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Research Article

Leavitt path algebras for order prime Cayley graphs of finite groups

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Abstract: In this paper, we generalize the concept of Cayley graphs associated to finite groups. The aim of this paper is the characterization of graph theoretic properties of new type of directed graph $\Gamma_P(G; S)$ and algebraic properties of Leavitt path algebra of order prime Cayley graph $O\Gamma(G; S)$, where G is a finite group with a generating set S. We show that the Leavitt path algebra of order prime Cayley graph $L_K(O\Gamma(G; S))$ of a non trivial finite group G with any generating set S over a field K is a purely infinite simple ring. Finally, we prove that the Grothendieck group of the Leavitt path algebra $L_K(\Gamma_P(D_n; S))$ is isomorphic to $\mathbb{Z}_{2n-1}$, where D_n is the dihedral group of degree n and $S = \{a, b\}$ is the generating set of D_n.

Keywords: group, directed Cayley graph, order prime Cayley graph, Grothendieck group.

AMS Subject classification: 05C25, 16E20, 16S99

1. Introduction

The notion of the Leavitt path algebra $L_K(E)$ of a directed graph E over a field K, natural generalization of the Leavitt algebra investigated by W. G. Leavitt in [12], was introduced and studied in [2] and [4] for row-finite graph. The study of algebraic structures of Leavitt path algebras, using the properties of directed graphs, has become an exciting research topic in the last few years, leading to many fascinating results and questions. There are many papers devoted on characterizing the algebraic

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structures of Leavitt path algebra of a directed Cayley graph associated to a group, for instance see, [3, 13, 14]. Recently, in [3], the authors studied Leavitt path algebra of Cayley graph of the finite cyclic group $\mathbb{Z}/n\mathbb{Z}$ with respect to the subset $\{\overline{1}, \overline{n-1}\}$. Then in [14], Nam and Phuc studied certain digraphs arising from groups such as Cayley graphs, power graphs and later in [15], they investigated Leavitt path algebras of Hopf graphs. In particular, they gave criteria for the Leavitt path algebra of a Cayley graph arising from a finite group to be purely infinite simple and also provided criteria for the Leavitt path algebra of a Hopf graph arising from an arbitrary group to be purely infinite simple. Moreover, they computed completely the stable rank of the Leavitt path algebra of a Hopf graph arising from an arbitrary group. In [7], we characterized algebraic properties of Leavitt path algebra of the directed power graph $\vec{\mathscr{P}}(G)$ as well as the directed punctured power graph $\vec{\mathscr{P}}^*(G)$ of a finite group G. Later in [8], we computed the Grothendieck group of purely infinite simple Leavitt path algebra of the directed punctured power graph of a finite group.

The idea of a graphical representation of a group, called as a Cayley digraph of a group, was introduced by Cayley in 1878.

Definition 1.   (Cayley digraph of a Group)[9] Let G be any finite group and let S be a generating set for G. We define a directed graph $\Gamma_1(G; S)$ associated to G, called the directed Cayley graph of G with generating set S, as follows:
(1) Each element of G is a vertex of $\Gamma_1(G; S)$.
(2) For $x, y \in G$, there is an arc from x to y if and only if $y = xs$, for some $s \in S$.

Throughout this paper, we denote the order of an element x of the group G as $o(x)$ and also denote the Cayley digraph as $\Gamma_1(G; S)$ because for $x, y \in G$, there is an arc from x to y if and only if $o(y^{-1}xs) = 1$, for some element $s \in S$.

Motivating by the definition of directed Cayley graph, we introduce a new type of directed graph and study its properties in this paper.

**Definition 2.   **For a finite group G with a generating set S, we define a directed graph $\Gamma_P(G; S)$ associated to G as follows :
(1) Each element of G is a vertex of $\Gamma_P(G; S)$.
(2) For $x, y \in G$, there is an arc from x to y if and only if there is some generator $s \in S$ such that $o(y^{-1}xs) = p$, for some prime p.
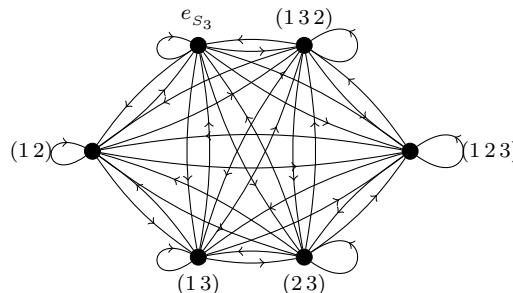
Figure 1. $\Gamma_P(S_3; \{(12), (123)\})$

It is worth mentioning the following note.

Remark 1. Between any two vertices x, y in $\Gamma_P(G; S)$, if $o(y^{-1}xs_i)$ are prime numbers for some $s_i \in S$, then we consider only one arc from x to y. Therefore, $\Gamma_P(G; S)$ is a directed graph containing no parallel arcs. To illustrate this, we first consider the graph $\Gamma_P(S_3; \{(12), (123)\})$ illustrated in Figure 1.

Remark 2. From Definition 2, it follows that for two vertices x and y in $\Gamma_P(G; S)$, there is an arc from x to y if and only if $y = xst$ for some $s \in S$ and $t \in G$ such that $o(t)$ is prime.

Now, for any finite group G along with a generating set S, we define another new directed graph $O\Gamma(G; S)$ associated to G, called *order prime Cayley graph*, as union of $\Gamma_1(G; S)$ and $\Gamma_P(G; S)$. Therefore, in $O\Gamma(G; S)$, for any two vertices x, y; there is an arc from x to y if and only if there is some generator $s \in S$ such that $o(y^{-1}xs) = 1$ or p, for some prime p. Thus, this new type of digraph can be considered as a generalization of Cayley digraph associated to finite group and adjacency of vertices are connected to prime order elements of associated group, that's why we have named this graph as order prime Cayley graph. One can easily verify that $\Gamma_P(\mathbb{Z}_2; \{\overline{1}\})$ is disconnected graph with loop at each vertex, whereas $\Gamma_1(\mathbb{Z}_2; \{\overline{1}\})$ is strongly connected having no loops. Thus, $\Gamma_1(G; S)$ and $\Gamma_P(G; S)$ may exhibit different graph theoretic properties and this observation motivates us to investigate various graph theoretic properties of $\Gamma_P(G; S)$.

This paper is organized as follows. First we recall the relevant background definitions and basic facts in the preliminary section. In Section 3, first we characterize strongly connectedness property for some well known finite groups together with their generating sets. Here we prove that $\Gamma_P(\mathbb{Z}_n; \{\overline{1}\})$ is strongly connected for all $n \geq 3$ (Theorem 3). Then we give sufficient conditions on the generating set S (Theorem 5, 6, 7) for the digraph $\Gamma_P(S_n; S)$ is strongly connected, where S_n is the symmetric group for every integer $n \geq 3$. For each positive integer $n \geq 4$, we show that $\Gamma_P(D_n; S)$ is a complete $2n$-graph with one loop at each vertex, where $S = \{a, b\}$ is the generating set of dihedral group D_n (Theorem 9). Again we prove that $\Gamma_P(Q_{4n}; S)$ is strongly connected, where $S = \{a, b\}$ is the generating set of dicyclic group Q_{4n} for all $n \geq 2$ (Theorem 10) and also, for each positive integer $k(\geq 2)$, $\Gamma_P(Q_{2^{k+1}}; S)$ is strongly connected, where $S = \{a, b\}$ is the generating set of generalized quaternian group $Q_{2^{k+1}}$ (Theorem 11). Another interesting result is that $\Gamma_P(SD_{2^m}; S)$ is strongly connected, where $S = \{a, b\}$ is the generating set of semi - dihedral group SD_{2^m} for all $m \geq 4$ (Theorem 12). In the remainder of the Section 3, we study various graph theoretic properties of $\Gamma_P(G; S)$ as well as $O\Gamma(G; S)$. Here we give the criteria for the digraphs $\Gamma_P(G; S)$ and $O\Gamma(G; S)$ satisfying Condition *(L)* (Theorem 18, Theorem 19). In Section 4, we study various algebraic properties of the Leavitt path algebras associated to order prime Cayley graphs of finite groups. Another interesting aspect of this section is to investigate the algebraic structures of the algebra $L_K(\Gamma_P(G; S))$. Here we characterize the stable rank, prime ring as well as the purely infinite simplicity of the Leavitt path algebras associated to order prime Cayley graphs of finite groups

(Theorem 21). Using the result (Theorem 9), we compute the Grothendieck group of the Leavitt path algebra $L_K(\Gamma_P(D_n; S))$ for every positive integer $n \geq 4$ (Theorem 24).

