

Maximal outerplanar graphs with semipaired domination number double the domination number

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Abstract: A subset S of vertices in a graph G is a dominating set if every vertex in $V(G) \setminus S$ is adjacent to a vertex in S . If the graph G has no isolated vertex, then a paired dominating set S of G is a dominating set of G such that $G[S]$ has a perfect matching. Further, a semipaired dominating set of G is a dominating set of G with the additional property that the set S can be partitioned into two element subsets such that the vertices in each subset are at most distance two apart. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G . Similarly, the paired (semipaired) domination number $\gamma_{\text{pr}}(G)$ ($\gamma_{\text{pr}2}(G)$) is the minimum cardinality of a paired (semipaired) dominating set of G . It is known that for a graph G , $\gamma(G) \leq \gamma_{\text{pr}2}(G) \leq \gamma_{\text{pr}}(G) \leq 2\gamma(G)$. In this paper, we characterize maximal outerplanar graphs G satisfying $\gamma_{\text{pr}2}(G) = 2\gamma(G)$. Hence, our result yields the characterization of maximal outerplanar graphs G satisfying $\gamma_{\text{pr}}(G) = 2\gamma(G)$.

Keywords: paired-domination, semipaired domination number, maximal outerplanar graphs

AMS Subject classification: 05C69

1. Introduction

A *dominating set* of a graph G is a set S of vertices of G such that every vertex not in S has a neighbor in S . The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set in G . A γ -*set* of G is a dominating set of G of

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minimum cardinality $\gamma(G)$. For recent books on domination in graphs, we refer the reader to [4–6].

An *isolate-free graph* is a graph that contains no isolated vertex. Paired domination was introduced in [13, 14] as a model for security applications involving backups for police officers. To model a backup, each vertex in the paired dominating set must be partnered with an adjacent vertex in the set. Formally, a *paired dominating set*, abbreviated PD-set, of an isolate-free graph G is a dominating set S of G with the additional property that the subgraph induced by S , denoted $G[S]$, contains a perfect matching. The *paired domination number*, denoted by $\gamma_{\text{pr}}(G)$, of G is the minimum cardinality of a PD-set of G . A γ_{pr} -set of G is a PD-set of G of minimum cardinality $\gamma_{\text{pr}}(G)$. For a state of the art on paired domination in graphs we refer the reader to the survey paper [3] and the book chapter [2].

A relaxed version of paired domination, called semipaired domination, was introduced by Haynes and Henning [8] and studied, for example, in [7, 9–12, 15, 16] and elsewhere. Following the notation introduced in [8], a set S of vertices in an isolate-free graph G is a *semipaired dominating set*, abbreviated semi-PD-set, of G if S is a dominating set of G and every vertex in S is paired with exactly one other vertex in S that is within at most distance 2 from it. Thus, the vertices in the dominating set S can be partitioned into 2-sets such that if $\{u, v\}$ is a 2-set, then $uv \in E(G)$ or the distance between u and v is 2. As defined in [8], we say that u and v are *S -paired* (or simply *paired* if the set S is clear from the context), and that u and v are *S -partners* (or simply *partners*), and we call such a pairing of the vertices of S a *semi-matching* in G . The *semipaired domination number*, denoted by $\gamma_{\text{pr}2}(G)$, is the minimum cardinality of a semi-PD-set of G . A semi-PD-set of G of cardinality $\gamma_{\text{pr}2}(G)$ is a $\gamma_{\text{pr}2}$ -set of G . Every semi-PD-set is a dominating set and every PD-set is a semi-PD-set. Hence, we have the following observation, where it is observed in [14] that $\gamma_{\text{pr}}(G) \leq 2\gamma(G)$ for every isolate-free graph G .

Observation 1. ([8]) If G is an isolate-free graph, then $\gamma(G) \leq \gamma_{\text{pr}2}(G) \leq \gamma_{\text{pr}}(G) \leq 2\gamma(G)$.

A *triangulated disc* is a (simple) planar graph all of whose inner faces are triangles. A *maximal outerplanar graph*, abbreviated *mop* in the literature, is a triangulated disc where the outer face contains all vertices. Thus, a maximal outerplanar graph can be embedded in the plane in such a way that all vertices are on the boundary of its outer face (the unbounded face) and all interior faces are triangles. We note that the addition of a single edge in a maximal outerplanar graph results in a graph that is not outerplanar.

In this paper, we characterize maximal outerplanar graphs G satisfying $\gamma_{\text{pr}2}(G) = 2\gamma(G)$. This yields the characterization of maximal outerplanar graphs G satisfying $\gamma_{\text{pr}2}(G) = \gamma_{\text{pr}}(G) = 2\gamma(G)$.

1.1. Notation

For notation and graph theory terminology, we in general follow [6]. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. Two vertices in G are *neighbors* if they are adjacent. The *open neighborhood* $N_G(v)$ of a vertex v in G is the set of neighbors of v , while the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N_G(v)$. We denote the *degree* of v in G by $\deg_G(v) = |N_G(v)|$. For a set $S \subseteq V(G)$, its *open neighborhood* is the set $N_G(S) = \cup_{v \in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. Moreover, the subgraph induced by S is denoted by $G[S]$. The graph $G - S$ is obtained from G by deleting all vertices in S (and all edges of G incident with vertices of S). If $S = \{v\}$, we denote $G - S$ simply by $G - v$ rather than $G - \{v\}$.

For a set $S \subseteq V(G)$ and a vertex $v \in S$, the *S -private neighborhood* $\text{pn}[v, S]$ of v is the set of vertices that are in the closed neighborhood of v but not in the closed neighborhood of the set $S \setminus \{v\}$, that is, $\text{pn}[v, S] = \{w \in V(G) : N_G[w] \cap S = \{v\}\}$. If $\text{pn}[v, S] \neq \emptyset$, then a vertex in $\text{pn}[v, S]$ is called an *S -private neighbor of v* . A set $B \subseteq V(G)$ is a *packing* of G if $N_G[u] \cap N_G[v] = \emptyset$ for any pair of distinct $u, v \in B$.

We denote a *path*, a *cycle*, and a *complete graph* on n vertices by P_n , C_n , and K_n , respectively. A complete graph K_3 we call a *triangle*. A *fan* of order $n \geq 5$, denoted F_n , is the graph obtained from a path P_{n-1} by adding a new vertex v and joining it to all vertices of the path. We say that the fan F_n is *centered at v* .

For an integer $k \geq 1$, we use the standard notation $[k] = \{1, \dots, k\}$ and $[k]_0 = [k] \cup \{0\} = \{0, 1, \dots, k\}$.

2. Main result

In this section, we characterize all mops G with $\gamma_{\text{pr2}}(G) = 2\gamma(G)$. We shall prove the following result, where the class $\mathcal{G}(\mathcal{F}, \mathcal{H})$ is defined in Section 4.

