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Short Note

A new construction of regular and quasi-regular self-complementary graphs

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Abstract: A graph G with a vertex set V and an edge set E is called regular if the degree of every vertex is the same. A quasi-regular graph is a graph whose vertices have one of two degrees r and $r-1$, for some positive integer r. A graph G is said to be self-complementary if G is isomorphic to it's complement $\bar{G}$. In this paper we give a new method for construction of regular and quasi-regular self-complementary graph.

Keywords: self-complementary graph, regular graph, quasi-regular graph.

AMS Subject classification: 05C07, 05C60

1. Introduction

The study of self-complementary graphs was initiated by Sachs in 1962 [5] and later but independently by Ringel [4]. Each presents a construction algorithm for self-complementary graphs. Sachs and Ringel also gave a construction algorithm for regular and quasi-regular self-complementary graphs. In 1972 R. Gibbs [3] gave a new algorithm for construction of self-complementary graphs. This algorithm provides a method for constructing all self-complementary graphs having a given complementing permutation σ with cycles of lengths that are powers of 2.

In this paper we present a new method for construction of regular self-complementary and quasi-regular self-complementary graphs. In section 2, we give some preliminary definitions and known results. In section 3, we introduce a new method for construction of regular self-complementary graphs and in section 4, we provide a new method for construction of quasi-regular self-complementary graphs.

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2. Preliminary Definitions and Results

In this section we give some preliminary definitions and results. Graphs considered here are simple. A graph $G = (V, E)$ is called *regular* if the degree of every vertex is the same. If G is a graph in which the degree of every vertex is k, then G is said to be a k-regular graph. A graph G is said to be *bi-regular* if there exist two distinct positive integers d_1 and d_2 such that the degree of each vertex is either d_1 or d_2. A graph G is said to be *quasi-regular* if the degree of each vertex is either r or $r-1$ for some positive integer r. A graph $G = (V, E)$ is called *self-complementary* if there exists a permutation $\sigma : V \to V$, called a complementing permutation, such that for every edge e of G, $e \in E$ if and only if $\sigma(e) \notin E$. We state below the most basic results on self-complementary graphs and regular graphs, ones included even in introductory courses on graph theory.

Result 1. [2, 3, 5] If G is a self-complementary graph on n vertices, then $n \equiv 0$ or $1 \pmod 4$.

Result 2. [2, 3, 5] A graph G is k-regular graph on n vertices if and only if kn is even.

Result 3. [1, 2] If $d_1 \geq d_2 \geq \cdots \geq d_n$ is the degree sequence of a self-complementary graph G then $d_i + d_{n+1-i} = n - 1$.

3. Constructing regular self-complementary graph

Theorem 4 gives a well known result for regular self-complementary graphs due to Sachs [5]. His proof involves first constructing a self-complementary graph G' on $4m$ vertices $v_1, v_2, \ldots, v_{4m}$, and then by adding a new vertex v_{4m+1} in the graph G' to get the required regular self-complementary graph G on the vertices $v_1, v_2, \ldots, v_{4m}, v_{4m+1}$. Two distinct vertices v_i, v_j in G' are joined if $i + j \equiv 0$ or $1 \pmod 4$ for $i, j = 1, 2, 3, \ldots, 4m$. In G', $d(v_i) = 2m$ if i is odd and $d(v_i) = 2m-1$ if i is even. Now the vertex v_{4m+1} is joined to all vertices v_i, with even i, $i = 1, 2, 3, \ldots, 4m$. The graph G so obtained is a regular self-complementary graph on the vertex set $V = \{v_1, v_2, \ldots, v_{4m}, v_{4m+1}\}$ with a complementing permutation, $\sigma : V \to V$ defined as $\sigma(v_i) = v_{i+1}, i = 1, 2, \ldots, 4m-1$, $\sigma(v_{4m}) = v_1, \sigma(v_{4m+1}) = v_{4m+1}$. There are several proofs for the theorem, and we introduce a new one.

Theorem 4. *There exists a regular self-complementary graph of order n if and only if $n \equiv 1 \pmod 4$.*

Proof. If G is a regular graph on n vertices, then from Result 3, degree of every vertex must be $r = \frac{n-1}{2}$. For r to be an integer, $n-1$ must be even, and since G is self-complementary, by Result 1, we get $n \equiv 1 \pmod 4$.

To prove the converse, we construct a regular self-complementary graph G of order n, where n is congruent to 1 modulo 4.
Let m be a positive integer and $V = \{u\} \cup V_0 \cup V_1 \cup V_2 \cup V_3$, where $V_i = \{v_j^i : j \in \mathbb{Z}_m\}$ for all $i \in \mathbb{Z}_4$. For pairwise distinct $i, i' \in \mathbb{Z}_4$, we define the following subsets of $V^{(2)}$, where $V^{(2)}$ denotes the set of all 2-subsets of V:

$$E_i = V_i^{(2)}, \quad E_{(i,i')} = \{\{v_{j_1}^i, v_{j_2}^{i'}\} : j_1, j_2 \in \mathbb{Z}_m\}, \quad E_i^u = \{\{u, v_j^i\} : j \in \mathbb{Z}_m\}.$$

Let $E = \bigcup_{i=0,1} (E_i \cup E_i^u) \cup E_{(0,3)} \cup E_{(2,3)} \cup E_{(1,2)}$ and let G be the graph with vertex set V and edge set E as defined above, having $n = 4m + 1$ vertices.