Throughout this paper, we assume that every directed graph contains no parallel arcs, the arc from a vertex a to another vertex b is denoted by $a \to b$, e_G denotes the identity element of the group G and K always denotes a field.

2. Preliminaries

In this section, for convenience of the reader and also for later use, we outline some definitions, notations and results concerning graph theory as well as Leavitt path algebras. For general notations, terminologies and results concerning Leavitt path algebras, the reader is referred to the book [1].

2.1. Directed Graph

In this subsection, we first recall some basic definitions and terminologies regarding graphs.

A *directed graph* is a quadruple $E = (E^0, E^1, r, s)$ consisting of two sets E^0 and E^1 together with two functions $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 are called *arcs*. For any arc e in E^1, $s(e)$ and $r(e)$ respectively denote the *source* and the *range* of e. Clearly, for any vertex $v \in E^0$, $s^{-1}(v)$ denotes the set of all arcs whose source is v, while $r^{-1}(v)$ denotes the set of all arcs whose range is v. A vertex $v \in E^0$ is called *sink* if $|s^{-1}(v)| = 0$ and also a vertex $v \in E^0$ is called *regular* if $0 < |s^{-1}(v)| < \infty$. A graph E is *row-finite* if $|s^{-1}(v)| < \infty$ for all vertices v of E.

A *path* $p = e_1 e_2 \cdots e_n$ in a directed graph is a sequence of arcs $e_1, e_2, \ldots, e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, 2, \ldots, n-1$. A path p is called a *closed path* based at v if $s(p) = r(p) = v$. Again, a closed path based at v is called a *closed simple path* at v if $s(e_i) \neq v$ for every $i > 1$. A *cycle* is a closed simple path which does not visit any of its vertices more than once. If c is a cycle with $s(c) = r(c) = v$, then c is said to be based at v. A directed graph containing no cycle is called *acyclic*. An arc e is an *exit* to a cycle $p = e_1 e_2 \cdots e_n$ if there exists some $i \in \{1, 2, \ldots, n\}$ such that $s(e_i) = s(e)$ but $e \neq e_i$. A graph E is said to satisfy *Condition (L)* if every cycle in E has an exit and a graph E is said to satisfy *Condition (K)* if for each $v \in E^0$ which lies on a closed simple path, there exist at least two distinct closed simple paths based at v.

A directed graph E is called strongly connected if, given any two vertices v, w of E, there exists a path p with $s(p) = v$ and $r(p) = w$. A directed graph E is called nontrivial if E does not consist solely of a single cycle. Again, a directed graph E is called essential if E contains no sources and no sinks. Unless otherwise stated, by a "graph" we always mean a directed graph.

2.2.   Leavitt path algebras

In this subsection, we recall some results related to Leavitt path algebra of a directed graph E over a field K which will be needed for our discussion.

Let K be a field and $E = (E^0, E^1, r, s)$ be a directed graph. The *Leavitt path algebra* of E with coefficients from K, denoted by $L_K(E)$, is the free associative K-algebra generated by the collection $\{v, e, e^* : v \in E^0, e \in E^1\}$ satisfying the following relations:

(A1) $uv = \delta_{u,v}u$, for all $u, v \in E^0$;

(A2) $s(e)e = er(e) = e$, $r(e)e^* = e^*s(e) = e^*$, for all $e \in E^1$;

(CK1) $e^*f = \delta_{e,f}r(e)$, for all $e, f \in E^1$;

(CK2) $v = \displaystyle\sum_{e \in s^{-1}(v)} ee^*$ for every regular vertex v.

The elements of E^1 are called *real arcs* and the elements of $(E^1)^* = \{e^* : e \in E^1\}$ are called *ghost arcs*, where $e \mapsto e^*$ is a bijective function between E^1 and $(E^1)^*$ with $s(e) = r(e^*)$ and $r(e) = s(e^*)$. We define $v^* = v$ for every vertex $v \in E^0$, and also define $(e^*)^* = e$ for all $e^* \in (E^1)^*$. Again, for any path $p = e_1 \cdots e_n$ $(e_1, \ldots, e_n \in E^1)$, we denote $p^* = e_n^* \cdots e_1^*$ as a *ghost path* with $s(p^*) = r(p)$ and $r(p^*) = s(p)$. Every element x of $L_K(E)$ can be expressed (though this representation is not unique) as $x = \displaystyle\sum_{i=1}^{n} k_i p_i q_i^*$, where $k_i \in K$, and paths p_i, q_i with $r(p_i) = r(q_i)$ for each $1 \le i \le n$.

3.   Graph theoretic properties of $\Gamma_P(G; S)$

In this section, we construct and discuss different graphs $\Gamma_P(G; S)$ associated to various kind of finite groups. Before going to discuss properties of $\Gamma_P(\mathbb{Z}_n; S)$, where $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$ is the additive group of integers modulo n and S is any generating set of $\mathbb{Z}_n$, we consider three examples, viz., $\Gamma_P(\mathbb{Z}_4; \{\overline{1}\})$, $\Gamma_1(\mathbb{Z}_4; \{\overline{1}\})$ and $\Gamma_P(\mathbb{Z}_5; \{\overline{1}\})$ which are given below.

$\Gamma_P(\mathbb{Z}_4; \{\overline{1}\})$ $\Gamma_1(\mathbb{Z}_4; \{\overline{1}\})$ $\Gamma_P(\mathbb{Z}_5; \{\overline{1}\})$
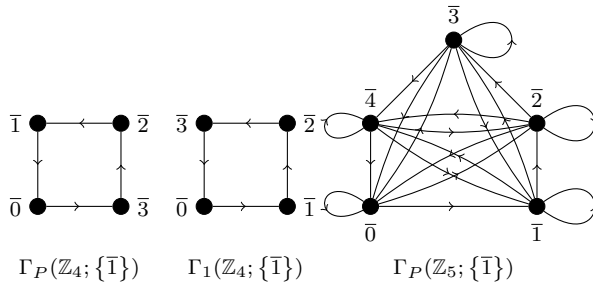
Figure 2.

From Figure 2, we observe the following interesting features.

Observation 1.   (1) Both $\Gamma_P(\mathbb{Z}_4; \{\overline{1}\})$ and $\Gamma_P(\mathbb{Z}_5; \{\overline{1}\})$ are strongly connected graphs.

(2) $\Gamma_P(\mathbb{Z}_4; \{\overline{1}\})$ has only one cycle, whereas $\Gamma_P(\mathbb{Z}_5; \{\overline{1}\})$ contains more than one cycle each of which consists of all the vertices.

(3) Here $\Gamma_P(\mathbb{Z}_4; \{\overline{1}\}) \neq \Gamma_1(\mathbb{Z}_4; \{\overline{1}\})$, but it is clear that $\Gamma_P(\mathbb{Z}_4; \{\overline{1}\}) = \Gamma_1(\mathbb{Z}_4; \{\overline{3}\})$.

The above observations motivate us to establish the following results.

Theorem 2. *For any positive integer $n \geq 2$, $\Gamma_P(\mathbb{Z}_{2^n}; \{\overline{1}\})$ is isomorphic to some Cayley digraph associated to $\mathbb{Z}_{2^n}$.*

Proof. First we show that $\Gamma_P(\mathbb{Z}_{2^n}; \{\overline{1}\})$ is a cycle of length 2^n, for any positive integer $n \geq 2$. Here $1 + 2^{n-1}$ is an odd number for $n \geq 2$ and hence $\left\{\overline{1 + 2^{n-1}}\right\}$ is another generating set of $\mathbb{Z}_{2^n}$. Thus $C_{\overline{0}} := (\overline{0} \to \overline{1 + 2^{n-1}} \to 2\overline{(1 + 2^{n-1})} \to \cdots \to k\overline{(1 + 2^{n-1})} \to \cdots \to 2^n\overline{(1 + 2^{n-1})} = \overline{0})$ is cycle based at $\overline{0}$ consisting of all 2^n elements. Again, $\mathbb{Z}_{2^n}$ has only one prime order element which is 2^{n-1}, so $|s^{-1}(v)| = 1$ for every $v \in \Gamma_P(\mathbb{Z}_{2^n}; \{\overline{1}\})$ and hence $\Gamma_P(\mathbb{Z}_{2^n}; \{\overline{1}\})$ consists solely of a single cycle of length 2^n. Consequently, $\Gamma_P(\mathbb{Z}_{2^n}; \{\overline{1}\}) = \Gamma_1(\mathbb{Z}_{2^n}; \left\{\overline{1 + 2^{n-1}}\right\})$. Hence the theorem. $\square$

Lemma 1. *For any positive integer n and any odd prime q, $\Gamma_P(\mathbb{Z}_{q^n}; \{\overline{1}\})$ is strongly connected.*

Proof. Clearly, either n is 1 or n is greater than 1. We consider the following cases.

