Communications in Combinatorics and Optimization
Vol. 10, No. 3 (2025), pp. 657-663
https://doi.org/10.22049/cco.2024.29118.1847

CCO
Commun. Comb. Optim.

Research Article

Some results on the complete sigraphs with exactly three non-negative eigenvalues

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Received: 30 October 2023; Accepted: 13 January 2024
Published Online: 20 January 2024

Abstract: Let (K_n, H^-) be a complete sigraph of order n whose negative edges induce a subgraph H. In this paper, we characterize (K_n, H^-) with exactly 3 non-negative eigenvalues, where H is a non-spanning two-cyclic subgraph of K_n.

Keywords: sigraph, complete graph, two-cyclic graph, non-negative eigenvalues.

AMS Subject classification: 05C22, 05C50, 15A18

1. Introduction

Let G be a simple graph. As usual, $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. If $V(G) = \{v_1, \ldots, v_n\}$, then $n = |V(G)|$ is called the *order* of G. The set of all neighbors of v_i in G is denoted by $N(v_i)$. A *pendant vertex* is a vertex of degree one. The girth of G, denoted by $\mathrm{gr}(G)$, is the order of the shortest cycle contained in G. A graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H with $|V(H)| \neq |V(G)|$ is said to be a *non-spanning subgraph* (briefly, ns-subgraph) of G. Also, a subgraph H of G is *induced* if $E(H)$ contains all edges of G that have both ends in $V(H)$. Let K_n, P_n and C_n denote the complete graph, the path and the cycle of order n, respectively. A two-cyclic graph is a connected graph with exactly two cycles.

A pair $\Gamma = (G, \sigma)$ is said to be a *signed graph* (called also sigraph), where $\sigma : E(G) \to \{-, +\}$ is a function defined on $E(G)$. The graph G is called the *underlying graph* of

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Γ, and σ is called the *signature*. We use (K_n, H^-) to denote a complete sigraph of order n whose negative edges induce a subgraph H. If H is a disjoint union of two graphs H_1 and H_2, then we denote (K_n, H^-) by $(K_n, H_1^- \cup H_2^-)$. Let $A(G) = (a_{ij})$ be the adjacency matrix of G. The *adjacency matrix* of a sigraph $\Gamma = (G, \sigma)$ is a matrix $A(\Gamma) = (a_{ij}^\sigma)$, where $a_{ij}^\sigma = \sigma(v_i v_j) a_{ij}$. The nullity of a graph G, denoted by $\mathrm{n}(G)$, is the nullity of $A(G)$. By $\varphi(A)$, we denote the characteristic polynomial of a square matrix A. If Γ is a sigraph, then we use $\varphi(\Gamma, \lambda)$ instead of $\varphi(A(\Gamma))$. The spectrum of $A(\Gamma)$ is referred to as the spectrum of Γ. The class of all sigraphs having exactly $r \geq 1$ non-negative eigenvalues (including their multiplicities) is denoted by $\mathcal{L}(r)$. Let $\lambda_1 > \cdots > \lambda_s$ be the distinct eigenvalues of a sigraph Γ with the corresponding multiplicities $m_\Gamma(\lambda_1), \ldots, m_\Gamma(\lambda_s)$. The spectrum of Γ is denoted by

$$\operatorname{Spec}\Gamma = \begin{pmatrix} \lambda_1 & \ldots & \lambda_s \\ m_\Gamma(\lambda_1) & \ldots & m_\Gamma(\lambda_s) \end{pmatrix}.$$

For some recent results on the spectra of sigraphs see [3, 5–7, 14, 15].

Let $\Gamma_1 = (G, \sigma)$ be a sigraph and $S \subset V(\Gamma_1)$. If Γ_2 is the sigraph obtained from Γ_1 by reversing the signs of all edges between S and $V(\Gamma_1) \setminus S$, then two graphs Γ_1 and Γ_2 are called *switching equivalent*, and denoted by $\Gamma_1 \sim \Gamma_2$. If two sigraphs Γ_1 and Γ_2 are switching equivalent, then they are cospectral, see [17].