Theorem 2. *If G is a mop, then $\gamma_{\text{pr2}}(G) \leq 2\gamma(G)$, with equality if and only if $G \in \mathcal{G}(\mathcal{F}, \mathcal{H})$.*

We proceed as follows. In Section 3, we present some preliminary results. In Section 4 we define the class $\mathcal{G}(\mathcal{F}, \mathcal{H})$ and prove that every graph G in the class $\mathcal{G}(\mathcal{F}, \mathcal{H})$ is a mop satisfying $\gamma_{\text{pr2}}(G) = 2\gamma(G)$. Finally, in Section 4 we prove that if G is a mop satisfying $\gamma_{\text{pr2}}(G) = 2\gamma(G)$, then $G \in \mathcal{G}(\mathcal{F}, \mathcal{H})$. The result of Theorem 2 follows immediately from Observation 1, and the results presented in Section 4 and Section 5.

3. Preliminary Results

In this section, we present some preliminary results on mops.

Observation 3. If v is a vertex in a mop G , then the induced subgraph $G[N_G[v]]$ is a fan centered at v , and so $G[N_G(v)]$ is a path.

We shall need the following property of mops observed by O'Rourke [18] (see Lemma 7.2, p. 169) and others.

Observation 4. ([18]) Every mop has a unique Hamiltonian cycle.

Following the notation of [17], for simplicity we refer to an edge that belongs to the Hamiltonian cycle of a mop as a *Hamiltonian edge*, and to every other edge of the mop as a *diagonal*. We shall also need the following property of mops due to Allgeier [1].

Lemma 1. ([1]) If H is a 2-connected induced subgraph of a mop G , then H is a mop.

4. The Class $\mathcal{G}(\mathcal{F}, \mathcal{H})$

Let $\ell \geq 2$ be an integer. Let F_i be a fan of order at least 5 and let x_i be the center of F_i for $i \in [\ell]$. Let $\mathcal{F}' = \{F_1, F_2, \dots, F_\ell\}$. We say that the fans in \mathcal{F}' are *linked* in a graph G if, for each $i \in [\ell]$, there exist exactly two consecutive vertices y_i and z_i on the path $F_i - x_i$ such that $\{y_i, z_i : i \in [\ell]\}$ induces a mop in G , which we call a *linked mop* associated with \mathcal{F}' . We say also that F_i is *linked* in \mathcal{F}' via the edge $y_i z_i$. Moreover, we call $y_i z_i$ the *linked edge* of F_i in \mathcal{F}' . For example, Figure 1 illustrates a

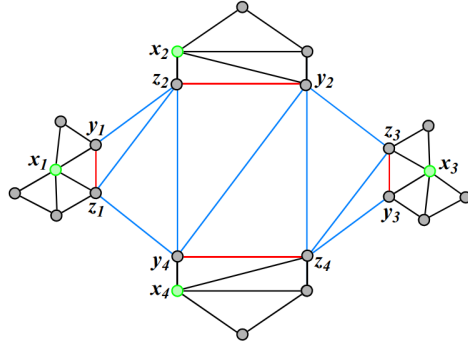


Figure 1. Linked fans

family \mathcal{F}' with four linked fans F_1, F_2, F_3, F_4 , where the linked edges of G are colored red and the center of each fan is colored green. Further the edges, different from the linked (red) edges, in the mop of G induced by the set $\{y_i, z_i : i \in [4]\}$ are colored blue in Figure 1. Thus the red and blue edges in Figure 1 form the linked mop of G associated with \mathcal{F}' .

Let $\mathcal{F}_1, \dots, \mathcal{F}_d$ be $d \geq 1$ families of linked fans in a mop G , and let F be a common fan that belongs to each family \mathcal{F}_j and is linked to the fans in \mathcal{F}_j by the linked edge $a_j b_j \in E(F)$ for all $j \in [d]$ (see Figure 2).

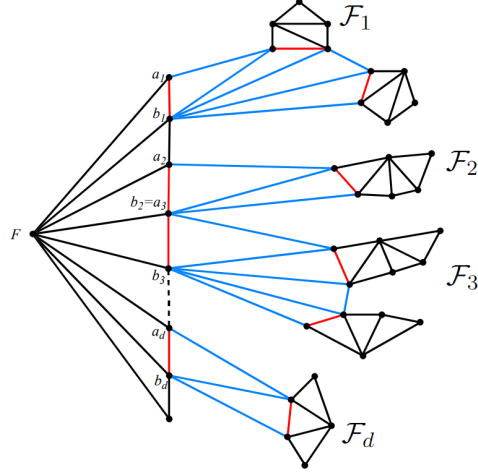


Figure 2. A common fan F in $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \dots, \mathcal{F}_d$

Further, let $\mathcal{F}_r \cap \mathcal{F}_s = \{F\}$ for all r and s where $1 \leq r < s \leq d$. Hence, all the fans from all the families $\mathcal{F}_1, \dots, \mathcal{F}_d$ are vertex disjoint, except for the fan F which belongs to each family \mathcal{F}_j for all $j \in [d]$. We define a *linked set* J_F of the fan F as follows:

- (a) If $a_j b_j$ is the edge of F that is linked to the family \mathcal{F}_j and $|\mathcal{F}_j| \geq 3$, then we add both a_j and b_j to J_F for all $j \in [d]$.
- (b) If $a_j b_j$ is the edge of F that is linked to the family \mathcal{F}_j and $|\mathcal{F}_j| \leq 2$, then we add only one of a_i or b_i to J_F .

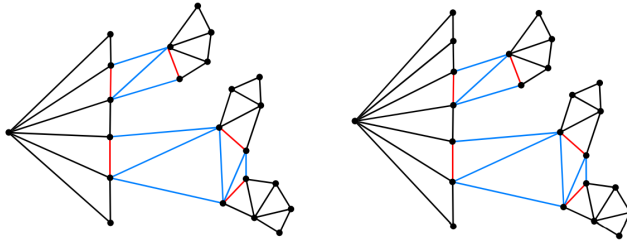


Figure 3. A fan without property \mathcal{J} (left) and a fan with property \mathcal{J} (right)

We say that the fan F has property \mathcal{J} if none of the linked sets J_F of F is a dominating set of F (see Figure 3).

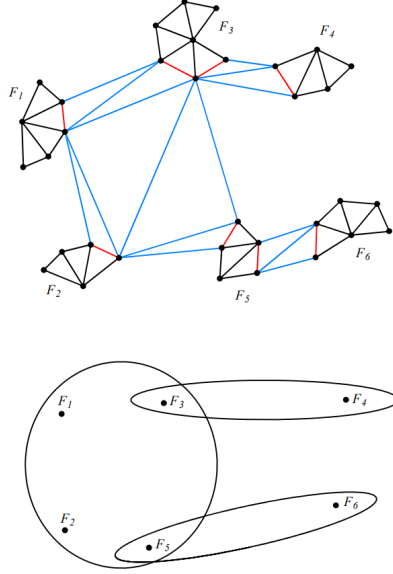


Figure 4. The hypergraph \mathcal{H} corresponding to the fans F_1, \dots, F_6

Let $\mathcal{F} = \{F_1, \dots, F_k\}$ be a family of vertex disjoint fans satisfying $|V(F_i)| \geq 5$ for all $i \in [k]$. Let \mathcal{H} be a hypergraph having vertex set $V(\mathcal{H}) = \mathcal{F}$ and edge set $E(\mathcal{H}) = \{e_1, e_2, \dots, e_m\}$ satisfying the following properties:

- (a) $\bigcup_{i=1}^m e_i = \mathcal{F}$;
- (b) $|e_i \cap e_j| \leq 1$ for $1 \leq i < j \leq m$;
- (c) The hypergraph \mathcal{H} contains no hypercycle, that is, \mathcal{H} is a linear hypertree where the vertices of \mathcal{H} correspond to the fans F_i of \mathcal{F} for $i \in [k]$ (see Figure 4).