Case 1. We first suppose that $n = 1$. Then we can find an element $\overline{t} \in \mathbb{Z}_q$ such that $\overline{t} \neq \overline{q-1}$. Thus $o(\overline{1+t}) = q$ and hence $C_{\overline{0}} := (\overline{0} \to \overline{1+t} \to 2\overline{(1+t)} \to \cdots \to q\overline{(1+t)} = \overline{0})$ is cycle based at $\overline{0}$ consisting of all q elements in $\mathbb{Z}_q$. Consequently, $\Gamma_P(\mathbb{Z}_q; \{\overline{1}\})$ is strongly connected.

Case 2. We now suppose that $n \geq 2$. Clearly, $\left\{\overline{1 + q^{n-1}}\right\}$ is another generating set of $\mathbb{Z}_{q^n}$. Thus, $C'_{\overline{0}} := (\overline{0} \to \overline{1 + q^{n-1}} \to 2\overline{(1 + q^{n-1})} \to \cdots \to k\overline{(1 + q^{n-1})} \to \cdots \to q^n\overline{(1 + q^{n-1})} = \overline{0})$ is cycle based at $\overline{0}$ consisting of all q^n elements in $\mathbb{Z}_{q^n}$. This implies $\Gamma_P(\mathbb{Z}_{q^n}; \{\overline{1}\})$ is strongly connected, where $n \geq 2$ is any positive integer. $\square$

Corollary 1. *For any positive integer $n \geq 2$, $\Gamma_P(\mathbb{Z}_{2^n}; S)$ is strongly connected, and also for any positive integer n and any odd prime q, $\Gamma_P(\mathbb{Z}_{q^n}; S)$ is strongly connected, where S is any generating set.*

Remark 3. Let q be any odd prime number and $n \geq 2$ be an integer. Then $C'_{\overline{0}} := (\overline{0} \to \overline{s + q^{n-1}} \to 2\overline{(s + q^{n-1})} \to \cdots \to k\overline{(s + q^{n-1})} \to \cdots \to q^n\overline{(s + q^{n-1})} = \overline{0})$ and $C''_{\overline{0}} := (\overline{0} \to \overline{s + 2q^{n-1}} \to 2\overline{(s + 2q^{n-1})} \to \cdots \to k\overline{(s + 2q^{n-1})} \to \cdots \to q^n\overline{(s + 2q^{n-1})} = \overline{0})$ are at least two different cycles consisting of all vertices of $\Gamma_P(\mathbb{Z}_{q^n}; S)$.

Lemma 2. *Let $n \geq 2$ be an integer with factorization $n = q_1^{k_1} q_2^{k_2} \cdots q_r^{k_r}$ in $\mathbb{Z}$, where $r \geq 2$ and $q_1, q_2, \ldots, q_r$ are distinct primes. Then $\Gamma_P(\mathbb{Z}_n; \{\overline{1}\})$ is strongly connected.*

Proof. We consider the following two cases.

Case 1. In this case, we let $n = q_1{}^{k_1} q_2{}^{k_2} \ldots q_r{}^{k_r}$ with at least one $k_s > 1$. Let $k_i > 1$ for some $i = 1, 2, \ldots, r$. Then for any $j = 1, 2, \ldots, r$; q_j does not divide $1 + q_1{}^{k_1} q_2{}^{k_2} \ldots q_i{}^{k_i-1} \ldots q_r{}^{k_r}$ and hence $o(\overline{1 + q_1{}^{k_1} q_2{}^{k_2} \ldots q_i{}^{k_i-1} \ldots q_r{}^{k_r}}) = n$, i.e., $\overline{1 + q_1{}^{k_1} q_2{}^{k_2} \ldots q_i{}^{k_i-1} \ldots q_r{}^{k_r}}$ is a generator of $\mathbb{Z}_n$ other than $\overline{1}$. Then by the similar argument as in the proof of Lemma 1, we can prove that $\Gamma_P(\mathbb{Z}_n; \{\overline{1}\})$ is strongly connected.

Case 2. Here we let $n = q_1{}^{k_1} q_2{}^{k_2} \ldots q_r{}^{k_r}$ such that $k_i = 1$, for all $i = 1, 2, \ldots, r$; i.e., $n = q_1 q_2 \ldots q_r$. Without any loss of generality, we assume that $q_1 < q_2 < \cdots < q_r$. Take $t = q_1 q_2 \ldots q_{r-1}$. If q_r does not divide $1 + t$, then for any $j = 1, 2, \ldots, r$; q_j does not divide $1 + t$ and hence $\overline{1 + t}$ is a generator of $\mathbb{Z}_n$ other than $\overline{1}$. Then by the similar argument as in the proof of Lemma 1, we can find a cycle consisting of all the vertices of $\Gamma_P(\mathbb{Z}_n; \{\overline{1}\})$. Otherwise, if q_r divides $1 + t$, then 1, t can be expressed as $1 = aq_r + x$ and $t = bq_r - x$, where $a, b \in \mathbb{Z}$ and q_r does not divide x. Then $1 + 2t = aq_r + x + 2bq_r - 2x = (a + 2b)q_r - x$. This implies q_r can not divide $1 + 2t$ and hence $\overline{1 + 2t}$ is a generator of $\mathbb{Z}_n$ other than $\overline{1}$. Then by the similar argument as in the proof of Lemma 1, we can find a cycle consisting of all the vertices of $\Gamma_P(\mathbb{Z}_n; \{\overline{1}\})$. Consequently, $\Gamma_P(\mathbb{Z}_n; \{\overline{1}\})$ is strongly connected. $\square$

Now, we are in a position to characterize the group $\mathbb{Z}_n$ for which $\Gamma_P(\mathbb{Z}_n; \{\overline{1}\})$ is strongly connected. Combining Theorem 2, Lemma 1 and Lemma 2, we have the following result.

Theorem 3. *Let $n \geq 3$ be an integer such that $n = q_1{}^{k_1} q_2{}^{k_2} \ldots q_r{}^{k_r}$, where $q_1, q_2, \ldots, q_r$ are distinct primes. Then $\Gamma_P(\mathbb{Z}_n; \{\overline{1}\})$ is strongly connected.*

Now, we focus on $\Gamma_P(S_n; S)$, where S_n is the permutation group on n symbols and S is a generating set of S_n. We know that $\{(1\,2), (1\,2\,\ldots\,n)\}$ is a generator of S_n. Now, we give an interesting result for $\Gamma_P(S_n; \{(1\,2), (1\,2\,\ldots\,n)\})$.

Theorem 4. *Let $S = \{(1\,2), (1\,2\,\ldots\,n)\}$ for all $n \geq 3$. Then the following statements are equivalent:*
(i) There exist two arcs; one from e_{S_n} to $(1\,2)$ and another from $(1\,2)$ to e_{S_n} in $\Gamma_P(S_n; S)$,
(ii) There exist two arcs; one from e_{S_n} to $(1\,2\,\cdots\,n)$ and another from $(1\,2\,\cdots\,n)$ to e_{S_n} in $\Gamma_P(S_n; S)$,
(iii) n is of the form $r + 1$, for some prime r.

Proof. $(i) \implies (ii)$: First we assume that there exist two arcs; one from e_{S_n} to $(1\,2)$ and another from $(1\,2)$ to e_{S_n} in $\Gamma_P(S_n; S)$. Then $o((1\,2)e_{S_n} s_1)$ and $o(e_{S_n}(1\,2)s_2)$ are prime numbers for some $s_1, s_2 \in \{(1\,2), (1\,2\,\ldots\,n)\}$. If $s_1 = s_2 = (1\,2)$, then $o((1\,2)e_{S_n} s_1) = 1$ and $o(e_{S_n}(1\,2)s_2) = 1$, a contradiction to the fact that $o((1\,2)e_{S_n} s_1)$ and $o(e_{S_n}(1\,2)s_2)$ are prime numbers. This implies we must have $s_1 = s_2 = (1\,2\,\ldots\,n)$. By the given conditions, it follows that $o((1\,2)(1\,2\,\ldots\,n))$ is prime. Let $o((2\,3\,\ldots\,n)) = q$, for some prime number q. Then

$o((2\,3\,\ldots\,n)^{-1}) = o((2\,n\,(n-1)\,\ldots\,3)) = q$. Therefore, $o((1\,2\,\ldots\,n)^{-1}e_{S_n}(1\,2)) = o((2\,n\,(n-1)\,\ldots\,3)) = q$ and $o(e_{S_n}(1\,2\,\ldots\,n)(1\,2)) = o((1\,2)(2\,3\,\ldots\,n)(1\,2)) = q$. Consequently, there exist two arcs; one from e_{S_n} to $(1\,2\,\ldots\,n)$ and another from $(1\,2\,\ldots\,n)$ to e_{S_n} in $\Gamma_P(S_n; S)$.