Characterizing graphs with a few non-negative eigenvalues has received a great deal of attention in literature. In [11–13], the authors characterized all graphs with exactly one or two non-negative eigenvalues. The authors in [9] determined all of the sigraphs (K_n, σ) belonging to $\mathcal{L}(1)$ or $\mathcal{L}(2)$. Also, in [9, 10], they provided a characterization of $(K_n, H^-) \in \mathcal{L}(3)$, where H is either a non-spanning tree or a unicyclic ns-subgraph of K_n. In this paper, we characterize $(K_n, H^-) \in \mathcal{L}(3)$, where H is a two-cyclic ns-subgraph of K_n. After our Theorem 4, the next natural step toward the complete structural characterization of complete sigraph in $\mathcal{L}(3)$ is to detect all (K_n, H^-) in that set with H being a θ-graph. We plan to attack this problem in a future paper.

2. Preliminaries

To prove the main theorem, we need the following results.

Theorem 1. (Interlacing Theorem [8, Theorem 1.3.11]) Let Γ be a sigraph with n vertices and eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, and let Γ' be an induced subgraph of Γ of order m. If $\lambda'_1 \geq \cdots \geq \lambda'_m$ are the eigenvalues of Γ', then

$$\lambda_{n-m+i} \leq \lambda'_i \leq \lambda_i \quad (i = 1, \ldots, m).$$

Theorem 2. [1, Corollary 1] *Let $\Gamma = (K_n, H^-)$ be a complete sigraph and $|V(H)| = t < n$. Then*

$$\varphi(\Gamma, \lambda) = (\lambda + 1)^{n-t-1} \varphi\left(\begin{bmatrix} A(K_t, H^-) & (n-t)J_{t\times 1} \\ J_{1\times t} & n-t-1 \end{bmatrix}\right),$$

and so $m_\Gamma(-1) \geq n - t - 1$.

Theorem 3. [2, Theorem 3] *Let* $\Gamma = (K_n, H^-)$ *be a complete sigraph and* $|V(H)| = t < n$. *Then* $m_\Gamma(-1) = n - t - 1 + \mathrm{n}(H)$.

Remark 1. Let H be a connected graph and consider the following equivalence relation on the vertex set $V(H)$: two vertices $v_i, v_j \in V(H)$ are related if and only if $N(v_i) = N(v_j)$. The corresponding quotient graph $C(H)$ is called the canonical graph of H. Let $n_+(H)$ and $n_-(H)$ denote the numbers of positive and negative eigenvalues of H, respectively. By [16, Proposition 1], we know that $n_+(H) = n_+(C(H))$ and $n_-(H) = n_-(C(H))$. Thus $\mathrm{n}(H) - \mathrm{n}(C(H)) = |V(H)| - |V(C(H))|$. If $\Gamma = (K_n, H^-)$ and $|V(H)| < n$, then by Theorem 3, we conclude that

$$m_\Gamma(-1) = n - 1 + \mathrm{n}(C(H)) - |V(C(H))|.$$

3. Main result

Let H be a two-cyclic ns-subgraph of K_n. In this section, we characterize $(K_n, H^-) \in \mathcal{L}(3)$. First, we have the next lemma.

Lemma 1. *Let* H *be a two-cyclic ns-subgraph of* K_n, *and let* C_g *be a cycle of* H. *If* $(K_n, H^-) \in \mathcal{L}(3)$, *then* $g \in \{3, 4\}$.