We call \mathcal{H} the linear hypertree associated with the family $\mathcal{F} = \{F_1, \dots, F_k\}$ of fans. Let G be the graph obtained from the vertex disjoint union of the fans F_1, \dots, F_k as follows. For each edge $e_i = \{F_{i_1}, \dots, F_{i_j}\}$, we add edges joining the fans in the family $\mathcal{F}_i = \{F_{i_1}, \dots, F_{i_j}\}$ in such a way that the fans in \mathcal{F}_i are linked and resulting linked mop consists of these added edges and the linked edges from each fan in the family \mathcal{F}_i , for all $i \in [m]$. Further, we construct the graph G in such a way that every fan has property \mathcal{J} . Moreover, for each $F_i \in \mathcal{F}$, if F_i is linked in the families \mathcal{F}_j and $\mathcal{F}_{j'}$ via the linked edges $y_j z_j$ and $y_{j'} z_{j'}$, respectively, then $y_j z_j \neq y_{j'} z_{j'}$. Let $\mathcal{G}(\mathcal{F}, \mathcal{H})$ be the class of all such graphs G constructed in this manner from the family of fans $\mathcal{F} = \{F_1, \dots, F_k\}$ and the linear hypertree \mathcal{H} associated with the family \mathcal{F} .

We will prove that every graph G in the class $\mathcal{G}(\mathcal{F}, \mathcal{H})$ is a mop satisfying $\gamma_{\text{pr2}}(G) = \gamma_{\text{pr}}(G) = 2\gamma(G)$.

Theorem 5. *If $G \in \mathcal{G}(\mathcal{F}, \mathcal{H})$, then G is a mop with $\gamma_{\text{pr2}}(G) = 2\gamma(G)$.*

Proof. Let $\mathcal{F} = \{F_1, \dots, F_k\}$ be a family of fans satisfying $|V(F_i)| \geq 5$ for all $i \in [k]$, and let \mathcal{H} be the linear hypertree associated with the family \mathcal{F} . Let x_i be the center of the fan F_i for $i \in [k]$. Let $G \in \mathcal{G}(\mathcal{F}, \mathcal{H})$. We proceed further by proving two claims.

Claim 1. *The graph G satisfies $\gamma_{\text{pr2}}(G) = 2\gamma(G)$.*

Proof. The center of the fans dominate the graph G , that is, $\{x_1, x_2, \dots, x_k\}$ is a dominating set of G . Thus, $\gamma(G) \leq k$, implying by Observation 1 that $\gamma_{\text{pr2}}(G) \leq 2\gamma(G) \leq 2k$. Hence it suffices for us to show that $\gamma_{\text{pr2}}(G) \geq 2k$. Among all γ_{pr2} -sets of G , let S be chosen so that $|S \cap \{x_1, x_2, \dots, x_k\}|$ is maximum.

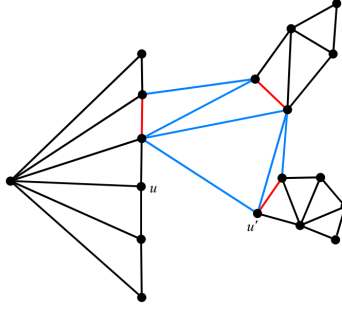


Figure 5. The vertices u and u'

We show that $\{x_1, x_2, \dots, x_k\} \subseteq S$. Suppose, to the contrary, that $x_i \notin S$ for some $i \in [k]$. If $S \cap V(F_i) \subseteq J_{F_i}$, then, by property \mathcal{J} , the set S does not dominate F_i , contradicting the fact that S is a semi-PD-set of G . Hence there exists a vertex $u \in S \cap V(F_i)$ such that $u \notin J_{F_i}$ (see Figure 5).

Let u' be the vertex that is S -paired with u . If $u' \in V(F_i)$, then $S' = (S \setminus \{u\}) \cup \{x_i\}$ is a semi-PD-set of G with u' S' -paired with x_i , and with the S' -pairings of all other pairs of vertices in S' unchanged from their S -pairings. We note that $|S'| = |S|$, and so S' is a γ_{pr2} -set of G . However, $|S' \cap \{x_1, x_2, \dots, x_k\}| > |S \cap \{x_1, x_2, \dots, x_k\}|$, contradicting our choice of the set S . Hence, $u' \notin V(F_i)$. Since $u \notin J_{F_i}$ and u' and u are S' -paired, it follows that $d_G(u', x_i) = 2$. As before, $S' = (S \setminus \{u\}) \cup \{x_i\}$ is a semi-PD-set of G with u' S' -paired with x_i , and with the S' -pairings of all other pairs of vertices in S' unchanged from their S -pairings. Thus, S' is a γ_{pr2} -sets of G and $|S' \cap \{x_1, x_2, \dots, x_k\}| > |S \cap \{x_1, x_2, \dots, x_k\}|$, a contradiction.

Hence, $\{x_1, x_2, \dots, x_k\} \subseteq S$. Let x'_i be the S -partner of x_i for $i \in [k]$, and so $d_G(x_i, x'_i) \leq 2$. The central vertices of the fans are pairwise at distance at least 3 apart, that is, $d_G(x_i, x_j) \geq 3$ for all i and j where $1 \leq i < j \leq k$. This implies that $x'_i \neq x'_j$ for $1 \leq i < j \leq k$. Therefore, $\{x'_1, x'_2, \dots, x'_k\} \subseteq S$ and $\{x_1, x_2, \dots, x_k\} \cap \{x'_1, x'_2, \dots, x'_k\} = \emptyset$, implying that $|S| \geq 2k$. \square