$(ii) \implies (iii)$: Now, we assume that there exists an arc from e_{S_n} to $(1\,2\,\cdots\,n)$ in $\Gamma_P(S_n; S)$. Thus $o((1\,2\,\ldots\,n)^{-1}e_{S_n}s)$ is a prime for some $s \in \{(1\,2),(1\,2\,\ldots\,n)\}$. Clearly, s must be equal to $(1\,2)$. Therefore, $o((1\,2\,\ldots\,n)^{-1}e_{S_n}(1\,2)) = o((2\,n\,(n-1)\,\ldots\,3))$ is r for some prime r. But being an $(n-1)$-cycle, $o((2\,n\,(n-1)\,\ldots\,3)) = n-1$. Therefore, $n = r+1$, where r is prime.

$(iii) \implies (i)$: Finally, we assume that $n = r+1$, for some prime r. Then $(1\,2) = e_{S_n}(1\,2\,\ldots\,n)(2\,n\,(n-1)\,\ldots\,3)$ and $e_{S_n} = (1\,2)(1\,2\,\ldots\,n)(2\,n\,(n-1)\,\ldots\,3)$, where $o((2\,n\,(n-1)\,\ldots\,3)) = n-1 = r$. Consequently, there is an arc from e_{S_n} to $(1\,2)$ and another arc from $(1\,2)$ to e_{S_n} in $\Gamma_P(S_n; S)$. $\qquad\square$

Theorem 5.   *Let $S = \{(1\,2),(1\,2\,\ldots\,n)\}$ for all $n \geq 3$. Then $\Gamma_P(S_n; S)$ is strongly connected.*

Proof.   First we take $n = 3$, then from the Figure 1, we have $\Gamma_P(S_3; \{(12),(123)\})$ is a complete graph with loop at each vertex. Now, we prove that for $n \geq 4$, any permutation $\alpha \in S_n$ is strongly connected with e_{S_n}. Actually, we prove this theorem by induction on m, the number of transpositions in α when α is expressed as product of transpositions. For $m = 1$, $\alpha \in S_n$ is a 2-cycle. If $\alpha \neq (1\,2)$, then $\alpha = e_{S_n}(1\,2)(1\,2)\alpha$ and $e_{S_n} = \alpha(1\,2)(1\,2)\alpha$, where e_{S_n} is the identity of S_n and $o((1\,2)\alpha)$ is either 2 or 3. This implies both $e_{S_n} \to \alpha$ and $\alpha \to e_{S_n}$ in $\Gamma_P(S_n; S)$. On the other hand, if $\alpha = (1\,2)$, then there exist two distinct elements a, b such that $a, b \notin \{1, 2\}$. Thus, there exists $(a\,b) \in S_n$ such that $(a\,b) = (1\,2)(1\,2)(a\,b)$ and $(1\,2) = (a\,b)(1\,2)(a\,b)$, which implies that $(1\,2) \to (a\,b)$ and $(a\,b) \to (1\,2)$ and hence $e_{S_n} \to (a\,b) \to (1\,2)$ and $(1\,2) \to (a\,b) \to e_{S_n}$ exist in $\Gamma_P(S_n; S)$. Therefore the result is true for $m = 2$.

We assume that the result is true for any permutation with at most $(m-1)\,(m \geq 2)$ transpositions when that permutation is expressed as product of transpositions, i.e., every permutation consisting of $(m-1)$ or less transpositions is strongly connected with e_{S_n} in $\Gamma_P(S_n; S)$. Let $\sigma = \tau(a\,b) \in S_n$ be a permutation consisting of m transpositions, where $\tau \in S_n$ is a permutation consisting of $(m-1)$ transpositions. So by induction hypothesis, τ is strongly connected with e_{S_n} in $\Gamma_P(S_n; S)$. Here two cases arise.

Case 1. We first suppose that $(a\,b) \neq (1\,2)$. Then $\sigma = \tau(1\,2)(1\,2)(a\,b)$ and $\tau = \sigma(1\,2)(1\,2)(a\,b)$ such that order of $(1\,2)(a\,b)$ is either 2 or 3. Therefore, there exist two arcs $\sigma \to \tau$ and $\tau \to \sigma$ in $\Gamma_P(S_n; S)$. Consequently, σ is strongly connected with e_{S_n} in $\Gamma_P(S_n; S)$.

Case 2. We assume that $(a\,b) = (1\,2)$. Since $n \geq 4$, so there exists $(c\,d) \in S_n$ such that a, b, c, d are distinct. Then $\tau(c\,d) = \sigma(1\,2)(c\,d)$ and $\sigma = \tau(c\,d)(1\,2)(c\,d)$ which implies that $\sigma \to \tau(c\,d)$ and $\tau(c\,d) \to \sigma$ are two paths in $\Gamma_P(S_n; S)$. Again, $\tau = \tau(c\,d)(1\,2)(1\,2)(c\,d)$ and $\tau(c\,d) = \tau(1\,2)(1\,2)(c\,d)$ such that order of $(1\,2)(a\,b)$

is either 2 or 3. Therefore, $\tau(c\,d) \to \tau$ and $\tau \to \tau(c\,d)$ are another two paths in $\Gamma_P(S_n; S)$. Consequently, σ is strongly connected with e_{S_n} in $\Gamma_P(S_n; S)$ and hence by the method of mathematical induction the result follows. $\square$

Using similar kind of argument as in the proof of Theorem 5, we can prove following two theorems.

Theorem 6. *For $n \geq 3$, the graph $\Gamma_P(S_n; S)$ is strongly connected, where the generating set S is given by $S = \{(a\,b), (1\,2\,\cdots\,n) : 1 \leq a < b \leq n \text{ satisfying } gcd(b-a, n) = 1\}$.*

Theorem 7. *Let S be the set of all 2-cycles in S_n ($n \geq 3$). Then $\Gamma_P(S_n; S)$ is strongly connected.*

Now, we characterize strongly connectedness property of $\Gamma_P(A_n; S)$, where A_n is the alternating group on n symbols and S is a generating set of A_n.

Theorem 8. *Let S be the set of all 3-cycles in the alternating group A_n on n symbols ($n \geq 3$). Then $\Gamma_P(A_n; S)$ is strongly connected.*

Proof. Let A_n be the alternating group on n symbols in $\{a_1, a_2, \ldots, a_n\}$. We now show that any nonidentity element $\sigma \in A_n$ is strongly connected with e_{A_n}, the identity of A_n.

Case 1. Let $\sigma = (a_i\,a_j\,a_k)$ be a 3-cycle. Then $\sigma = e_{A_n}(a_i\,a_k\,a_j)(a_i\,a_k\,a_j)$ and $e_{A_n} = \sigma(a_i\,a_j\,a_k)(a_i\,a_j\,a_k)$. Therefore, $e_{A_n} \to \sigma$ and $\sigma \to e_{A_n}$ are two arcs in $\Gamma_P(A_n; S)$.

