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Research Article

A classification of graphs through quadratic embedding constants and clique graph insights

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Abstract: The quadratic embedding constant (QEC) of a graph G is a new numeric invariant, which is defined in terms of the distance matrix and is denoted by $\mathrm{QEC}(G)$. By observing graph structure of the maximal cliques (clique graph), we show that a graph G with $\mathrm{QEC}(G) < -1/2$ admits a "cactus-like" structure. We derive a formula for the quadratic embedding constant of a graph consisting of two maximal cliques. As an application we discuss characterization of graphs along the increasing sequence of $\mathrm{QEC}(P_d)$, where P_d is the path on d vertices. In particular, we study graphs G satisfying $\mathrm{QEC}(G) < \mathrm{QEC}(P_5)$.

Keywords: cactus-like graph, clique graph, distance matrix, quadratic embedding constant.

AMS Subject classification: 05C50, 05C12, 05C76

1. Introduction

In the recent paper [22] the *quadratic embedding constant* (*QE constant* for short) of a finite connected graph $G = (V, E)$ with $|V| \geq 2$ is defined by

$$\mathrm{QEC}(G) = \max\{\langle f, Df \rangle \,;\, f \in C(V),\, \langle f, f \rangle = 1,\, \langle 1, f \rangle = 0\}, \tag{1.1}$$

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where $C(V)$ is the space of all $\mathbb{R}$-valued functions on V, $\in C(V)$ the constant function taking value 1, and $\langle\cdot,\cdot\rangle$ the canonical inner product. The QE constant is profoundly related to the quadratic embedding of a graph in a Euclidean space [10, 12, 25, 26] or more generally to Euclidean distance geometry [1, 3, 6, 14]. In fact, it is essential to note that a graph G admits a quadratic embedding in Euclidean space if and only if $\mathrm{QEC}(G) \le 0$. In recent years, the QE constant has garnered growing interest as a new numeric invariant of graphs. The QE constants of graphs of particular classes are known explicitly, see [5, 11, 17, 19–23], and the formulas in relation to graph operations are established in [16, 18]. Moreover, a table of the QE constants of graphs on $n \le 5$ vertices is available [22], where the value of the graph No. 12 on $n = 5$ vertices is wrong and the correction is found in [4].

With the above mentioned background, we are naturally led to the forward-thinking project of classifying graphs by means of the QE constants. In [4] we initiated an attempt to classify graphs along with $\mathrm{QEC}(P_d)$, the QE constant of the path P_d on $d \ge 2$ vertices, which forms an increasing sequence as

$$-1 = \mathrm{QEC}(P_2) < \mathrm{QEC}(P_3) < \cdots < \mathrm{QEC}(P_d) < \cdots \to -\frac{1}{2}\,. \tag{1.2}$$

In fact, it is known [17] that

$$\mathrm{QEC}(P_d) = -\left(1 + \cos\frac{\pi}{d}\right)^{-1}, \qquad d \ge 2. \tag{1.3}$$

In this paper, we study graphs $G = (V, E)$ with $\mathrm{QEC}(G) < -1/2$ by means of the *clique graph* $\Gamma(G)$. Here the clique graph is a graph $\Gamma(G) = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of maximal cliques of G and $\{H_1, H_2\} \in \mathcal{E}$ if and only if $H_1 \ne H_2$ and $H_1 \cap H_2 \ne \emptyset$. A key result is that the clique graph of a graph G with $\mathrm{QEC}(G) < -1/2$ is a tree. Then, combining the result on forbidden subgraphs [4], we conclude that a graph G with $\mathrm{QEC}(G) < -1/2$ consists of maximal cliques which form a "cactus-like" structure, for the precise statement see Theorem 1. As an application we discuss graphs G satisfying $\mathrm{QEC}(G) < \mathrm{QEC}(P_5)$.