Proof. If $g \geq 5$, then (K_n, H^-) contains $(K_7, P_4^- \cup K_2^-)$ as an induced subgraph. By a computer search, one can see that

$$\operatorname{Spec}(K_7, P_4^- \cup K_2^-) = \begin{pmatrix} 4.01 & 2.24 & 1 & 0.09 & -1.58 & -2.24 & -3.52 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Note that the values in the spectrum are approximate. So $(K_7, P_4^- \cup K_2^-) \in \mathcal{L}(4)$ and hence by Theorem 1, we deduce that $(K_n, H^-) \in \mathcal{L}(r)$ for some $r \geq 4$, a contradiction. $\square$

Let $q \geq 1$ be an integer. Let $H(q)$ be the graph with $q + 7$ vertices obtained by two quadrangles sharing a vertex u_1, by attaching q pendant vertices to u_1. Note that $C(H(q)) \cong T$, see Figure 1.

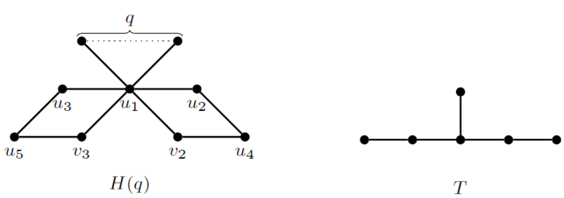

Figure 1. **The two-cyclic graph $H(q)$ and its canonical graph T.**

Now, we prove the main result of the paper.

Theorem 4. *Let (K_n, σ) be a complete sigraph and $(K_n, \sigma) \sim (K_n, H^-)$, where H is a two-cyclic ns-subgraph of K_n. Then $(K_n, H^-) \in \mathcal{L}(3)$ if and only if one of the next assertions holds:*

1. *$H \cong Q_1$ for $n = 7$ or $H \cong Q_2$ for $n > 7$ or $H \cong Q_3$ for $n > 8$, see Fig. 2.*
2. *$H \cong H(1)$ for $9 \leq n \leq 12$ or $H \cong H(2)$ for $n = 10$.*

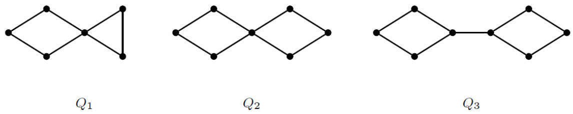

Figure 2. **The two-cyclic graphs Q_1, Q_2 and Q_3.**

Proof. First we consider the following cases:

1. Let $H \cong Q_1$, depicted in Figure 2, and $n = 7$. By a computer search, we find the spectrum of (K_7, Q_1^-) as follows:

$$\mathrm{Spec}\,(K_7, Q_1^-) = \begin{pmatrix} 3.86 & 2.33 & 1 & -0.02 & -1 & -2.54 & -3.63 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Hence $(K_7, Q_1^-) \in \mathcal{L}(3)$.

Now, let $H \cong Q_2$, shown in Figure 2, and $\Gamma = (K_n, Q_2^-)$, where $n > 7$. Since $\mathrm{n}(Q_2) = 3$, by Theorem 3, we have $m_\Gamma(-1) = n - 5$. The following are the eigenvalues of (K_8, Q_2^-):

$$\mathrm{Spec}\,(K_8, Q_2^-) = \begin{pmatrix} 4.46 & 3 & 1.83 & -1 & -2.46 & -3.83 \\ 1 & 1 & 1 & 3 & 1 & 1 \end{pmatrix}.$$

So (K_8, Q_2^-) has three positive eigenvalues and two negative eigenvalues smaller than -1. The sigraph $\Gamma = (K_n, Q_2^-)$ contains (K_8, Q_2^-) as an induced subgraph, for each $n \geq 8$. By Theorem 1, we conclude that $\Gamma = (K_n, Q_2^-) \in \mathcal{L}(3)$, for each $n > 7$.