Case 2. Let $\sigma(\neq e_{A_n}) \in A_n$ be an element such that σ is not a 3-cycle. Expressing σ as a product of 3-cycles, we get $\sigma = (a_{i_1}\;a_{i_2}\;a_{i_3})$ $(a_{i_4}\;a_{i_5}\;a_{i_6})\,\cdots\,(a_{i_{k-8}}\;a_{i_{k-7}}\;a_{i_{k-6}})\,(a_{i_{k-5}}\;a_{i_{k-4}}\;a_{i_{k-3}})\,(a_{i_{k-2}}\;a_{i_{k-1}}\;a_{i_k})$ $= e_{A_n}\,(a_{i_1}\;a_{i_2}\;a_{i_3})(a_{i_4}\;a_{i_5}\;a_{i_6})\,\cdots\,(a_{i_{k-8}}\;a_{i_{k-7}}\;a_{i_{k-6}})(a_{i_{k-5}}\;a_{i_{k-4}}\;a_{i_{k-3}})(a_{i_{k-2}}$ $a_{i_{k-1}}\;a_{i_k}) = \tau_1\,(a_{i_{k-5}}\;a_{i_{k-4}}\;a_{i_{k-3}})(a_{i_{k-2}}a_{i_{k-1}}\;a_{i_k})$, where $\tau_1 = e_{A_n}(a_{i_1}\;a_{i_2}\;a_{i_3})(a_{i_4}$ $a_{i_5}\;a_{i_6})\,\cdots\,(a_{i_{k-8}}\;a_{i_{k-7}}\;a_{i_{k-6}})$. Again, from the relation between τ_1 and σ, we have $\tau_1 = \sigma(a_{i_{k-2}}\;a_{i_k}\;a_{i_{k-1}})(a_{i_{k-5}}\;a_{i_{k-3}}\;a_{i_{k-4}})$. Therefore, there exist two arcs $\tau_1 \to \sigma$ and $\sigma \to \tau_1$ in $\Gamma_P(A_n; S)$. Now, τ_1 is either the identity permutation or 3-cycle or a product of more than one 3-cycles. If $\tau_1 = e_{A_n}$, then we have done. Again, if τ_1 is a 3-cycle, then similar to Case 1, we have two paths $\sigma \to \tau_1 \to e_{A_n}$ and $e_{A_n} \to \tau_1 \to \sigma$ in $\Gamma_P(A_n; S)$. Finally, if τ_1 is a product of more than one 3-cycles, then we continue the same process as in Case 2 and after a finite number of steps (t steps say), we get τ_t such that either τ_t is the identity permutation or a 3-cycle. Therefore, in this case, there exist two paths one from σ to e_{A_n} and another from e_{A_n} to σ in $\Gamma_P(A_n; S)$. $\square$

For each positive integer $n \geq 4$, the dihedral group of degree n, denoted by D_n, is defined by $D_n = \{\langle a, b \rangle : o(a) = n, o(b) = 2, ba = a^{-1}b\}$. We know that D_n is a noncommutative group containing exactly $2n$ elements.

Now, we study graph theoretic properties of $\Gamma_P(D_n; S)$.

Theorem 9. *For each positive integer $n \geq 4$, $\Gamma_P(D_n; S)$ is a complete $2n$-graph with one loop at each vertex, where $S = \{a, b\}$ is the generating set of D_n.*

Proof. Let $H = \langle a \rangle$. Then H is a subgroup of D_n of order n. Moreover, $D_n = H \cup K$, where $K = Hb$ is a right coset of H different from H. Then all the n elements of K are of order 2. Let $x \in D_n$ be an arbitrary element. Since $x = xbb$, so there is a loop at each vertex in $\Gamma_P(D_n; S)$. To show $\Gamma_P(D_n; S)$ is complete, let x, y be any two distinct vertices in $\Gamma_P(D_n; S)$. Then the following four cases arise.

Case 1. Let $x, y \in H$. Then $x = a^{t_1}$ and $y = a^{t_2}$ for some positive integers t_1, t_2. Since $y = a^{t_2} = a^{t_1} b b a^{t_2 - t_1} = x b b a^{t_2 - t_1}$ and $o(b a^{t_2 - t_1}) = 2$, so there is an arc from x to y in $\Gamma_P(D_n; S)$.

Case 2. Let $x, y \in K$. Then $x = a^{s_1} b$ and $y = a^{s_2} b$ for some positive integers s_1, s_2. Since $y = a^{s_2} b = a^{s_1} b b a^{s_2 - s_1} b = x b a^{s_2 - s_1} b$ and $o(a^{s_2 - s_1} b) = 2$, so there is an arc from x to y in $\Gamma_P(D_n; S)$.

Case 3. Let $x \in H$ and $y \in K$. Then $x = a^{r_1}$ and $y = a^{r_2} b$ for some positive integers r_1, r_2. Since $y = a^{r_2} b = a^{r_1} a a^{r_2 - r_1 - 1} b = x a a^{r_2 - r_1 - 1} b$ and $o(a^{r_2 - r_1 - 1} b) = 2$, so there is an arc from x to y in $\Gamma_P(D_n; S)$.

Case 4. Let $x \in K$ and $y \in H$. Then $x = a^{l_1} b$ and $y = a^{l_2}$ for some positive integers l_1, l_2. Since $y = a^{l_2} = a^{l_1} b a b a^{l_2 - l_1 + 1} = x a b a^{l_2 - l_1 + 1}$ with $o(b a^{l_2 - l_1 + 1}) = 2$, it follows that there is an arc from x to y in $\Gamma_P(D_n; S)$.

Considering all the cases, we conclude that $\Gamma_P(D_n; S)$ is a complete $2n$-graph with loop at each vertex. $\qquad \square$

A dicyclic group Q_{4n} is a group of order $4n$ with generators a and b such that the group has the presentation $Q_{4n} = \big\{ \langle a, b \rangle : o(a) = 2n, a^n = b^2, aba = b \big\}$, where $n > 1$ is a positive integer.

Theorem 10. *For each positive integer $n \geq 2$, $\Gamma_P(Q_{4n}; S)$ is strongly connected, where $S = \{a, b\}$ is the generating set of Q_{4n}.*

Proof. Clearly, $Q_{4n} = \big\{ \langle a, b \rangle : o(a) = 2n, a^n = b^2, aba = b \big\} = \{ e_{Q_{4n}}, a, \ldots, a^{2n-1},$ $b, ba, \ldots, ba^{2n-1} \} = \big\{ e_{Q_{4n}}, a, a^2, \ldots, a^{2n-1} \big\} \cup \big\{ b, ba, ba^2, \ldots, ba^{2n-1} \big\} = U \cup V$, where $e_{Q_{4n}}$ is the identity element of Q_{4n} and $U = \langle a \rangle$ is the cyclic subgroup of Q_{4n} generated by a and also $V = bU$ is a left coset of U different from U. Since $\Gamma_P(U; \{a\}) \cong \Gamma_P(\mathbb{Z}_{2n}; \{\bar{1}\})$, so by the similar argument as in the proof of Lemma 2, we can prove that $\Gamma_P(U; \{a\})$ is strongly connected. Since $o(a) = o(a^{n+1})$ and $aba = b$, so we have a cycle $C_b := (b \to ba^{n+1} \to ba^{2(n+1)} \to \cdots \to ba^{i(n+1)} \to \cdots \to ba^{(2n-1)(n+1)} \to b)$ based at b consisting of all $2n$ elements of V. Since $e_{Q_{4n}} = bbb^2$ and $ba^n = e_{Q_{4n}} ba^n$, so we have two arcs one from b to $e_{Q_{4n}}$ and another from $e_{Q_{4n}}$ to ba^n in $\Gamma_P(Q_{4n}; S)$. Therefore $\Gamma_P(Q_{4n}; S)$ is strongly connected for all $n \geq 2$. $\qquad \square$

A generalized quaternion group $Q_{2^{k+1}}$ is a group of order 2^{k+1} with generators a and b such that the group has the presentation $Q_{2^{k+1}} =$

$\left\{\langle a, b\rangle : o(a) = 2^k, a^{2^{k-1}} = b^2, aba = b\right\}$. So, it is the dicyclic group with parameter 2^{k-1}. But here we discuss strongly connectedness property of $\Gamma_P(Q_{2^{k+1}}; S)$ independent from Theorem 10.