It is noteworthy that the QE constant provides additional information to the distance spectra and raises interesting questions, for the distance spectra see e.g., [2, 3, 9, 15]. In fact, it is known [16] that $\delta_2(G) \le \mathrm{QEC}(G) < \delta_1(G)$, where $\delta_1(G)$ and $\delta_2(G)$ are the largest and the second largest eigenvalues of the distance matrix of G, respectively. It is straightforward to see that $\delta_2(G) = \mathrm{QEC}(G)$ holds if the distance matrix of G has a constant row sum (in some literatures, such a graph is called *transmission regular*). But the converse is not true as the paths P_n with even n are counter-examples [17]. In this aspect characterization of graphs satisfying $\delta_2(G) = \mathrm{QEC}(G)$ is an interesting question. On the other hand, the second largest eigenvalue $\delta_2(G)$ has been adopted for classifying graphs, in particular, the distance-regular graphs G with $\delta_2(G) \le 0$ are classified [13]. In the recent paper [7] the bicyclic graphs and the split graphs G with

$\delta_2(G) < -1/2$ are characterized. A detailed comparison is naturally expected to be very interesting and will appear elsewhere.

The paper is organized as follows. In Section 2 we assemble some basic notations for the QE constants. In Section 3 we examine some properties of the clique graph $\Gamma(G)$ and show a relation between the diameters of G and $\Gamma(G)$ (Propositions 5 and 6). In Section 4 we prove the main result (Theorem 1). In Section 5 we determine a graph with exactly two maximal cliques. In Section 6 we discuss graphs G satisfying $\mathrm{QEC}(G) < \mathrm{QEC}(P_5)$. In Appendix we derive a formula for the QE constant of a graph with exactly two maximal cliques.

2. Quadratic Embedding Constants

Throughout the paper a graph $G = (V, E)$ is a pair, where V is a non-empty finite set and E is a set of two-element subsets $\{x, y\} \subset V$. As usual, elements of V and E are respectively called a *vertex* and an *edge*. Two vertices $x, y \in V$ are called *adjacent* if $\{x, y\} \in E$, and we also write $x \sim y$. In that case we have $x \neq y$ necessarily.
For $s \geq 0$ a sequence of vertices $x = x_0, x_1, \dots, x_s = y \in V$ is called a *walk* connecting x and y of length s if they are successively adjacent:

$$x = x_0 \sim x_1 \sim \cdots \sim x_s = y. \tag{2.1}$$

A graph G is called *connected* if any pair of vertices are connected by a walk. In this paper a graph is always assumed to be connected unless otherwise stated. The length of a shortest walk connecting two vertices $x, y \in V$ is called the *graph distance* between x and y in G and is denoted by $d_G(x, y)$. A walk in (2.1) is called a *shortest path* connecting x and y if $s = d_G(x, y)$ holds. In that case $x_0, x_1, \dots, x_s$ are mutually distinct.

As is defined in (1.1), the *QE constant* of a graph $G = (V, E)$, denoted by $\mathrm{QEC}(G)$, is the conditional maximum of the quadratic function $\langle f, Df\rangle$ associated to the distance matrix $D = [d_G(x, y)]$ subject to two constraints $\langle f, f\rangle = 1$ and $\langle 1, f\rangle = 0$.
In this section we assemble some basic results, for more details see e.g., [4, 18, 22]. We first recall a computational formula based on the standard method of Lagrange's multipliers.

Proposition 1 ([22]). *Let D be the distance matrix of a graph $G = (V, E)$ on $n = |V|$ vertices with $n \geq 3$. Let $\mathcal{S}$ be the set of all stationary points (f, λ, μ) of*

$$\varphi(f, \lambda, \mu) = \langle f, Df\rangle - \lambda(\langle f, f\rangle - 1) - \mu\langle 1, f\rangle, \tag{2.2}$$

where $f \in C(V) \cong \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. Then we have

$$\mathrm{QEC}(G) = \max\{\lambda\,;\,(f, \lambda, \mu) \in \mathcal{S}\}.$$

In general, a graph $H = (V', E')$ is called a *subgraph* of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. If both G and H are connected, they have their own graph distances. If they coincide in such a way that

$$d_H(x, y) = d_G(x, y), \qquad x, y \in V',$$

we say that H is *isometrically embedded* in G. The next assertions are immediate from definition but useful.

Proposition 2 ([20, 21]). *Let $G = (V, E)$ be a graph and $H = (V', E')$ a subgraph.*

(1) *If H is isometrically embedded in G, then H is an induced subgraph of G.*

(2) *If H is an induced subgraph of G and*

$$\mathrm{diam}\,(H) = \max\{d_H(x, y)\,;\, x, y \in V'\} \le 2,$$

then H is isometrically embedded in G.