Next, suppose that $H \cong Q_3$ (shown in Figure 2) and $\Gamma = (K_n, Q_3^-)$, where $n > 8$. By Theorem 2, we find that

$$\varphi(\Gamma, \lambda) = (\lambda + 1)^{n-9} \varphi\left(\begin{bmatrix} A(K_8, Q_3^-) & (n-8)J_{8\times 1} \\ J_{1\times 8} & n-9 \end{bmatrix}\right) = (\lambda + 1)^{n-7} g(\lambda),$$

where $g(\lambda) = \lambda^7 + (7-n)\lambda^6 + (21-6n)\lambda^5 + (21n-133)\lambda^4 + (124n-829)\lambda^3 + (805-119n)\lambda^2 + (3751-502n)\lambda + 217 - 29n$. It is easy to check that $g(-1) \neq 0$ and also $g(0) = 217 - 29n < 0$, for each $n > 8$. On the other hand, we have $\mathrm{Spec}\,(K_9, Q_3^-)$ as follows:

$$\begin{pmatrix} 4.46 & 3.69 & 2.56 & -0.06 & -1 & -1.56 & -2.46 & -4.63 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

Thus (K_9, Q_3^-) contains three positive eigenvalues and three negative eigenvalues smaller than -1. Since (K_9, Q_3^-) is an induced subgraph of $\Gamma = (K_n, Q_3^-)$, by Theorem 1, we deduce that $\Gamma \in \mathcal{L}(3)$ or $\Gamma \in \mathcal{L}(4)$. If $\lambda_1, \ldots, \lambda_7$ are the roots of $g(\lambda)$, then $g(0) = -\prod_{i=1}^{7} \lambda_i$. Now, $g(0) < 0$ yields that $\Gamma = (K_n, Q_3^-) \in \mathcal{L}(3)$, for each $n > 8$.

2. Let $H \cong H(q)$ and $\Gamma = (K_n, H(q)^-)$, where $m = n - (q+7) > 0$. We have $C(H(q)) \cong T$ (cf. Figure 1) and $\mathrm{n}(T) = 0$. By Remark 1, we find that

$$m_\Gamma(-1) = n - 1 + \mathrm{n}(T) - |V(T)| = n - 7.$$

The spectrum of $(K_9, H(1)^-)$ is as follows:

$$\begin{pmatrix} 5.62 & 3.14 & 1.83 & -0.22 & -1 & -1.95 & -2.58 & -3.83 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

Hence $(K_9, H(1)^-)$ has 3 positive eigenvalues and 3 negative eigenvalues smaller than -1. Since Γ has $(K_9, H(1)^-)$ as an induced subgraph, by Theorem 1, $\Gamma \in \mathcal{L}(3)$ or $\Gamma \in \mathcal{L}(4)$. Now, we compute $\varphi(A(\Gamma))$. Suppose that $V(H(q))$ is partitioned into the parts $X_1 = \{u_1\}, X_2 = \{u_2, v_2\}, X_3 = \{u_3, v_3\}, X_4 = \{u_4\}$, and $X_5 = \{u_5\}$, see Fig. 1. Let X_6 be the set of pendant vertices of $H(q)$ and $X_7 = V(K_n) \setminus V(H(q))$. Note that $|X_6| = q$ and $|X_7| = m = n - q - 7$. If B is the quotient matrix of $A(\Gamma)$ related to the equitable partition $\sqcap = \{X_1, \ldots, X_7\}$ of $V(\Gamma)$, then

$$B = \begin{bmatrix} 0 & -2 & -2 & 1 & 1 & -q & m \\ -1 & 1 & 2 & -1 & 1 & q & m \\ -1 & 2 & 1 & 1 & -1 & q & m \\ 1 & -2 & 2 & 0 & 1 & q & m \\ 1 & 2 & -2 & 1 & 0 & q & m \\ -1 & 2 & 2 & 1 & 1 & q-1 & m \\ 1 & 2 & 2 & 1 & 1 & q & m-1 \end{bmatrix}.$$