Theorem 11. *For each positive integer $k(\geq 2)$, $\Gamma_P(Q_{2^{k+1}}; S)$ is strongly connected, where $S = \{a, b\}$ is the generating set of $Q_{2^{k+1}}$.*

Proof. Clearly, $Q_{2^{k+1}} = U \cup V$, where $U = \langle a\rangle$ is the cyclic subgroup of order 2^k in $Q_{2^{k+1}}$ generated by a and also $V = Ub$ is a right coset of U different from U. Since $o(a)$ is 2^k, so $o(a^{2^{k-1}+1})$ is 2^k and hence $U = \langle a^{2^{k-1}+1}\rangle$. Therefore $C_e := (e_{Q_{2^{k+1}}} \to a^{2^{k-1}+1} \to a^2 \to \cdots \to a^{i2^{k-1}+i} \to \cdots \to a^{2^k-2^{k-1}-1} \to e_{Q_{2^{k+1}}})$ is cycle based at $e_{Q_{2^{k+1}}}$ consisting of all 2^k elements of U. Since $a^{2^{k-1}+1}$ is generator of U and $aba = b$, so we have a cycle $C_b := (b \to a^{2^{k-1}-1}b \to a^{-2}b \to \cdots \to a^{i2^{k-1}+i}b \to \cdots \to a^{2^k-2^{k-1}-1}b \to b)$ based at b consisting of all 2^k elements of V. Also we have two arcs one from b to $e_{Q_{2^{k+1}}}$ and another from $e_{Q_{2^{k+1}}}$ to $a^{2^{k-1}}b$ in $\Gamma_P(Q_{2^{k+1}}; S)$. Therefore $\Gamma_P(Q_{2^{k+1}}; S)$ is strongly connected for all $k \geq 2$. $\square$

A semi-dihedral group SD_{2^m} is a group of order 2^m with generators a and b such that the group has the presentation $SD_{2^m} = \{\langle a, b\rangle : o(a) = 2^{m-1}, o(b) = 2, ba = a^{2^{m-2}-1}b\}$, where $m \geq 4$ is any positive integer. Here $SD_{2^m} = \{e_{SD_{2^m}}, a, \ldots, a^{2^{m-1}-1}, b, ab, \ldots, a^{2^{m-1}-1}b\} = \{e_{SD_{2^m}}, a, \ldots, a^{2^{m-1}-1}\} \cup \{b, ab, \ldots, a^{2^{m-1}-1}b\} = H \cup K$, where $e_{SD_{2^m}}$ is the identity element of SD_{2^m} and $H = \langle a\rangle$ is the cyclic subgroup of SD_{2^m} generated by a and also $K = Hb$ is a right coset of H different from H.

Here $(a^ib)(a^ib) = a^ia^{i(2^{m-2}-1)} = a^{i2^{m-2}(mod\ 2^{m-1})}$, for all $i \in \{1, 2, \ldots, 2^{m-1}\}$. Hence for all $i \in \{1, 2, \ldots, 2^{m-1}\}$,

$$o(a^ib) = \begin{cases} 2, & \text{if } i \text{ is even,} \\ 4, & \text{otherwise.} \end{cases}$$

Thus the order of any element of K is either 2 or 4 and also $o(a^{2^{m-2}}) = 2$. Therefore prime order elements of SD_{2^m} are $a^{2^{m-2}}, b, a^2b, a^4b, \ldots, a^{2^{m-1}-2}b$.

Theorem 12. *For each positive integer $m \geq 4$, $\Gamma_P(SD_{2^m}; S)$ is strongly connected, where $S = \{a, b\}$ is the generating set of SD_{2^m}.*

Proof. Clearly, $SD_{2^m} = H \cup K$, where $H = \langle a\rangle$ is the cyclic subgroup of order 2^{m-1} in SD_{2^m} and also $K = Hb$ is a right coset of H different from H. Since $o(a) = o(a^{2^{m-2}+1})$, so by the similar argument as in the proof of Theorem 11, we have a cycle C_e based at $e_{SD_{2^m}}$ consisting of all 2^{m-1} elements of H. Since $a^{2^{m-2}+1}$ is a generator of H and $ba = a^{2^{m-2}-1}b$, so by the similar argument as in the proof of Theorem 11, we have a cycle C_b based at b consisting of all 2^{m-1} elements of

K. Also we have two arcs one from $e_{SD_{2^m}}$ to ab and another from ab to $a^{2^{m-2}+1}$ in $\Gamma_P(SD_{2^m}; S)$. Consequently, $\Gamma_P(SD_{2^m}; S)$ is strongly connected for all $m \geq 4$. $\qquad\square$

In [10], the authors described the groups U_{6n} and V_{8n} for even positive integer n; and in [6], the authors further studied the group V_{8n}. According to them, U_{6n} and V_{8n} are described as follows :

$$U_{6n} = \left\{ \langle a, b \rangle : o(a) = 2n, o(b) = 3, a^{-1}ba = b^{-1} \right\},$$
$$V_{8n} = \left\{ \langle a, b \rangle : o(a) = 2n, o(b) = 4, aba = b^{-1}, ab^{-1}a = b \right\},$$

where n is a positive integer.

Theorem 13. *For each positive integer n, $\Gamma_P(U_{6n}; S)$ is strongly connected, where $S = \{a, b\}$ is the generating set of U_{6n}.*

Proof. Clearly, $U_{6n} = \left\{ \langle a, b \rangle : o(a) = 2n, o(b) = 3, a^{-1}ba = b^{-1} \right\} = \{e_{U_{6n}}, a, a^2, \ldots, a^{2n-1}, b, ba, ba^2, \ldots, ba^{2n-1}, b^2, b^2a, b^2a^2, \ldots, b^2a^{2n-1}\} = \{e_{U_{6n}}, a, a^2, \ldots, a^{2n-1}\} \cup \{b, ba, ba^2, \ldots, ba^{2n-1}\} \cup \{b^2, b^2a, b^2a^2, \ldots, b^2a^{2n-1}\} = W_1 \cup W_2 \cup W_3$, where $e_{U_{6n}}$ is the identity element of U_{6n}, $W_1 = \langle a \rangle$ is the cyclic subgroup of U_{6n} generated by a, $W_2 = bW_1$ and $W_3 = b^2W_1$ are both left cosets of W_1. Since $\Gamma_P(W_1; \{a\}) \cong \Gamma_P(\mathbb{Z}_{2n}; \{\overline{1}\})$, so by the similar argument as in the proof of Lemma 2, we can prove that $\Gamma_P(W_1; \{a\})$ is strongly connected. Since $o(a) = o(a^{n+1})$ and $a^{-1}ba = b^{-1}$, so by the similar argument as in the proof of Theorem 10, we have two cycles; one cycle C_b containing every $2n$ elements of W_2 and another cycle C_{b^2} consisting of all $2n$ elements of W_3. Also, we have four arcs; $e_{U_{6n}} \to ba^n$, $b \to e_{U_{6n}}$, $b^2 \to b$, $b \to b^2a^n$ in $\Gamma_P(U_{6n}; S)$. Consequently, $\Gamma_P(U_{6n}; S)$ is strongly connected for every positive integer n. $\qquad\square$

By similar kind of argument as in the proof of Theorem 13, we can prove the following theorem.

Theorem 14. *For every positive integer n, $\Gamma_P(V_{8n}; S)$ is strongly connected, where $S = \{a, b\}$ is the generating set of V_{8n}.*

Now, we investigate various graph theoretic properties of $\Gamma_P(G; S)$ of a finite group G with a generating set S. We know that if S contains the identity element of a finite group G, then in $\Gamma_1(G; S)$ and in $O\Gamma(G; S)$ there is a loop at every vertex. But $\Gamma_P(G; S)$ may not contain any loop even if the identity element belongs to the generating set S of G. In fact, the graph $\Gamma_P(\mathbb{Z}_4; \{\overline{0}, \overline{1}\})$ is such a graph which does not contain any loop.

Theorem 15. *For a finite group G and its any generating set S, $\Gamma_P(G; S)$ has one loop at each vertex if and only if generating set S contains a prime order element.*

Proof. First we assume that $\Gamma_P(G; S)$ has a loop at any vertex x. Then there exists an element $s \in S$ such that $o(x^{-1}xs)$ is prime, i.e., $o(s)$ is prime. Therefore, S contains an element of prime order.

Conversely, suppose that generating set S contains an element b of prime order. Let $x \in G$ be an arbitrary element. Then $x = xbb^{-1}$ such that $o(b^{-1})$ is prime. This implies there is a loop at each vertex in $\Gamma_P(G; S)$. $\qquad\square$

Theorem 16. *If G is a finite group and S is a generating set of G such that $S = S^{-1}$, then $\Gamma_P(G; S)$ is undirected, i.e., for two distinct vertices a, b, if $a \to b$ is an arc in $\Gamma_P(G; S)$, then $b \to a$ is also an arc in $\Gamma_P(G; S)$.*

Proof. Let $a, b \in G$ such that $a \to b$ is an arc in $\Gamma_P(G; S)$. Then $o(b^{-1}as) = p$, for some prime p. This implies $o((b^{-1}as)^{-1}) = p$, i.e., $o(s^{-1}a^{-1}b) = p$, i.e., $o(a^{-1}bs^{-1}) = p$. Now, $S = S^{-1}$ implies $s^{-1} \in S$ and hence $b \to a$ is an arc in $\Gamma_P(G; S)$. $\qquad\square$

Theorem 17. *The graph $\Gamma_P(G; S)$ of a non trivial finite group G together with its any generating set S is an essential graph. Moreover, the order prime Cayley graph $O\Gamma(G; S)$ of a non trivial finite group G together with its any generating set S is an essential graph.*

Proof. By Theorem 15, it follows that $\Gamma_P(\mathbb{Z}_2; \{\bar{1}\})$ has loop at each vertex and also $\Gamma_P(\mathbb{Z}_2; \{\bar{0}, \bar{1}\}) \cong O\Gamma(\mathbb{Z}_2; \{\bar{1}\}) \cong O\Gamma(\mathbb{Z}_2; \{\bar{0}, \bar{1}\})$ is the complete graph with one loop at each vertex. Therefore, both $\Gamma_P(\mathbb{Z}_2; \{\bar{1}\})$ and $\Gamma_P(\mathbb{Z}_2; \{\bar{0}, \bar{1}\})$ contain no sources and no sinks and hence they are all essential graphs.