Proposition 3 ([22]). *Let $G = (V, E)$ and $H = (V', E')$ be two graphs with $|V| \ge 2$ and $|V'| \ge 2$. If H is isometrically embedded in G, we have*

$$\mathrm{QEC}(H) \le \mathrm{QEC}(G). \tag{2.3}$$

In particular, (2.3) holds if H is an induced subgraph of G and $\mathrm{diam}\,(H) \le 2$.

Since any graph $G = (V, E)$ with $|V| \ge 2$ contains at least one edge, it has a subgraph K_2 isometrically embedded in G. It then follows from Proposition 3 that

$$\mathrm{QEC}(G) \ge \mathrm{QEC}(K_2) = -1.$$

Moreover, we have the following assertion, see also Proposition 11.

Proposition 4 ([4]). *For a graph G we have* $\mathrm{QEC}(G) = -1$ *if and only if G is a complete graph.*

Thus, in order to determine $\mathrm{QEC}(G)$ of a graph G which is not a complete graph, it is sufficient to seek out the stationary points of $\varphi(f, \lambda, \mu)$ with $\lambda > -1$ and then to specify the maximum of λ appearing therein.

3. Clique Graphs

Most of this section follows a standard argument; however, to avoid ambiguity, we present some basic properties of clique graphs.

Let $G = (V, E)$ be a graph (always assumed to be finite and connected). For a non-empty subset $H \subset V$, the subgraph induced by H is denoted by $\langle H \rangle$. By definition the vertex set of $\langle H \rangle$ is H itself and two-element subset $\{x, y\} \subset H$ belongs to the edge set of $\langle H \rangle$ if and only if $\{x, y\} \in E$. A non-empty subset $H \subset V$ is called a *clique* of G if $\langle H \rangle$ is a complete graph. A clique is called *maximal* if it is maximal in the family of cliques with respect to the inclusion relation. Except for notation, 'clique' may also refer to the subgraph it induces.

Obviously, for a clique H_0 there exists a maximal clique H such that $H_0 \subset H$. In particular, for two vertices $a \sim b$ there exists a maximal clique containing $\{a, b\}$.

Lemma 1. *Let H_1 and H_2 be maximal cliques of a graph G such that $H_1 \neq H_2$. Then $H_1 \backslash H_2 \neq \emptyset$ and $H_2 \backslash H_1 \neq \emptyset$. Moreover, there exist $a \in H_1 \backslash H_2$ and $b \in H_2 \backslash H_1$ such that $a \not\sim b$.*

Proof. Suppose that $H_1 \backslash H_2 = \emptyset$ or $H_2 \backslash H_1 = \emptyset$. If the former occurs, we have $H_1 \subset H_2$. Since both H_1 and H_2 are maximal and $H_1 \neq H_2$ by assumption, we come to a contradiction.

For the second half of the assertion, suppose that any pair of $x \in H_1 \backslash H_2$ and $y \in H_2 \backslash H_1$ are adjacent. We will show that $H_1 \cup H_2$ is a clique. In fact, take a pair of distinct vertices $x, y \in H_1 \cup H_2$. If $x, y \in H_1$ or $x, y \in H_2$, they are adjacent since H_1 and H_2 are cliques. If otherwise, we have $x \in H_1 \backslash H_2$ and $y \in H_2 \backslash H_1$ or vice versa, and hence $x \sim y$ by assumption. Consequently, for any pair of distinct vertices $x, y \in H_1 \cup H_2$ we have $x \sim y$, namely, $H_1 \cup H_2$ becomes a clique. Since $H_1 \cup H_2$ contains H_1 and H_2 properly, we come to a contradiction. $\square$

For a graph $G = (V, E)$ let $\mathcal{V}$ be the set of all maximal cliques and $\mathcal{E}$ the set of two-element subsets $\{H_1, H_2\} \subset \mathcal{V}$ such that $H_1 \cap H_2 \neq \emptyset$. Then $\Gamma(G) = (\mathcal{V}, \mathcal{E})$ becomes a (in fact, connected) graph, which is called the *clique graph* of G. Accordingly, for two maximal cliques H_1 and H_2 of G we write $H_1 \sim H_2$ if $H_1 \neq H_2$ and $H_1 \cap H_2 \neq \emptyset$. For more information on the clique graph, see e.g., [24, 27].