Claim 2. *The graph G is a mop.*

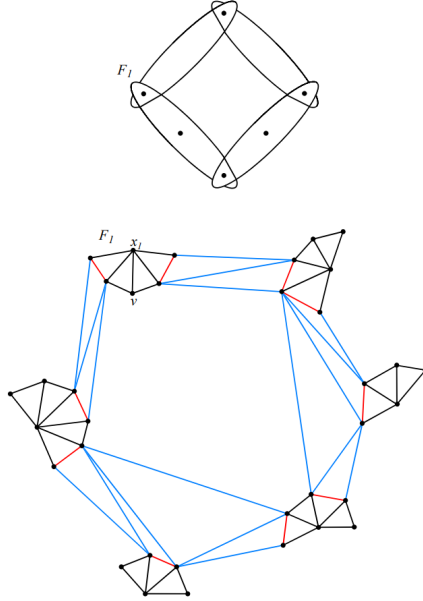


Figure 6. The corresponding hypergraph which is a hypercycle

Proof. Suppose, to the contrary, that G is not a mop. By construction, G is a planar graph and all interior faces of G are triangles. Thus, G has a vertex v which does not lie on the outer face. Renaming vertices if necessary, we may assume that $v \in V(F_1) \setminus \{x_1\}$. Since v does not lie on the outer face, it follows that v is enclosed by maximal outerplanar subgraphs corresponding to the edges e_1, e_2, \dots, e_d of \mathcal{H} where $F_1 \in e_1 \cap e_d$ and $e_i \cap e_{i+1} \neq \emptyset$ for $i \in [d-1]$. If $d \geq 3$, then \mathcal{H} has a hypercycle (see Figure 6). Thus, $d = 2$. In this case, v still lies on the outer face if $e_1 \cap e_2 = \{F_1\}$. Thus, $|e_1 \cap e_2| \geq 2$, contradicting the fact that \mathcal{H} is linear. \square

The proof of Theorem 5 now follows from Claims 1 and 2. \square

5. Mops G satisfying $\gamma_{\text{pr2}}(G) = \gamma(G)$

In this section, we will show that if G is a mop satisfying $\gamma_{\text{pr2}}(G) = 2\gamma(G)$, then $G \in \mathcal{G}(\mathcal{F}, \mathcal{H})$.

Theorem 6. *If G is a mop satisfying $\gamma_{\text{pr2}}(G) = 2\gamma(G)$, then $G \in \mathcal{G}(\mathcal{F}, \mathcal{H})$.*

Proof. Let G be a mop satisfying $\gamma_{\text{pr2}}(G) = 2\gamma(G) = 2k$. If $k = 1$, then the graph G contains a dominating vertex x (that is adjacent to every other vertex in G , and $G = G[N_G[x]]$ is a fan centered at x). In this case, trivially $G \in \mathcal{G}(\mathcal{F}, \mathcal{H})$. Hence we may assume that $k \geq 2$. Let $S = \{x_1, x_2, \dots, x_k\}$ be a γ -set of G . Since $\gamma_{\text{pr2}}(G) = 2k$,

it follows that $d_G(x_i, x_j) \geq 3$ for all $1 \leq i < j \leq k$. Thus, $N_G[x_i] \cap N_G[x_j] = \emptyset$. Clearly, $G[N_G[x_i]]$ is a fan centered at x_i . Let $F_i = G[N_G[x_i]]$ for $i \in k$ and let $\mathcal{F} = \{F_i : i \in [k]\}$. Further, let $V(F_i) = \{x_i, x_i^1, x_i^2, \dots, x_i^{\ell_i}\}$ where the vertices are labelled clockwise (see Figure 7). Moreover, we let $P(F_i) = x_i^1 x_i^2 \dots x_i^{\ell_i}$ be the path

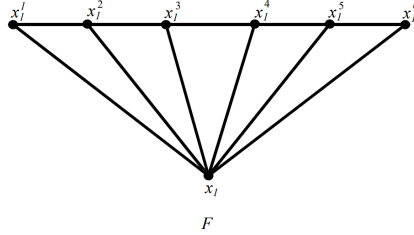


Figure 7. The clockwise labelling

$F_i - x_i$. Since S is a dominating set and $N_G[x_i] \cap N_G[x_j] = \emptyset$, it follows that S is a packing of G . Moreover since G is connected, there are edges of G that join vertices in one fan F_i to a different fan F_j for some i and j where $1 \leq i < j \leq k$. We proceed further by proving a series of claims establishing properties of the graph G .

Claim 3. $\deg_G(x_i) = \deg_{F_i}(x_i)$ for all $i \in [k]$.

Proof. Suppose that $\deg_G(x_i) > \deg_{F_i}(x_i)$ for all $i \in [k]$. Thus, the vertex x_i is adjacent to a vertex of F_j for some j with $i \neq j$ and $j \in [k]$, implying that $d_G(x_i, x_j) \leq 2$, contradicting our earlier observation that the set S is a packing in G . \square

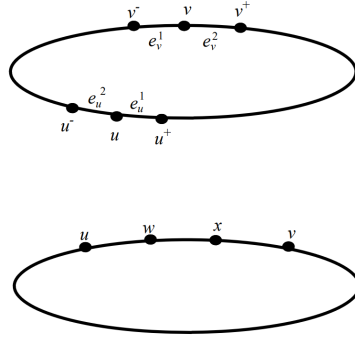


Figure 8. The edges $e_v^1 = v^-v$ and $e_v^2 = vv^+$ (top) and the path $u\overrightarrow{C}v = uwxv$ (bottom)

By Claim 3, the vertex x_i is not adjacent to any vertex of F_j for all $i, j \in [k]$ where $i \neq j$. Before we prove the following claims, we introduce some additional terminology and notation. Let C be the unique Hamiltonian cycle of G . We write \overrightarrow{C} to indicate the clockwise orientation on C . Moreover, the *successor* of a vertex v on \overrightarrow{C} we denote

by v^+ and the *predecessor* we denote by v^- . Observe that every vertex v is incident to exactly two Hamiltonian edges v^-v and vv^+ . For notational convenience, we denote $e_v^1 = v^-v$ and $e_v^2 = vv^+$. For two vertices u and v on C , we use $u\overrightarrow{C}v$ to indicate the (u, v) -path that follows the orientation on \overrightarrow{C} and all edges of the path are Hamiltonian edges (see Figure 8). A cycle in a mop is *alternating* if its edges alternate between Hamiltonian edges and diagonal.

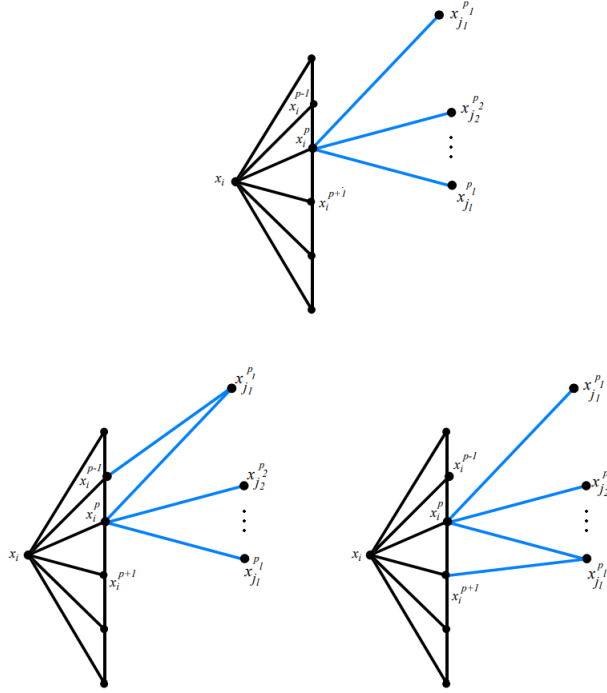


Figure 9. The graph when $x_i^{p-1}x_{j_1}^{p_1} \in E(G)$ or $x_i^{p+1}x_{j_\ell}^{p_\ell} \in E(G)$

Claim 4. Let x_i^p be a vertex of F_i which is adjacent to vertices $x_{j_1}^{p_1}, x_{j_2}^{p_2}, \dots, x_{j_\ell}^{p_\ell} \in V(G) \setminus V(F_i)$ where these vertices occur in the clockwise orientation on \overrightarrow{C} . (It is possible that $j_s = j_{s+1}$ for some s , implying that in this case the vertices $x_{j_s}^{p_s}$ and $x_{j_{s+1}}^{p_{s+1}}$ are in the same fan). Then, $x_i^{p-1}x_{j_1}^{p_1} \in E(G)$ or $x_i^{p+1}x_{j_\ell}^{p_\ell} \in E(G)$ (see Figure 9).

