb{F}_3+u^2\mathbb{F}_3$. Then $A_u=\{u,2u\}$, $A_{u^2}=\{u^2,2u^2\}$, $A_{u+u^2}=\{u+u^2,2u+2u^2,u+2u^2,2u+u^2\}$. The number of vertices is 8 and the number of edges is 13.

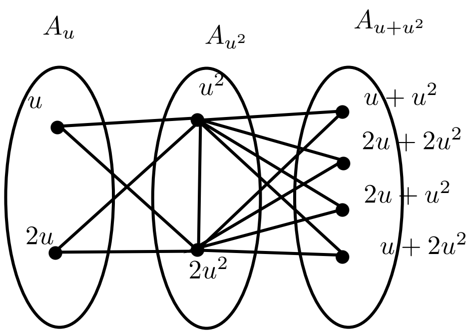

Figure 2. **Zero-divisor graph of** $R=\mathbb{F}_3+u\mathbb{F}_3+u^2\mathbb{F}_3$

Theorem 2. *The diameter of the zero-divisor graph* $diam(\Gamma(R))=2$.

Proof. From the Figure 1, we can see that the distance between any two distinct vertices are either 1 or 2. Therefore, the maximum of distance between any two distinct vertices is 2. Hence, $diam(\Gamma(R))=2$. □

Theorem 3. *The clique number* $\omega(\Gamma(R))$ *of* $\Gamma(R)$ *is* p.

Proof. From the Figure 1, A_{u^2} is a complete subgraph(clique) in $\Gamma(R)$. If we add exactly one vertex v from either A_u or A_{u+u^2}, then resulting subgraph form a complete subgraph(clique). Then $A_{u^2}\cup\{v\}$ forms a complete subgraph with maximum vertices. Therefore, the clique number of $\Gamma(R)$ is $\omega(\Gamma(R))=|A_{u^2}\cup\{v\}|=p-1+1=p$. □

Theorem 4. *The chromatic number* $\chi(\Gamma(R))$ *of* $\Gamma(R)$ *is* p.

Proof. Since A_{u^2} is a complete subgraph with $p-1$ vertices in $\Gamma(R)$, then at least $p-1$ different colors needed to color the vertices of A_{u^2}. And no two vertices in A_u are adjacent then one color different from previous $p-1$ colors is enough to color all vertices in A_u. We take the same color in A_u to color vertices of A_{u+u^2} as there is no direct edge between A_u and A_{u+u^2}. Therefore, minimum p different colors required for proper coloring. Hence, the chromatic number $\chi(\Gamma(R))$ is p. □

The above two theorems show that the clique number and the chromatic number of our graph are same.

Theorem 5. *The girth of the graph $\Gamma(R)$ is 3.*

Proof. Since $p \geq 3$, we have $\Gamma(R)$ contains a cycle of length 3. Hence, the result follows from the definition of girth. □

Theorem 6. *The vertex connectivity $\kappa(\Gamma(R))$ of $\Gamma(R)$ is $p-1$.*

Proof. As the minimum degree $\delta(\Gamma(R))$ of $\Gamma(R)$ is $p-1$, $\kappa(\Gamma(R)) \leq \delta(\Gamma(R)) = p-1$. Note that, every vertex of $A_u \cup A_{u+u^2}$ is adjacent to every vertex of A_{u^2}. Hence there is no vertex cut of cardinality $p-2$ and therefore the result follows. □

Theorem 7. *The edge connectivity $\lambda(\Gamma(R))$ of $\Gamma(R)$ is $p-1$.*

Proof. As $\Gamma(R)$ connected graph, $\kappa(\Gamma(R)) \leq \lambda(\Gamma(R)) \leq \delta(\Gamma(R))$. Since $\kappa(\Gamma(R)) = p-1$ and $\delta(\Gamma(R)) = p-1$, then $\lambda(\Gamma(R)) = p-1$. □

3. Some Topological Indices of $\Gamma(R)$

In this section, we find the Wiener index, first Zagreb index, second Zagreb index and Randić index of the zero divisor graph $\Gamma(R)$.