Lemma 2. *The clique graph $\Gamma(G)$ of a graph G (always assumed to be connected) is connected.*

Proof. Let H_1, H_2 be two maximal cliques such that $H_1 \neq H_2$. By Lemma 1 we may choose $a \in H_1 \backslash H_2$ and $b \in H_2 \backslash H_1$. Since G is connected, there exists a walk connecting a and b, say,

$$a = x_0 \sim x_1 \sim \cdots \sim x_s = b,$$

where $s \geq 1$. For $1 \leq i \leq s$ take a maximal clique J_i containing $\{x_{i-1}, x_i\}$. Then $x_i \in J_i \cap J_{i+1}$ implies that $J_i = J_{i+1}$ or $J_i \sim J_{i+1}$. Moreover, it follows from $a = x_0 \in H_1 \cap J_1$ that $H_1 = J_1$ or $H_1 \sim J_1$. Similarly, $H_2 = J_s$ or $H_2 \sim J_s$. In any case, H_1 and H_2 are connected by a walk consisting of $J_1, J_2, \dots, J_s$. □

Example 1. For a complete graph K_n with $n \geq 1$, a path P_n with $n \geq 2$, and a cycle C_n with $n \geq 3$ we have

$$\Gamma(K_n) = K_1, \qquad \Gamma(P_n) = P_{n-1}, \qquad \Gamma(C_n) = C_n\,.$$

If every maximal clique of a graph G is K_2, the clique graph $\Gamma(G)$ is nothing else but the line graph of G. Examples of this type are $G = P_n$ and $G = C_n$.

Lemma 3. *For $d \geq 1$ let*

$$x_0 \sim x_1 \sim \cdots \sim x_d\,, \tag{3.1}$$

be a shortest path connecting x_0 and x_d, that is, $d(x_0, x_d) = d$. For $1 \leq i \leq d$ let H_i be a maximal clique containing $\{x_{i-1}, x_i\}$. Then,

$$H_1 \sim H_2 \sim \cdots \sim H_d \tag{3.2}$$

and $d(H_1, H_d) = d - 1$. Hence (3.2) is a shortest path connecting H_1 and H_d, and $H_1, H_2, \dots, H_d$ are mutually distinct.

Proof. For $1 \leq i \leq d-1$ we have $x_i \in H_i \cap H_{i+1}$ by definition, and hence $H_i = H_{i+1}$ or $H_i \sim H_{i+1}$. Suppose that $H_i = H_{i+1}$ occurs. Then $x_{i-1}, x_i, x_{i+1} \in H_i$ and these three vertices are mutually distinct because (3.1) is a shortest path. Since H_i is a clique, we have $x_{i-1} \sim x_{i+1}$, which contradicts to that (3.1) is a shortest path. Thus we obtain a walk as in (3.2).
We next prove that (3.2) gives rise to a shortest path. Let $s = d(H_1, H_d)$ and take a shortest path

$$H_1 = J_0 \sim J_1 \sim \cdots \sim J_s = H_d\,.$$

In that case we have

$$s \leq d - 1. \tag{3.3}$$

For $1 \leq i \leq s$ we take $y_i \in J_{i-1} \cap J_i$. Then $y_1 \sim y_2 \sim \cdots \sim y_s$. Moreover, since $x_0, y_1 \in H_1 = J_0$ we have $x_0 = y_1$ or $x_0 \sim y_1$. Similarly, $x_d = y_s$ or $x_d \sim y_s$. Thus we obtain a walk connecting x_0 and x_d whose length is $s-1$, s or $s+1$. Hence $d = d(x_0, x_d) \leq s+1$. Combining (3.3) we obtain $s = d-1$ and hence $d(H_1, H_d) = d-1$ as desired. □

Proposition 5. *Let G be a graph and $\Gamma(G)$ its clique graph. Then*

$$\mathrm{diam}\,(G) - 1 \leq \mathrm{diam}\,(\Gamma(G)). \tag{3.4}$$

Proof. It is sufficient to show the assertion for a graph G with $d = \mathrm{diam}\,(G) \ge 1$. We take a shortest path $x_0 \sim x_1 \sim \cdots \sim x_d$ such that $d(x_0, x_d) = d$. Define a sequence of maximal cliques $H_1, \ldots, H_d$ as in Lemma 3. Then we have

$$d - 1 = d(H_1, H_d) \le \mathrm{diam}\,(\Gamma(G)),$$

which completes the proof of (3.4). □

Lemma 4. *For $d \ge 1$ let*

$$H_0 \sim H_1 \sim \cdots \sim H_d \tag{3.5}$$

be a shortest path connecting H_0 and H_d, that is $d(H_0, H_d) = d$. For $1 \le i \le d$ take a vertex $x_i \in H_{i-1} \cap H_i$. Then

$$x_1 \sim x_2 \sim \cdots \sim x_d \tag{3.6}$$

and $d(x_1, x_d) = d-1$. Hence (3.6) is a shortest path connecting x_1 and x_d, and $x_1, x_2, \ldots, x_d$ are mutually distinct.