Now, let G be a finite group with $|G| > 2$ and S be a generating set of G. Also, let $g \in G$ be an arbitrary element. Since G contains at least one element a (say) of prime order, so for an element $s \in S$ with $o(sa) = n$, clearly $(C_g : g \to g(sa) \to g(sa)^2 \to \cdots \to g(sa)^n = g)$ is a cycle based at g in $\Gamma_P(G; S)$. Therefore, $\Gamma_P(G; S)$ contain no sources and no sinks and hence $\Gamma_P(G; S)$ is essential graph. Consequently, $O\Gamma(G; S)$ is also an essential graph. $\qquad\square$

Corollary 2. *The graph $\Gamma_P(G; S)$ of a non trivial finite group G together with its any generating set S cannot be an acyclic graph. Moreover, the order prime Cayley graph $O\Gamma(G; S)$ of a non trivial finite group G together with its any generating set S cannot be acyclic.*

We know that the Cayley graph $\Gamma_1(G; S)$ of a finite cyclic group $G = \langle g \rangle$ with a generating set $S = \{g\}$ does not satisfy Condition *(L)* and hence does not satisfy Condition *(K)*. Again, the graph $\Gamma_P(\mathbb{Z}_2; \{\bar{1}\})$ does not satisfy Condition *(L)*.

Theorem 18. *The graph $\Gamma_P(G; S)$ of a non trivial finite group G together with minimal generating set S satisfies Condition (L) if and only if G is not isomorphic to a cyclic 2-group. Moreover, $\Gamma_P(G; S)$ satisfies Condition (L) if and only if every vertex in $\Gamma_P(G; S)$ is the base of two or more cycles.*

Proof. First we suppose that $G \not\cong \mathbb{Z}_{2^n}$ for any positive integer n. Then two cases arise.

Case 1. If $|G|$ has an odd prime factor, then G contents at least two elements a and b (say) of prime order and thus it follows that $|s^{-1}(v)| \geq 2$ for every $v \in \Gamma_P(G; S)$. Hence $\Gamma_P(G; S)$ satisfies *Condition (L)*. To show every vertex in $\Gamma_P(G; S)$ is the base of two or more cycles, let g be any element in $\Gamma_P(G; S)$ such that there exists a cycle P_g' based at g. Then for an element $s \in S$, either $g \to gsa$ is a subpath of P_g' or $g \to gsa$ is not a subpath of P_g'. If $g \to gsa$ is a subpath of P_g', then there exists a cycle $(C_1 : g \to g(sb) \to g(sb)^2 \to \cdots \to g(sb)^n = g)$ based at g. On the other hand, if $g \to gsa$ is not a subpath of P_g', then there exists a cycle $(C_2 : g \to g(sa) \to g(sa)^2 \to \cdots \to g(sa)^n = g)$ based at g.

Case 2. If $|G|$ has no odd prime factor, then $|G| = 2^m$ for some positive integer m. Moreover, since $G \not\cong \mathbb{Z}_{2^n}$ for any positive integer n, it follows that G is non-cyclic. Thus, minimal generating set of G contains at least two elements s_1, s_2 (say). Therefore, $|s^{-1}(v)| \geq 2$ for every $v \in \Gamma_P(G; S)$ and hence $\Gamma_P(G; S)$ satisfies Condition *(L)*. To show $\Gamma_P(G; S)$ has two or more cycles at every vertex, let P_{g_1}'' be a cycle in $\Gamma_P(G; S)$ based at any vertex g_1. Also, let $d \in G$ be such that $o(d) = 2$ and $o(s_1 d) = n_1$, $o(s_2 d) = n_2$. Then either $g_1 \to g_1 s_1 d$ is a subpath of P_{g_1}'' or $g_1 \to g_1 s_1 d$ is not a subpath of P_{g_1}''. If $g_1 \to g_1 s_1 d$ is a subpath of P_{g_1}'', then there exists a cycle $(C_2' : g_1 \to g_1 s_2 d \to g_1(s_2 d)^2 \to \cdots \to g_1(s_2 d)^{n_2} = g_1)$ based at g_1. On the other hand, if $g_1 \to g_1 s_1 d$ is not a subpath of P_{g_1}'', then there exists a cycle $(C_1' : g_1 \to g_1 s_1 d \to g_1(s_1 d)^2 \to \cdots \to g_1(s_1 d)^{n_1} = g_1)$ based at g_1. Hence every vertex in $\Gamma_P(G; S)$ is the base of two or more cycles.

Conversely, suppose that $G \cong \mathbb{Z}_{2^k}$ for some positive integer k. Then minimal generating set of G contains exactly one element and also G contains a unique element of prime order. Then from the proof of Theorem 2, it follows that $\Gamma_P(G; S)$ is isomorphic to a cycle of length 2^n and hence $\Gamma_P(G; S)$ does not satisfy Condition *(L)*. □

From the above theorem, it is clear that $\Gamma_P(G; S)$ of a finite group G satisfies Condition *(K)* if and only if $G \not\cong \mathbb{Z}_{2^k}$.

The next theorem describes when $O\Gamma(G; S)$ of a finite group G with any generating set S satisfies Condition *(L)*.

Theorem 19. *Let G be a finite group with at least 3 elements and S be a generating set of G. Then the order prime Cayley graph $O\Gamma(G; S)$ satisfies Condition (L).*

Proof. If G is a non-cyclic group containing at least 3 elements, then clearly $|S| \geq 2$ and so there are at least two arcs emitted from every vertex of $O\Gamma(G; S)$. Consequently $O\Gamma(G; S)$ satisfies Condition *(L)*. On the other hand, if G is a cyclic group of order $n > 2$ and $|S| \geq 2$, then by the similar argument, we conclude that $O\Gamma(G; S)$ satisfies Condition *(L)*.

Finally, we consider the case when G is cyclic and S contains exactly one element. Then two cases arise.

Case 1. Suppose $|G| = n(> 2)$ is a prime number. Then there is an arc between any two vertices of $O\Gamma(G; S)$ and therefore, $O\Gamma(G; S)$ satisfies Condition *(L)*.

Case 2. Assume that $|G| = n(> 2)$ is a composite number. Then there exists at least one element $a \in G$ such that $o(a)$ is prime. Clearly, $a \notin S = \{s\}$ and therefore, it follows that $|s^{-1}(v)| \geq 2$ for every $v \in \Gamma_P(G; S)$. Consequently, $O\Gamma(G; S)$ satisfies Condition *(L)*. □

Remark 4. By similar argument as in Theorem 18, we can show that the order prime Cayley graph $O\Gamma(G; S)$ satisfies Condition *(K)*, where G is a finite group containing at least 3 elements and S is a generating set of G. Also, $O\Gamma(\mathbb{Z}_2; \{\overline{1}\})$ satisfies Condition *(L)* as well as Condition *(K)*.

Theorem 20. *Let G be a nontrivial finite group and S be a minimal generating set. Then $\Gamma_P(G; S)$ is a nontrivial graph if and only if $G \not\cong \mathbb{Z}_{2^n}$ for any positive integer $n > 1$.*

Proof. Clearly, the graph $\Gamma_P(\mathbb{Z}_2; \{\overline{1}\})$ is a disconnected graph and has loop at each of its vertices. Therefore, $\Gamma_P(\mathbb{Z}_2; \{\overline{1}\})$ is a nontrivial graph. Now, we assume that $G \not\cong \mathbb{Z}_{2^n}$ for any positive integer $n > 1$. Then by Theorem 18, it follows that every vertex in $\Gamma_P(G; S)$ is the base of two or more cycles. Consequently, $\Gamma_P(G; S)$ is a nontrivial graph.