Proof. Suppose $x_i^{p-1}x_{j_1}^{p_1} \notin E(G)$ and $x_i^{p+1}x_{j_\ell}^{p_\ell} \notin E(G)$. Since G is 2-connected, there exists an $(x_i^{p-1}, x_{j_1}^{p_1})$ -path or an $(x_{j_\ell}^{p_\ell}, x_i^{p+1})$ -path, neither of which contains the vertex x_i^p . Suppose, without loss of generality, that there exists an $(x_i^{p-1}, x_{j_1}^{p_1})$ -path P that does not contain the vertex x_i^p (see Figure 10). Clearly, $G' = G[V(P) \cup \{x_i^p\}]$ is a 2-connected subgraph. By Lemma 1, the graph G' is a mop. Because $x_i^{p-1}x_{j_1}^{p_1} \notin E(G)$, it follows that x_i^p is adjacent to a vertex in $V(P) \setminus \{x_i^{p-1}, x_{j_1}^{p_1}\}$, contradicting the fact

that $x_{j_1}^{p_1}$ is the first vertex that x_i^p is adjacent to on $C - V(F_i)$. Thus, $x_i^{p-1}x_{j_1}^{p_1} \in E(G)$. The case $x_i^{p+1}x_{j_\ell}^{p_\ell} \in E(G)$ can be proved similarly and this completes the proof. \square

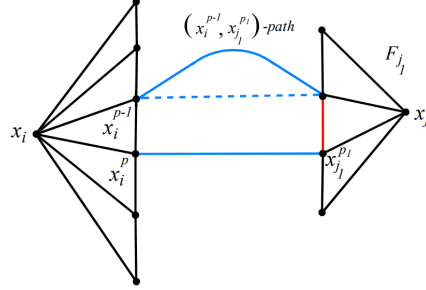


Figure 10. An $(x_i^{p-1}, x_{j_1}^{p_1})$ -path P that does not contain the vertex x_i^p

By Claim 4, renaming vertices if necessary, we may assume that $x_i^{p-1}x_{j_1}^{p_1} \in E(G)$. With this assumption, we have the following claim.

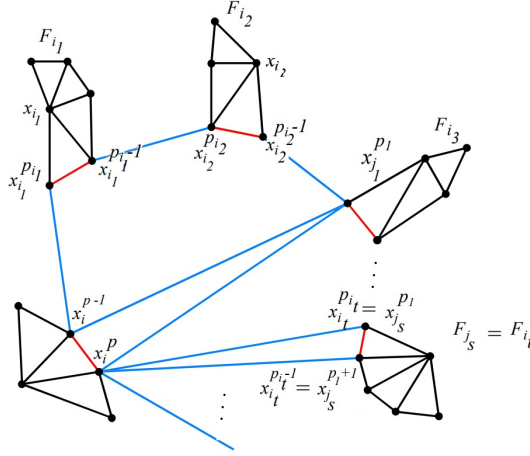


Figure 11. The mop induced by $\{x_i^p, x_i^{p-1}, x_{i_1}^{p_{i_1}}, x_{i_1}^{p_{i_1}-1}, x_{i_2}^{p_{i_2}}, x_{i_2}^{p_{i_2}-1}, \dots, x_{i_t}^{p_{i_t}}, x_{i_t}^{p_{i_t}-1}\}$

Claim 5. If $x_i^{p-1}x_{j_1}^{p_1} \in E(G)$, then there exist fans $F_{i_1}, F_{i_2}, \dots, F_{i_t}$ such that $F_{i_t} = F_{j_s}$ for some $s \in [\ell]$ such that, for each fan F_{i_q} , there are exactly two vertices $x_{i_q}^{p_{i_q}}$ and $x_{i_q}^{p_{i_q}-1}$ of F_{i_q} such that $\{x_i^p, x_i^{p-1}, x_{i_1}^{p_{i_1}}, x_{i_1}^{p_{i_1}-1}, x_{i_2}^{p_{i_2}}, x_{i_2}^{p_{i_2}-1}, \dots, x_{i_t}^{p_{i_t}}, x_{i_t}^{p_{i_t}-1}\}$ induced a mop (see Figure 11).

Proof. We will find an alternating cycle by the following method. We start the cycle from the path $x_i^p x_i^{p-1}$. From the vertex x_i^{p-1} , we follow the Hamiltonian edge $e_{x_i^{p-1}}^2$.

Since S is a packing in G , the edge $e_{x_i^{p-1}}^2$ is incident to a vertex of some fan in \mathcal{F} , F_{i_1} say. Because $d_G(x_i, x_j) \geq 3$ for all $1 \leq i < j \leq k$, the edge $e_{x_i^{p-1}}^2$ is incident to a vertex $x_{i_1}^{p_{i_1}}$ which is not the center of F_{i_1} (see Figure 12). We first construct the

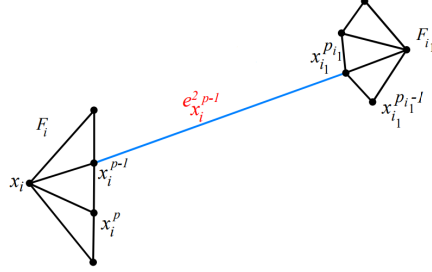


Figure 12. The edge $e_{x_i^{p-1}}^2$

alternating path

$$x_i^p x_i^{p-1} x_{i_1}^{p_{i_1}} x_{i_1}^{p_{i_1}-1}.$$

From $x_{i_1}^{p_{i_1}-1}$, again, we follow the Hamiltonian edge $e_{x_{i_1}^{p_{i_1}-1}}^2$. If $x_{i_1}^{p_{i_1}-1}$ is not adjacent to any vertex in $V(G) \setminus V(F_{i_1})$, then, by the orientation,

$$e_{x_{i_1}^{p_{i_1}-1}}^2 = x_{i_1}^{p_{i_1}-1} x_{i_1}^{p_{i_1}}$$

but $x_{i_1}^{p_{i_1}}$ has occurred once on the edge $e_{x_i^{p-1}}^2$. This contradicts the fact that C is a Hamiltonian cycle. Thus, $e_{x_{i_1}^{p_{i_1}-1}}^2$ is incident to a vertex $x_{i_2}^{p_{i_2}}$ which is not the center of a fan $F_{i_2} \in \mathcal{F} \setminus \{F_{i_1}\}$. We now add $x_{i_2}^{p_{i_2}}$ and $x_{i_2}^{p_{i_2}-1}$ to the path (see Figure 13), yielding the alternating path

$$x_i^p x_i^{p-1} x_{i_1}^{p_{i_1}} x_{i_1}^{p_{i_1}-1} x_{i_2}^{p_{i_2}} x_{i_2}^{p_{i_2}-1}.$$