Theorem 8. *The Wiener index of the zero-divisor graph $\Gamma(R)$ of R is $W(\Gamma(R)) = \frac{p(2p^3-2p^2-7p+5)}{2}$.*

Proof. Consider,

$$\begin{aligned}
W(\Gamma(R)) &= \sum_{\substack{x,y\in Z^*(R)\\ x\neq y}} d(x,y)\\
&= \sum_{\substack{x,y\in A_u\\ x\neq y}} d(x,y) + \sum_{\substack{x,y\in A_{u^2}\\ x\neq y}} d(x,y) + \sum_{\substack{x,y\in A_{u+u^2}\\ x\neq y}} d(x,y)\\
&\quad + \sum_{\substack{x\in A_u\\ y\in A_{u^2}}} d(x,y) + \sum_{\substack{x\in A_u\\ y\in A_{u+u^2}}} d(x,y) + \sum_{\substack{x\in A_{u^2}\\ y\in A_{u+u^2}}} d(x,y)\\
&= (p-1)(p-2) + \frac{(p-1)(p-2)}{2} + p(p-2)(p-1)^2\\
&\quad + (p-1)^2 + 2(p-1)^3 + (p-1)^3
\end{aligned}$$

$$= (p-1)^2 + 3(p-1)^3 + \frac{(p-1)(p-2)}{2} + (p-1)(p-2)(p^2-p+1)$$
$$= \frac{p(2p^3-2p^2-7p+5)}{2}.$$

□

Denote $[A, B]$ be the set of edges between the subset A and B of V. For any $a \in A_u$, $d_a = p-1$, for any $a \in A_{u^2}, d_a = p^2-2$ and any $a \in A_{u+u^2}, d_a = p-1$.

Theorem 9. *The Randić index of the zero-divisor graph $\Gamma(R)$ of R is*

$$R(\Gamma(R)) = \frac{(p-1)}{2(p^2-2)}\left[2p\sqrt{(p-1)(p^2-2)} + (p-2)\right].$$

Proof. Consider,

$$\begin{aligned}
R(\Gamma(R)) &= \sum_{(a,b)\in E} \frac{1}{\sqrt{d_a d_b}} \\
&= \sum_{(a,b)\in[A_u, A_{u^2}]} \frac{1}{\sqrt{d_a d_b}} + \sum_{(a,b)\in[A_{u^2}, A_{u^2}]} \frac{1}{\sqrt{d_a d_b}} + \sum_{(a,b)\in[A_{u^2}, A_{u+u^2}]} \frac{1}{\sqrt{d_a d_b}} \\
&= (p-1)^2 \frac{1}{\sqrt{(p-1)(p^2-2)}} + \frac{(p-1)(p-2)}{2} \frac{1}{\sqrt{(p^2-2)(p^2-2)}} \\
&\quad + (p-1)^3 \frac{1}{\sqrt{(p^2-2)(p-1)}} \\
&= \frac{(p-1)^2}{\sqrt{(p-1)(p-2)}}[p(p-1)] + \frac{(p-1)(p-2)}{2(p^2-2)} \\
&= \frac{p(p-1)^2}{\sqrt{(p-1)(p^2-2)}} + \frac{(p-1)(p-2)}{2(p^2-2)} \\
&= \frac{(p-1)}{2(p^2-2)}\left[2p\sqrt{(p-1)(p^2-2)} + (p-2)\right]
\end{aligned}$$

□

Theorem 10. *The first Zagreb index of the zero-divisor graph $\Gamma(R)$ of R is $M_1(\Gamma(R)) = (p-1)[p^4+p^3-4p^2+p+4]$.*

Proof. Consider,

$$\begin{aligned}
M_1(\Gamma(R)) &= \sum_{a\in Z^*(R)} d_a^2 \\
&= \sum_{a\in A_u} d_a^2 + \sum_{a\in A_{u^2}} d_a^2 + \sum_{a\in A_{u+u^2}} d_a^2 \\
&= (p-1)(p-1)^2 + (p-1)(p^2-2)^2 + (p-1)^2(p-1)^2 \\
&= (p-1)^3 + (p-1)^4 + (p^2-2)^2(p-1) \\
&= p(p-1)^3 + (p-1)(p^2-2) \\
&= (p-1)[p^4+p^3-4p^2+p+4].
\end{aligned}$$

□

Theorem 11. *The second Zagreb index of the zero-divisor graph $\Gamma(R)$ of R is*

$$M_2(\Gamma(R)) = \frac{1}{2}[3p^6 - 9p^5 + 22p^3 - 16p^2 - 8p + 8].$$

Proof. Consider,

$$\begin{aligned}
M_2(\Gamma(R)) &= \sum_{(a,b)\in E} d_a d_b \\
&= \sum_{(a,b)\in[A_u,A_{u^2}]} d_a d_b + \sum_{(a,b)\in[A_{u^2},A_{u^2}]} d_a d_b + \sum_{(a,b)\in[A_{u^2},A_{u+u^2}]} d_a d_b \\
&= (p-1)^2(p-1)(p^2-2) + \frac{(p-1)(p-2)}{2}(p^2-2)(p^2-2) \\
&\qquad + (p-1)^3(p^2-2)(p-1) \\
&= \frac{(p-1)(p^2-2)}{2}[3p^3 - 6p^2 + 4] \\
&= \frac{1}{2}[3p^6 - 9p^5 + 22p^3 - 16p^2 - 8p + 8].
\end{aligned}$$