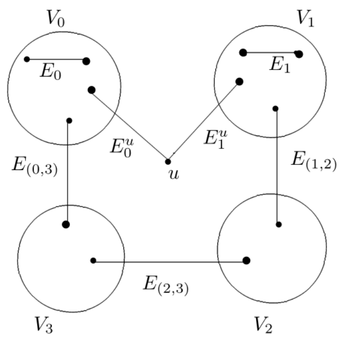

Figure 1. **The types of edges of the graph G**

Figure 1 explains the construction of the graph G in another way.
First we show that G is regular. Take any vertex v_j^i. Then, for fixed i, the vertex v_j^i lies in $m - 1$ subsets of E_i, m subsets of $E_{(i,i')}$ and one subset of E_i^u.
Hence, for every vertex v_j^i in G with $i \in \{0, 1\}$, we have $\mathrm{deg}(\mathrm{v_j^i}) = \mathrm{m} - 1 + \mathrm{m} + 1 = 2\mathrm{m}$, and for every vertex v_j^i in G with $i \in \{2, 3\}$, we have $\mathrm{deg}(\mathrm{v_j^i}) = \mathrm{m} + \mathrm{m} = 2\mathrm{m}$. Furthermore, $\mathrm{deg(u)} = \mathrm{m} + \mathrm{m} = 2\mathrm{m}$. We conclude that G is regular.
Define a bijection $\phi : V \to V$ as $\phi(u) = u, \phi(v_j^0) = v_j^3, \phi(v_j^1) = v_j^2, \phi(v_j^2) = v_j^0$, and $\phi(v_j^3) = v_j^1$, for all $j \in \mathbb{Z}_m$. It can be easily checked that G is self-complementary, with ϕ as its complementing permutation. □

4. Constructing quasi-regular self-complementary graph

The known result for quasi-regular self-complementary graphs due to Sachs [5] is as follows. There are several proofs for the theorem, and we provide a new one.

Theorem 5. *There exists a quasi-regular self-complementary graph of order n if and only $n \equiv 0 \pmod 4$.*

Proof. Let G be a quasi-regular self-complementary graph on n vertices. By Result 3, $s + (s - 1) = 2s - 1 = n - 1$. Then $n = 2s$, and since G is self-complementary, by

Result 1, it follows that $n \equiv 0 \pmod 4$.

To prove the converse, we construct a graph G of order congruent to 0 modulo 4, which is quasi-regular and self-complementary.
Let m be a positive integer and $V = V_0 \cup V_1 \cup V_2 \cup V_3$, where $V_i = \{v_j^i : j \in \mathbb{Z}_m\}$ for all $i \in \mathbb{Z}_4$. For pairwise distinct $i, i' \in \mathbb{Z}_4$, define the following subsets of $V^{(2)}$ where $V^{(2)}$ denotes the set of all 2-subsets of V:

$$E_i = V_i^{(2)}, \quad E_{(i,i')} = \{\{v_{j_1}^i, v_{j_2}^{i'}\} : j_1, j_2 \in \mathbb{Z}_m\}.$$

Let $E = \bigcup_{i=0,1} (E_i) \cup E_{(0,3)} \cup E_{(2,3)} \cup E_{(1,2)}$ and let G be the graph with vertex set V

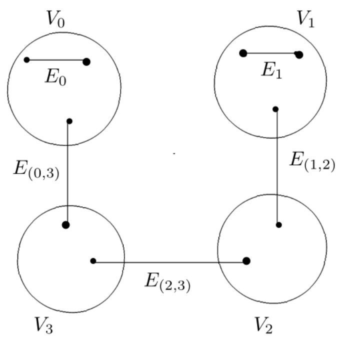

Figure 2. **The types of edges of the graph G**

and edge set E as defined above having $n = 4m$ vertices.
Figure 2 explains the construction of the graph G in another way.
First we show that G is quasi-regular. Take any vertex v_j^i. Then, for fixed i, the vertex v_j^i lies in $m-1$ subsets of E_i and m subsets of $E_{(i,i')}$. Hence, for every vertex v_j^i in G with $i \in \{0,1\}$, we have $\mathrm{deg(v_j^i) = m - 1 + m = 2m - 1}$, and for every vertex v_j^i in G with $i \in \{2,3\}$, we have $\mathrm{deg(v_j^i) = m + m = 2m}$. Therefore, there are $2m$ vertices having degree $2m-1$ and $2m$ vertices of degree $2m$.
We conclude that G is quasi-regular.
Define a bijection $\phi : V \to V$ as $\phi(v_j^0) = v_j^3, \phi(v_j^1) = v_j^2, \phi(v_j^2) = v_j^0$, and $\phi(v_j^3) = v_j^1$, for all $j \in \mathbb{Z}_m$. It can be easily checked that G is self-complementary, with ϕ as its complementing permutation. □

Conflict of Interest: The authors declare no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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