If $h(\lambda) = \varphi(B)$, then $h(\lambda) = \lambda^7 + (7-n)\lambda^6 + (21-6n)\lambda^5 + (17m + 9q + 4mq - 6)\lambda^4 + (108m + 76q + 16mq + 87)\lambda^3 + (q - 15m - 40mq + 44)\lambda^2 + (-262m - 166q - 112mq - 99)\lambda + 196mq - 105q - 161m - 70$. By [4, Lemma 2.3.1], $h(\lambda)$ divides $\varphi(A(\Gamma))$. A direct check shows that if $h(-1) = 0$, then $mq = 0$, a contradiction. Hence, $\varphi(\Gamma, \lambda) = (\lambda+1)^{n-7} h(\lambda)$. Since $h(0) = (196q - 161)m - 105q - 70$, so if $m < \frac{105q+70}{196q-161}$, then $\Gamma = (K_n, H(q)^-) \in \mathcal{L}(3)$. Otherwise, $\Gamma = (K_n, H(q)^-) \in \mathcal{L}(4)$. The function $f(q) := \frac{105q+70}{196q-161}$ is strictly decreasing for $q \geq 1$. Moreover, $f(1) = 5$ and $f(3) < 1 < f(2) < 2$. This means that only $H(1)$ and $H(2)$ can possibly satisfy the conditions $m < f(q)$ and $m = n - (q+7) > 0$. Hence, $(K_n, H(1)^-) \in \mathcal{L}(3)$ for $9 \leq n \leq 12$, and $(K_n, H(2)^-) \in \mathcal{L}(3)$ for $n = 10$.

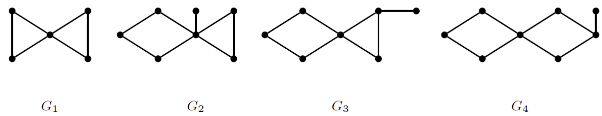

Figure 3. **The two-cyclic graphs G_1, G_2, G_3 and G_4.**

Conversely, assume that $\Gamma = (K_n, H^-) \in \mathcal{L}(3)$, where H is a two-cyclic ns-subgraph of K_n. Since two sigraphs $(K_5, C_3^- \cup K_2^-)$ and $(K_7, P_4^- \cup K_2^-)$ belong to the class $\mathcal{L}(4)$, so they cannot appear as induced subgraphs of Γ. By Lemma 1, H has no cycle of length greater than 4. First, suppose that $\mathrm{gr}(H) = 3$. It is not difficult to verify that $H \cong Q_1$ or one of the graphs G_1, G_2, G_3 (shown in Figure 3) is an induced subgraph of H, for otherwise the sigraphs $(K_5, C_3^- \cup K_2^-)$ or $(K_7, P_4^- \cup K_2^-)$ will appear as induced subgraphs of $\Gamma = (K_n, H^-)$. A direct check shows that the graphs (K_8, Q_1^-), (K_6, G_1^-), (K_8, G_2^-), and (K_8, G_3^-) belong to the class $\mathcal{L}(4)$. Thus $H \cong Q_1$ and $n = 7$. Next, assume that $\mathrm{gr}(H) = 4$. Again, to avoid $(K_7, P_4^- \cup K_2^-)$ as an induced subgraph, one can deduce that $H \cong Q_2$ or $H \cong Q_3$ or $H \cong H(q)$ (for some positive integer q) or the two-cyclic graph G_4 (shown in Figure 3) is an induced subgraph of H. It is easy to check that $(K_9, G_4^-) \in \mathcal{L}(4)$. As we saw above, if $H \cong H(q)$, then $q = 1$ and $9 \leq n \leq 12$ or $q = 2$ and $n = 10$. Also, the sigraphs $\Gamma = (K_n, Q_2^-)$, for each $n > 7$, and $\Gamma = (K_n, Q_3^-)$, for each $n > 8$, belong to the class $\mathcal{L}(3)$. $\square$

Acknowledgements. The authors would like to express their deep gratitude to the referee for his/her careful reading and constructive comments.

Conflict of interest. The authors declare that there is no competing interest related to this paper.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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