□

4. Codes from Incidence Matrix of $\Gamma(R)$

In this section, we find the incidence matrix of the graph $\Gamma(R)$ and we find the parameters of the linear code generated by the rows of incidence matrix $Q(\Gamma(R))$. The incidence matrix $Q(\Gamma(R))$ is given below

$$Q(\Gamma(R)) = \begin{array}{c} \\ A_u \\ A_{u^2} \\ A_{u+u^2} \end{array} \begin{array}{c} \begin{array}{ccc} [A_u, A_{u^2}] & [A_{u^2}, A_{u^2}] & [A_{u^2}, A_{u+u^2}] \end{array} \\ \begin{pmatrix} D^{(p-1)}_{(p-1)\times(p-1)^2} & \mathbf{0}_{(p-1)\times\frac{(p-1)(p-2)}{2}} & \mathbf{0}_{(p-1)\times(p-1)^3} \\ J_{(p-1)\times(p-1)^2} & J_{(p-1)\times\frac{(p-1)(p-2)}{2}} & J_{(p-1)\times(p-1)^3} \\ \mathbf{0}_{(p-1)^2\times(p-1)^2} & \mathbf{0}_{(p-1)^2\times\frac{(p-1)(p-2)}{2}} & D^{(p-1)}_{(p-1)^2\times(p-1)^3} \end{pmatrix} \end{array},$$

where J is a all one matrix, $\mathbf{0}$ is a zero matrix with appropriate order, $\mathbf{1}_{(p-1)}$ is a all one $1\times(p-1)$ row vector and $D^{(p-1)}_{k\times l} = \begin{pmatrix} \mathbf{1}_{(p-1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{(p-1)} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{1}_{(p-1)} \end{pmatrix}_{k\times l}$.

Example 2. The incidence matrix of the zero-divisor graph $\Gamma(R)$ given in the Example 1 is

$$Q(\Gamma(R)) = \begin{array}{c} u \\ 2u \\ u^2 \\ 2u^2 \\ u+u^2 \\ 2u+2u^2 \\ 2u+u^2 \\ u+2u^2 \end{array} \left(\begin{array}{cccc|c|cccccccc}
1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&0&0&0\\ \hline
1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1\\ \hline
0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1
\end{array}\right)_{8\times 13}.$$

The number of linearly independent rows is 7 and hence the rank of the matrix $Q(\Gamma(R))$ is 7. The rows of the incidence matrix $Q(\Gamma(R))$ is generate a $[n = 13, k = 7, d = 2]_2$ code over $\mathbb{F}_2$.

The edge connectivity of the zero-divisor graph $\Gamma(R)$ is $p-1$, then we have the following theorem:

Theorem 12. *The linear code generated by the incidence matrix $Q(\Gamma(R))$ of the zero-divisor graph $\Gamma(R)$ is a $C_2(\Gamma(R)) = [\frac{1}{2}(2p^3 - 3p^2 - p + 2), p^2 - 2, p - 1]_2$ linear code over the finite field $\mathbb{F}_2$.*

5. Adjacency and Laplacian Matrices of $\Gamma(R)$

In this section, we find the eigenvalues, energy and spectral radius of both adjacency and Laplacian matrices of $\Gamma(R)$.

If μ is an eigenvalue of matrix A then $\mu^{(k)}$ means that μ is an eigenvalue with multiplicity k.

The vertex set partition into A_u, A_{u^2} and A_{u+u^2} of cardinality $p-1, p-1$ and $(p-1)^2$, respectively. Then the adjacency matrix of $\Gamma(R)$ is

$$A(\Gamma(R)) = \begin{array}{c} \\ A_u \\ A_{u^2} \\ A_{u+u^2} \end{array} \begin{array}{c} \begin{array}{ccc} A_u & A_{u^2} & A_{u+u^2} \end{array} \\ \begin{pmatrix} \mathbf{0}_{p-1} & J_{p-1} & \mathbf{0}_{(p-1)\times(p-1)^2} \\ J_{p-1} & J_{p-1} - I_{p-1} & J_{(p-1)\times(p-1)^2} \\ \mathbf{0}_{(p-1)^2\times(p-1)} & J_{(p-1)^2\times(p-1)} & \mathbf{0}_{(p-1)^2} \end{pmatrix} \end{array},$$

where J_k is an $k \times k$ all one matrix, $J_{n\times m}$ is an $n \times m$ all matrix, $\mathbf{0}_k$ is an $k \times k$ zero matrix, $\mathbf{0}_{n\times m}$ is an $n \times m$ zero matrix and I_k is an $k \times k$ identity matrix.