Converse part follows from Theorem 2. □

4. Leavitt path algebras of order prime Cayley graphs

In this section, we study some interesting properties of the Leavitt path algebra of the order prime Cayley graph of a finite group G together with a generating set S of G.

The authors (in their paper [5]) proved that the only possible values for the stable rank of the Leavitt path algebras of row-finite graphs are 1, 2 and ∞. Later, in [11], Larki and Riazi extended this result to an arbitrary graph. In [15], Nam and Phuc computed the stable rank of Leavitt path algebras of Hopf graphs via ramification datas and proved that possible values for the stable rank of the Leavitt path algebras of Hopf graphs are 1, 2 and ∞. In the following theorem, we prove that the stable rank of Leavitt path algebra $L_K(O\Gamma(G; S))$ of non trivial finite group G together its generating set S is 2. Moreover, we establish the following theorem which lists a number of important properties of the Leavitt path algebra $L_K(O\Gamma(G; S))$ that will be useful in our further discussion.

Theorem 21. *Let G be a non trivial finite group and S be any generating set. Then*
(i) The Leavitt path algebra $L_K(O\Gamma(G; S))$ is a purely infinite simple ring,
(ii) The Leavitt path algebra $L_K(O\Gamma(G; S))$ is a prime ring,
(iii) The stable rank of $L_K(O\Gamma(G; S))$ is 2.

Proof. (i) By Theorem 19 and Remark 4, it is clear that $O\Gamma(G; S)$ satisfies *Condition (L)*. Since $\langle S \rangle = G$, Cayley graph $\Gamma_1(G; S)$ is strongly connected. Here Cayley graph $\Gamma_1(G; S)$ is a subdirected graph of $O\Gamma(G; S)$. Hence for any two vertices $v_1, v_2 \in (O\Gamma(G; S))^0$, there exists a path from v_1 to v_2 and vice versa and therefore, the hereditary and saturated subsets of $(O\Gamma(G; S))^0$ are $\emptyset$ and $G = (O\Gamma(G; S))^0$. Also, strongly connectedness property of $O\Gamma(G; S)$ implies $O\Gamma(G; S)$ contains a cycle. Therefore, by [1, Theorem 3.1.10], it follows that $L_K(O\Gamma(G; S))$ is purely infinite simple.

(ii) Since $O\Gamma(G; S)$ is strongly connected, so $O\Gamma(G; S)$ must be downward directed. Therefore, by [1, Theorem 4.1.5], it follows that the Leavitt path algebra $L_K(O\Gamma(G; S))$ is a prime ring.

(iii) It is clear that $O\Gamma(G; S)$ is not an acyclic graph. Also, the hereditary and saturated subsets of $(O\Gamma(G; S))^0$ are $\emptyset$ and $G = (O\Gamma(G; S))^0$. Therefore, by [1, Theorem 4.4.19], we conclude that the stable rank of $L_K(O\Gamma(G; S))$ is 2. $\square$

Next we mention some interesting note.

Remark 5. (1) By Theorem 2 and [1, Corollary 2.5.15], it is clear that $L_K(\Gamma_P(\mathbb{Z}_{2^n}; \{\overline{1}\}))$ is graded simple. Also, from [1, Corollary 4.2.14], it follows that $L_K(\Gamma_P(\mathbb{Z}_{2^n}; \{\overline{1}\}))$ is isomorphic to $M_{2^n}(K[x, x^{-1}])$.

(2) From Theorem 3 and Theorem 18, it is clear that $L_K(\Gamma_P(\mathbb{Z}_n; \{\overline{1}\}))$ is purely infinite simple if and only if n is of the form pm, where p is any odd prime and m is any positive integer.

(3) From Fig.1, Theorem 5 and Theorem 18, it follows that $L_K(\Gamma_P(S_n; S))$ is purely infinite simple, where $S = \{(12), (12 \cdots n)\}$. Also, by Theorem 8 and Theorem 18, we conclude that $L_K(\Gamma_P(A_n; S))$ is purely infinite simple, where S is the set of all 3-cycles in A_n.

(4) From Theorem 10 and Theorem 18, $L_K(\Gamma_P(Q_{4n}; S))$ is purely infinite simple, where $S = \{a, b\}$ is the generating set of Q_{4n}.

(5) From Theorem 11 and Theorem 18, $L_K(\Gamma_P(Q_{2^{k+1}}; S))$ is purely infinite simple, where $S = \{a, b\}$ is the generating set of $Q_{2^{k+1}}$.

(6) From Theorem 12 and Theorem 18, $L_K(\Gamma_P(SD_{2^m}; S))$ is purely infinite simple, where $S = \{a, b\}$ is the generating set of SD_{2^m}.

(7) From Theorem 13 and Theorem 18, $L_K(\Gamma_P(U_{6n}; S))$ is purely infinite simple, where $S = \{a, b\}$ is the generating set of U_{6n}.

The Cuntz-Krieger Uniqueness theorem for Leavitt path algebra in [1, Theorem 2.2.16] states that a ring homomorphism $\varphi : L_K(E) \to A$, from a Leavitt path algebra $L_K(E)$ to an K-algebra A is injective if the graph E satisfies *Condition (L)* and $\varphi(v) \neq 0$ for all vertices v in E. For the Leavitt path algebra $L_K(O\Gamma(G; S))$, we have the following stronger version of the uniqueness theorem.

Theorem 22. *Let G be a finite group with a generating set S and A be any K-algebra. Then a ring homomorphism $\varphi : L_K(O\Gamma(G; S)) \to A$ is injective if and only if $\varphi(e_G) \neq 0$, where e_G is the identity element of G.*

Proof. Suppose $\varphi(e_G) \neq 0$. Then for any vertex $a \in (O\Gamma(G; S))^0$, $\varphi(a) \neq 0$ as $a^n = e_G$ for some positive integer n. Also, by Theorem 19 and Remark 4, $O\Gamma(G; S)$ satisfies *Condition (L)*. So by the Cuntz-Krieger Uniqueness theorem, we conclude that φ is injective. The converse part is obvious. □

Now, we calculate socle of the Leavitt path algebra $L_K(\Gamma_P(G; S))$ for any non trivial finite group G together with its any generating set S.

Theorem 23. *Let G be a non trivial finite group and S be its generating set. Then socle of the Leavitt path algebra $L_K(\Gamma_P(G; S))$ is $\{0\}$, i.e., $L_K(\Gamma_P(G; S))$ contains no minimal one sided left ideal.*

Proof. Every vertex of $\Gamma_P(G; S)$ is a base vertex of some cycle in $\Gamma_P(G; S)$. Thus $\Gamma_P(G; S)$ contains no line point. So by [1, Theorem 2.6.14], we conclude that socle of the Leavitt path algebra $L_K(\Gamma_P(G; S))$ is $\{0\}$. □

We know that finding out the Grothendieck group of a purely infinite simple Leavitt path algebra is an important area of the study of Leavitt path algebra. If we consider the Leavitt path algebras $L_K(\Gamma_P(Q_8; S))$, $L_K(\Gamma_P(Q_{16}; S))$ and $L_K(\Gamma_P(D_4; S))$, then one can find out that their Grothendieck groups K_0 are given by $K_0(L_K(\Gamma_P(Q_8; S))) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, $K_0(L_K(\Gamma_P(Q_{16}; S))) \cong \mathbb{Z}_7 \oplus \mathbb{Z}_{21}$ and $K_0(L_K(\Gamma_P(D_4; S))) \cong \mathbb{Z}_7$. Moreover, using the following theorem one can find out the Grothendieck group of the Leavitt path algebra $L_K(\Gamma_P(D_n; S))$ for any positive integer $n \geq 4$.

Theorem 24. *For any positive integer $n \geq 4$, the Leavitt path algebra $L_K(\Gamma_P(D_n; S))$ is isomorphic to the Leavitt algebra $L(1, 2n)$, where $S = \{a, b\}$ is the generating set of D_n. Moreover, the Grothendieck group of the Leavitt path algebra $L_K(\Gamma_P(D_n; S))$ is isomorphic to $\mathbb{Z}_{2n-1}$.*

Proof. By Theorem 9, we have $\Gamma_P(D_n; S)$ is the complete $2n$-graph with one loop at each vertex. Then from [13, Theorem 4.2], it follows that $L_K(\Gamma_P(D_n; S))$ is isomorphic to Leavitt algebra $L(1, 2n)$. Since the Grothendieck group $K_0(L(1, 2n))$ is isomorphic to $\mathbb{Z}_{2n-1}$, so we have $K_0(L_K(\Gamma_P(D_n; S))) \cong \mathbb{Z}_{2n-1}$. □

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