Similarly, the vertex $x_{i_2}^{p_{i_2}-1}$ is adjacent to a vertex $x_{i_3}^{p_{i_3}} \in V(F_{i_3})$ via the Hamiltonian edge $e_{x_{i_2}^{p_{i_2}-1}}^2$ where $x_{i_3}^{p_{i_3}}$ is not the center of F_{i_3} and we add $x_{i_3}^{p_{i_3}}$ and $x_{i_3}^{p_{i_3}-1}$ to the path. We keep applying this method until we meet the fan F_i again. Because C is the (unique) Hamiltonian cycle in the mop G , by this choice, the path will return to F_i . If the path contains a Hamiltonian edge that joins the vertex $x_{i_t}^{p_{i_t}-1}$ to the vertex in F_i which is not x_i^p , then x_i^p does not lie on the outer face, contradicting the fact that G is an outerplanar graph. Thus, $x_{i_t}^{p_{i_t}-1}$ is adjacent to x_i^p via $e_{x_{i_t}^{p_{i_t}-1}}^2$. Thus,

$$C': x_i^p x_i^{p-1} x_{i_1}^{p_{i_1}} x_{i_1}^{p_{i_1}-1} x_{i_2}^{p_{i_2}} x_{i_2}^{p_{i_2}-1} \dots x_{i_t}^{p_{i_t}} x_{i_t}^{p_{i_t}-1} x_i^p$$

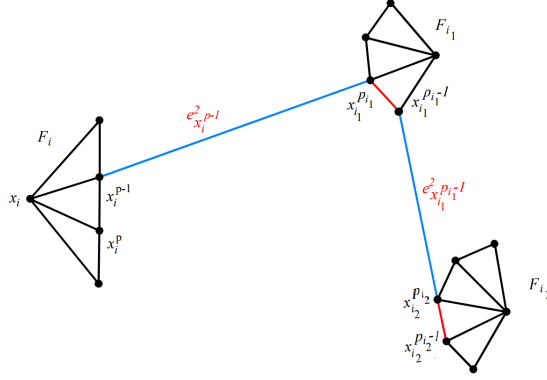


Figure 13. The alternating path $x_i^p x_i^{p-1} x_{i_1}^{p_{i_1}-1} x_{i_1}^{p_{i_1}-1} x_{i_2}^{p_{i_2}-1}$

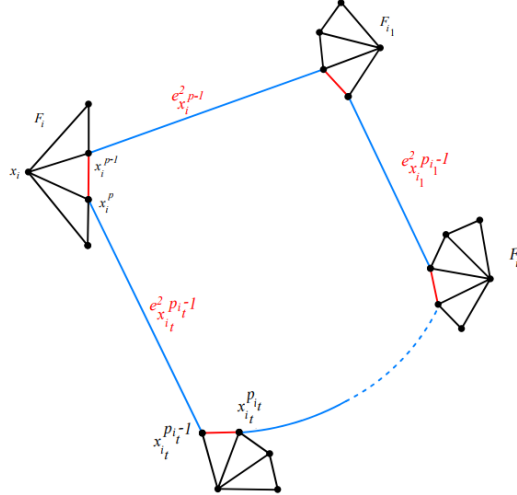


Figure 14. The alternating cycle $C' : x_i^p x_i^{p-1} x_{i_1}^{p_{i_1}-1} x_{i_1}^{p_{i_1}-1} x_{i_2}^{p_{i_2}-1} x_{i_2}^{p_{i_2}-1} \dots x_{i_t}^{p_{i_t}-1} x_{i_t}^{p_{i_t}-1} x_i^p$

is an alternating cycle as required (see Figure 14). Since $G[V(C')]$ induced a 2-connected subgraph of G , by Lemma 1 the graph $G[V(C')]$ is a mop. This completes the proof of Claim 5. \square

We now use Claim 5 to construct a hypergraph. For $F_{i_1}, F_{i_2}, \dots, F_{i_\ell}$, if there are two vertices $x_{i_j}^{p_j}, x_{i_j}^{p_j-1} \in V(F_{i_j})$ for $j \in [\ell]$ such that $G[\{x_{i_j}^{p_j}, x_{i_j}^{p_j-1} : j \in [\ell]\}]$ is a mop, then we say that $F_{i_1}, F_{i_2}, \dots, F_{i_\ell}$ are *linked*. In the following, we construct the corresponding hypergraph H_G of a mop G satisfying $2\gamma(G) = \gamma_{\text{pr2}}(G)$. Let $V(H_G) = \{F_i : i \in [k]\}$ and $E(H_G) = \{e = \{F_{i_1}, F_{i_2}, \dots, F_{i_r}\} : F_{i_1}, F_{i_2}, \dots, F_{i_r} \text{ are linked in } G\}$. We call H_G the *linked hypergraph* of G . Let B be a mop and let C_B

denote the (unique) Hamiltonian cycle in B , and let $a, b \in V(B)$. For notational convenience, we let aBb denote the path $a\overrightarrow{CB}b$.

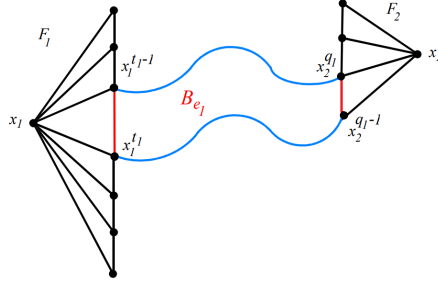


Figure 15. The mop B_{e_1}

Claim 6. If G is a mop with H_G as the linked hypergraph, then H_G is a linear hypertree.

Proof. We first show that H_G is linear. Suppose to the contrary that there exists two hyperedges $e_1, e_2 \in E(H_G)$ such that $|e_1 \cap e_2| \geq 2$. Renaming vertices if necessary, we may assume that $\{F_1, F_2\} \subseteq e_1 \cap e_2$. Thus, F_1 and F_2 are linked twice by two different maximal outerplanar subgraphs. So, there exist $x_1^{t_1}, x_1^{t_1-1} \in V(F_1)$ and $x_2^{q_1}, x_2^{q_1-1} \in V(F_2)$ such that, for some $W_1 \subseteq V(G) \setminus (V(F_1) \cup V(F_2))$, the graph $G[\{x_1^{t_1}, x_1^{t_1-1}, x_2^{q_1}, x_2^{q_1-1}\} \cup W_1]$ is a mop corresponding to the hyperedge e_1 and we call this mop B_{e_1} (see Figure 15).