All the rows in A_{u^2} are linearly independent and all the rows in A_u and A_{u+u^2} are linearly dependent. Therefore, $p - 1 + 1 = p$ rows are linearly independent. So, the rank of $A(\Gamma(R))$ is p. By Rank-Nullity theorem, nullity of $A(\Gamma(R)) = p^2 - p - 1$. Hence, zero is an eigenvalue with multiplicity $p^2 - p - 1$.
For $p = 3$, the adjacency matrix of $\Gamma(R)$ is

$$A(\Gamma(R)) = \left(\begin{array}{cc|cc|cccc} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right)_{8\times 8}.$$

The eigenvalues of $A(\Gamma(R))$ are $0^{(5)}, 4^{(1)}, (-1)^{(1)}$ and $(-3)^{(1)}$. For $p = 5$, the eigenvalues of $A(\Gamma(R))$ are $0^{(19)}, 10^{(1)}, (-1)^{(3)}$ and $(-7)^{(1)}$.

Theorem 13. *The energy of the adjacency matrix $A(\Gamma(R))$ is $\varepsilon(\Gamma(R)) = 6p - 10$.*

Proof. For any odd prime p, the eigenvalues of $A(\Gamma(R))$ are $0^{(p^2-p-1)}$, $(3p-5)^{(1)}$, $(-1)^{(p-2)}$, $(3-2p)^{(1)}$. The energy of adjacency matrix $A(\Gamma(R))$ is the sum of the absolute values of all eigenvalues of $A(\Gamma(R))$. That is,

$$\begin{aligned}\varepsilon(\Gamma(R)) &= \sum_{i=1}^{p^2-1}|\lambda_i| \qquad \text{where } \lambda_i\text{'s are eigenvalues of } A(\Gamma(R))\\ &= |3p-5|+(p-2)|-1|+|3-2p|\\ &= 3p-5+p-2+2p-3 \qquad \text{since } p>2\\ &= 6p-10.\end{aligned}$$

□

The degree matrix of the graph $\Gamma(R)$ is

$$D(\Gamma(R)) = \begin{array}{c} \\ A_u \\ A_{u^2} \\ A_{u+u^2}\end{array}\begin{array}{c} \begin{array}{ccc} A_u & A_{u^2} & A_{u+u^2}\end{array}\\ \begin{pmatrix} (p-1)I_{p-1} & \mathbf{0}_{p-1} & \mathbf{0}_{(p-1)\times(p-1)^2}\\ \mathbf{0}_{p-1} & (p^2-2)I_{p-1} & \mathbf{0}_{(p-1)\times(p-1)^2}\\ \mathbf{0}_{(p-1)^2\times(p-1)} & \mathbf{0}_{(p-1)^2\times(p-1)} & (p-1)I_{(p-1)^2}\end{pmatrix}\end{array}.$$

The Laplacian matrix $L(\Gamma(R))$ of $\Gamma(R)$ is defined by $L(\Gamma(R)) = D(\Gamma(R)) - A(\Gamma(R))$. Therefore,

$$L(\Gamma(R)) = \begin{array}{c} \\ A_u \\ A_{u^2} \\ A_{u+u^2}\end{array}\begin{array}{c} \begin{array}{ccc} A_u & A_{u^2} & A_{u+u^2}\end{array}\\ \begin{pmatrix} (p-1)I_{p-1} & -J_{p-1} & \mathbf{0}_{(p-1)\times(p-1)^2}\\ -J_{p-1} & (p^2-1)I_{p-1}-J_{p-1} & -J_{(p-1)\times(p-1)^2}\\ \mathbf{0}_{(p-1)^2\times(p-1)} & -J_{(p-1)^2\times(p-1)} & (p-1)I_{(p-1)^2}\end{pmatrix}\end{array}.$$