Similarly, there exist $x_1^{t_2}, x_1^{t_2-1} \in V(F_1)$ and $x_2^{q_2}, x_2^{q_2-1} \in V(F_2)$ such that the induce subgraph $G[\{x_1^{t_2}, x_1^{t_2-1}, x_2^{q_2}, x_2^{q_2-1}\} \cup W_2]$ for some $W_2 \subseteq V(G) \setminus (V(F_1) \cup V(F_2))$ is a mop corresponding to the edge e_2 and we call this mop B_{e_2} . Without loss of generality, let $t_1 < t_2$.

First, we may assume that $q_1 < q_2$. Recall that aGb is a path from the vertex a to the vertex b passing Hamiltonian edges of G in clockwise direction. Clearly, x_1 is enclosed by the cycle

$$x_1^{t_2-1} B_{e_2} x_2^{q_2} F_2 x_2 F_2 x_2^{q_1-1} B_{e_1} x_1^{t_1} F_1 x_1^{t_2-1} \text{ (see Figure 16)}$$

or x_2 is enclosed by the cycle

$$x_2^{q_2-1} B_{e_2} x_1^{t_2} F_1 x_1 F_1 x_1^{t_1-1} B_{e_1} x_1^{q_1} F_2 x_2^{q_2-1} \text{ (see Figure 17)}.$$

Therefore, x_1 or x_2 does not lie on the outer face, contradicting the fact that G is an outerplanar graph. Thus, we may assume that $q_2 < q_1$. Clearly, x_1 and x_2 are enclosed by the cycle

$$x_1^{t_2-1} B_{e_2} x_2^{q_2} F_2 x_2^{q_1-1} B_{e_1} x_1^{t_1} F_1 x_1^{t_2-1} \text{ (see Figure 18)}$$

or $x_1^{t_1}, x_1^{t_2-1}, x_2^{q_2}, x_2^{q_1-1}$ are enclosed by the cycle

$$x_1^{t_1-1} B_{e_1} x_2^{q_1} F_2 x_2 F_2 x_2^{q_2-1} B_{e_2} x_1^{t_2} F_1 x_1 F_1 x_1^{t_1-1} \text{ (see Figure 19)}.$$

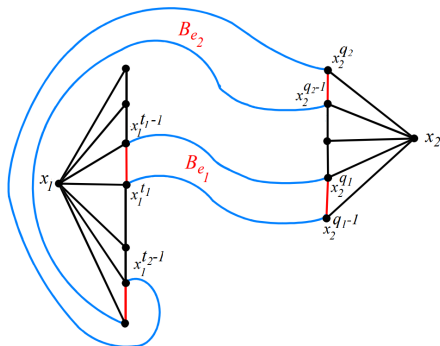


Figure 16. The cycle that encloses x_1

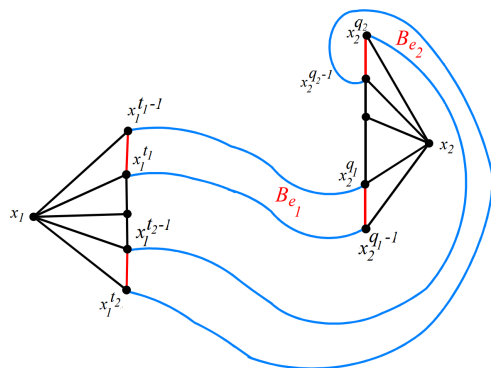


Figure 17. The cycle that encloses x_2

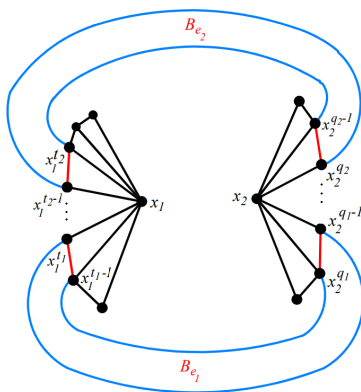


Figure 18. The cycle that encloses x_1 and x_2

Therefore, x_1 and x_2 or $x_1^{t_1}, x_1^{t_2-1}, x_2^{q_2}, x_2^{q_1-1}$ do not lie on the outer face, contradicting that fact that G is an outerplanar graph. Hence, H_G is linear. We show next that

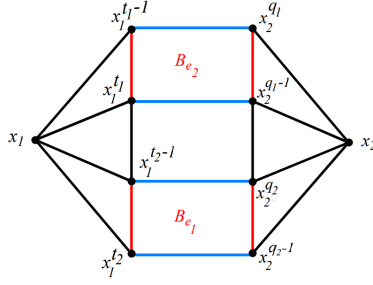


Figure 19. The cycle that encloses $x_1^{t_1}, x_1^{t_2-1}, x_2^{q_2}, x_2^{q_1-1}$

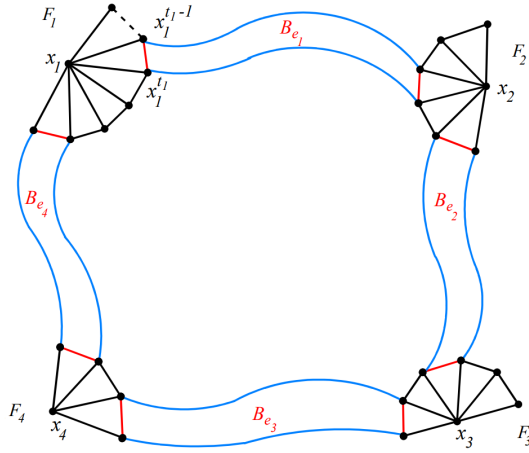


Figure 20. The cycle that encloses $x_1^{t_1}$

H_G is a hypertree. Suppose, to the contrary, that H_G is not a hypertree. Thus, there exists a hypercycle C_{H_G} which is a subhypergraph of H_G . We may let $E(C_{H_G}) = \{e_1, e_2, \dots, e_\ell\}$, $\{F_{i+1}\} = e_i \cap e_{i+1}$ for all $i \in [\ell - 1]$ and $\{F_1\} = e_\ell \cap e_1$. By the definition of e_i , there exists the maximal outerplanar subgraph B_{e_i} which contains a fan from each of F_i and F_{i+1} , and a maximal outerplanar subgraph B_{e_ℓ} which contains a fan from each of F_ℓ and F_1 . Renaming vertices if necessary, we may let $x_i^{t_i-1}, x_i^{t_i} \in V(B_{e_i}) \cap V(F_i)$ and $x_{i+1}^{p_{i+1}-1}, x_{i+1}^{p_{i+1}} \in V(B_{e_i}) \cap V(F_{i+1})$ and $x_\ell^{t_\ell-1}, x_\ell^{t_\ell} \in V(B_{e_\ell}) \cap V(F_\ell)$ and $x_1^{p_1}, x_1^{p_1-1} \in V(B_{e_\ell}) \cap V(F_1)$. Since x_1, x_2, \dots, x_ℓ are in the Hamiltonian cycle C , the vertex $x_1^{t_1}$ is enclosed by the cycle

$$x_1 F_1 x_1^{t_1-1} B_{e_1} x_2^{p_1} F_2 x_2 F_2 x_2^{t_2-1} B_{e_2} x_3^{p_2} \dots x_i^{t_i-1} B_{e_i} x_{i+1}^{p_i} F_{i+1} x_{i+1} F_{i+1} \dots x_\ell^{t_\ell-1} B_{e_\ell} x_1^{p_\ell} F_1 x_1,$$