Since each row sum is zero, zero is one of the eigenvalues of $L(\Gamma(R))$. By Lemma 1, the second smallest eigenvalue of $L(\Gamma(R))$ is positive as $\Gamma(R)$ is connected. Hence zero is an eigenvalue with multiplicity one, and all other eigenvalues are positive.
For $p = 3$, the Laplacian matrix is

$$L(\Gamma(R)) = \begin{pmatrix} 2 & 0 & -1 & -1 & 0 & 0 & 0 & 0\\ 0 & 2 & -1 & -1 & 0 & 0 & 0 & 0\\ -1 & -1 & 7 & -1 & -1 & -1 & -1 & -1\\ -1 & -1 & -1 & 7 & -1 & -1 & -1 & -1\\ 0 & 0 & -1 & -1 & 2 & 0 & 0 & 0\\ 0 & 0 & -1 & -1 & 0 & 2 & 0 & 0\\ 0 & 0 & -1 & -1 & 0 & 0 & 2 & 0\\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 2\end{pmatrix}_{8\times 8}.$$

The eigenvalues of $L(\Gamma(R))$ are $0^{(1)}, 8^{(2)}, 2^{(5)}$.
For $p = 5$, the eigenvalues of $L(\Gamma(R))$ are $0^{(1)}, 24^{(4)}, 4^{(19)}$.
For any prime p, the eigenvalues of $L(\Gamma(R))$ are $0^{(1)}, (p^2-1)^{(p-1)}, (p-1)^{(p^2-p-1)}$.

Theorem 14. *The Laplacian energy of* $\Gamma(R)$ *is* $LE(\Gamma(R)) = \dfrac{2p^5-6p^4+6p^3-4p+1}{p^2-1}$.

Proof. Let $|V| = n$ and $|E| = m$. Let $\mu_1, \mu_2, \ldots, \mu_n$ are eigenvalues of $L(\Gamma(R))$. Then the Laplacian energy $LE(\Gamma(R))$ is given by

$$LE(\Gamma(R)) = \sum_{i=1}^{n} \Big|\mu_i - \frac{2m}{n}\Big|.$$

We know that the eigenvalues of $L(\Gamma(R))$ are $0^{(1)}, (p^2-1)^{(p-1)}, (p-1)^{(p^2-p-1)}$. Then

$$\begin{aligned} LE(\Gamma(R)) &= \sum_{i=1}^{n} \Big|\mu_i - \frac{2m}{n}\Big| \\ &= \sum_{i=1}^{n} \Big|\mu_i - \frac{2p^3-3p^2-p+2}{p^2-1}\Big| \\ &= \Big|0 - \frac{2p^3-3p^2-p+2}{p^2-1}\Big| + (p-1)\Big|(p^2-1) - \frac{2p^3-3p^2-p+2}{p^2-1}\Big| \\ &\qquad + (p^2-p-1)\Big|(p-1) - \frac{2p^3-3p^2-p+2}{p^2-1}\Big| \\ &= \frac{2p^5-6p^4+6p^3-4p+1}{p^2-1} \qquad \text{since } p \geq 2. \end{aligned}$$

□

We denote by $\rho(\Gamma(R))$ the largest eigenvalue in absolute of $A(\Gamma(R))$ and call it the spectral radius of $\Gamma(R)$; we denote by $\mu(\Gamma(R))$ the largest eigenvalue in absolute of $L(\Gamma(R))$ and call it the Laplacian spectral radius of $\Gamma(R)$.

Theorem 15. *For any odd prime p, $\rho(\Gamma(R)) = 3p-5$ and $\mu(\Gamma(R)) = p^2-1$.*

Proof. The eigenvalues of the adjacency matrix $A(\Gamma(R))$ are $0^{(p^2-p-1)}$, $(3p-5)^{(1)}$, $(-1)^{(p-2)}$ and $(3-2p)^{(1)}$. Then the largest eigenvalue in absolute is $3p-5$ as $p > 2$. That is, $\rho(\Gamma(R)) = 3p-5$.

The eigenvalues of the Laplacian matrix $L(\Gamma(R))$ are $0^{(1)}$, $(p^2-1)^{(p-1)}$ and $(p-1)^{(p^2-p-1)}$. Then the largest eigenvalue in absolute is p^2-1. That is, $\mu(\Gamma(R)) = p^2-1$. □

Conclusion

In this article, we discussed the zero-divisor graph of a commutative ring with identity $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$ where $u^3 = 0$ and p is an odd prime. We find the clique number, chromatic number, vertex connectivity, edge connectivity, diameter and girth of a zero-divisor graph associated with the ring. We find some of topological indices and the main parameters of the code derived from the incidence matrix of the zero-divisor graph $\Gamma(R)$. Also, we find the eigenvalues, energy and spectral radius of both adjacency and Laplacian matrices of $\Gamma(R)$.

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