(see Figure 20) contradicting that fact that G is an outerplanar graph. This establishes Claim 6. \square

Claim 7. *Each F_i has at least five vertices.*

Proof. Suppose, to the contrary, that $|V(F_i)| \leq 4$. Let $F_j \in e \setminus \{F_i\}$ be a fan which is linked with F_i in e . We pair x_j with x_i^2 and each vertex $x_q \in S \setminus \{x_j, x_i\}$ with its own private neighbor x'_q in F_q . Thus, $(S \setminus \{x_i\}) \cup \{x_i^2\} \cup \{x'_q : q \in [k] \setminus \{i\}\}$ is a semi-PD-set of G . Clearly, $\gamma_{\text{pr2}}(G) < 2k$, a contradiction. Therefore, $|V(F_i)| \geq 5$. \square

The following claim follows readily from the property that G is a mop.

Claim 8. *If F_i is linked in e_j and $e_{j'}$ by the edges $x_i^{t-1}x_i^t$ and $x_i^{p-1}x_i^p$, respectively, then $x_i^{t-1}x_i^t \neq x_i^{p-1}x_i^p$.*

Finally, we need only prove that every fan F_i has property \mathcal{J} defined in Section 4.

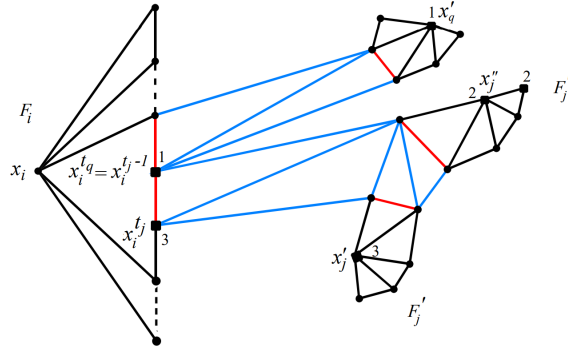


Figure 21. Pairing when $x_i^{t_j-1} = x_i^{t_q}$ and $x_i^{t_q}$ is already paired with x'_q

Claim 9. *Every fan F_i has property \mathcal{J} .*

Proof. Suppose, to the contrary, that there exists a linked set J_{F_i} such that J_{F_i} dominates F_i . Let $e_1, e_2, \dots, e_p \in E(H_G)$ be all subfamilies of \mathcal{F} that are linked to F_i . Renaming hyperedges if necessary, we assume that $|e_j| = 2$ for all $j \in [r]$ and $|e_i| \geq 3$ for all $r < j \leq p$. We, further, let $x_i^{t_j-1}x_i^{t_j}$ be the linked edge of F_i in e_j . For the case when $1 \leq j \leq r$, we let $\{F'_j\} = e_j \setminus \{F_i\}$. By the construction on J_{F_i} in Section 4(b), we let $\{v_j\} = \{x_i^{t_j-1}, x_i^{t_j}\} \cap J_{F_i}$. For the case when $r < j \leq p$, we focus on the clockwise orientation on B_{e_j} , and we let F'_j and F''_j be the fans in $e_j \setminus \{F_i\}$ that occur consecutively before and after F_i , respectively. Clearly,

$$J_{F_i} = \{v_j : j \in [r]\} \cup \{x_i^{t_j-1}, x_i^{t_j} : r < j \leq p\}.$$

We let x'_j and x''_j be the centers of F'_j and F''_j , respectively. We pair x'_j with v_j for all $j \in [r]$ and we pair $x_i^{t_j}$ and $x_i^{t_j-1}$ with x'_j and x''_j , respectively. We remark that if $x_i^{t_j-1} = x_i^{t_q}$ for some $q \in [\ell]$ and $x_i^{t_q}$ is already paired with x'_q , then we pair

x_j'' with its own private neighbor in F_j'' and still pair $x_i^{t_j}$ with x_j' (see Figure 21). We note from the figure that the vertices that are assigned the same number are paired. We let Z be the set of these private neighbors. Observe that, if all vertices in $\{v_j: j \in [r]\} \cup \{x_i^{t_j-1}, x_i^{t_j}: r < j \leq p\}$ are distinct, then $Z = \emptyset$. Moreover, $|J_{F_i} \cup Z| = r + 2(p - r) = 2p - r$. We let $X = \{x_j': j \in [p]\} \cup \{x_j'': r < j \leq p\}$.

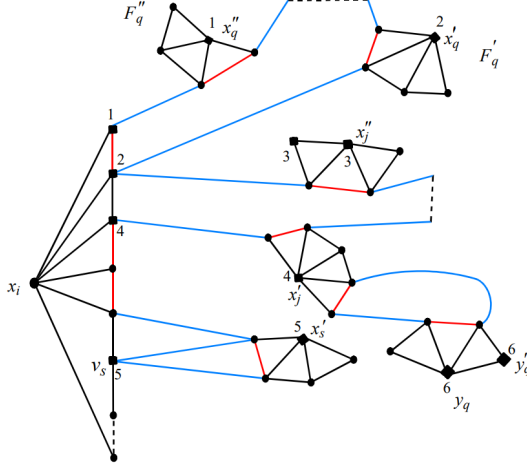


Figure 22. Pairing y_q with own neighbor y'_q

Now we have that $|S \setminus (X \cup \{x_i\})| = k - (r + 2(p - r) + 1) = k - 2p + r - 1$. For notational convenience, we rename the vertices in $S \setminus (X \cup \{x_i\})$ to be y_1, y_2, \dots, y_ℓ where $\ell = k - 2p + r - 1$. We pair each vertex $y_q \in S \setminus (X \cup \{x_i\})$ with its own private neighbor y'_q in the same fan (see Figure 22). We let $Y = \{y'_q: q \in [\ell]\}$, and note that $(S \setminus \{x_i\}) \cup Y \cup (J_{F_i} \cup Z)$ is a semi-PD-set of G . Moreover,

$$\gamma_{\text{pr2}}(G) \leq |(S \setminus \{x_i\}) \cup Y \cup (J_{F_i} \cup Z)| = (k - 1) + \ell + 2p - r = 2k - 2,$$

contradicting the fact that $\gamma_{\text{pr2}}(G) = 2\gamma(G)$. Thus, F_i has the property \mathcal{J} and this completes the proof of Claim 9. \square

By the properties of the graph G established in Claims 3–9, we infer that $G \in \mathcal{G}(\mathcal{F}, \mathcal{H})$. This completes the proof of Theorem 6. \square

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Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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