Proof. For $1 \le i \le d-1$ we have $x_i, x_{i+1} \in H_i$ and hence $x_i = x_{i+1}$ or $x_i \sim x_{i+1}$. Suppose that $x_i = x_{i+1}$ occurs. Then $x_i = x_{i+1} \in H_{i-1} \cap H_i \cap H_{i+1}$ and hence $H_{i-1} \cap H_{i+1} \ne \emptyset$, from which we obtain $H_{i-1} = H_{i+1}$ or $H_{i-1} \sim H_{i+1}$. In any case we come to a contradiction because (3.5) is a shortest path. We have thus obtained a walk as in (3.6).
We next prove that (3.6) gives rise to a shortest path. We set $s = d(x_1, x_d)$ and take a shortest path connecting x_1 and x_d, say,

$$x_1 = y_0 \sim y_1 \sim \cdots \sim y_s = x_d.$$

In that case we have

$$s \le d - 1. \tag{3.7}$$

For $1 \le i \le s$ let J_i be a maximal clique containing $\{y_{i-1}, y_i\}$. It then follows from Lemma 3 that $d(J_1, J_s) = s - 1$. Since $x_1 = y_0 \in H_0 \cap J_1$, we have $H_0 = J_1$ or $H_0 \sim J_1$. Similarly, we have $H_d = J_s$ or $H_d \sim J_s$. Thus, we obtain a walk connecting H_0 and H_d of which length is $s-1$, s or $s+1$. Hence $d = d(H_0, H_d) \le s + 1$. Combining (3.7), we obtain $s = d - 1$ and hence $d(x_1, x_d) = d - 1$ as desired. □

Proposition 6. *Let G be a graph and $\Gamma(G)$ its clique graph. If $\Gamma(G)$ is a tree, we have*

$$\mathrm{diam}\,(G) - 1 = \mathrm{diam}\,(\Gamma(G)). \tag{3.8}$$

Proof. Set $d = \mathrm{diam}\,(\Gamma(G))$ and take a shortest path

$$H_0 \sim H_1 \sim \cdots \sim H_d, \tag{3.9}$$

where $d(H_0, H_d) = d$. For $1 \le i \le d$ take $x_i \in H_{i-1} \cap H_i$. By Lemma 4 we have a shortest path $x_1 \sim x_2 \sim \cdots \sim x_d$. Moreover, we take $x_0 \in H_0 \backslash H_1$ and $x_{d+1} \in H_d \backslash H_{d-1}$. Thus we obtain a walk

$$x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_d \sim x_{d+1} \tag{3.10}$$

whose length is $d + 1$.
We shall prove that (3.10) is a shortest path. Set $s = d(x_0, x_{d+1})$ and take a shortest path, say,

$$x_0 = y_0 \sim y_1 \sim y_2 \sim \cdots \sim y_s = x_{d+1} \tag{3.11}$$

For $1 \le i \le s$ let J_i be a maximal clique containing $\{y_{i-1}, y_i\}$. By Lemma 3.2 we obtain a shortest path $J_1 \sim \cdots \sim J_s$, namely,

$$d(J_1, J_s) = s - 1. \tag{3.12}$$

Now note that $x_0 = y_0 \in H_0 \cap J_1$. Then we have $H_0 = J_1$ or $H_0 \sim J_1$. Since $\Gamma(G)$ is a tree, the ends of a diameter (3.9) are pending vertices. Hence $H_0 \sim J_1$ implies that $J_1 = H_1$. In that case we have $x_0 = y_0 \in H_0 \cap J_1 = H_0 \cap H_1$. On the other hand, we chose $x_0 \in H_0 \backslash H_1$, which is a contradiction. Therefore, $H_0 \sim J_1$ does not occur and we have $H_0 = J_1$. In a similar manner, we see that $H_d = J_s$. Consequently, combining (3.12) we come to

$$d = d(H_0, H_d) = d(J_1, J_s) = s - 1.$$

Thus,

$$\operatorname{diam}(\Gamma(G)) = d = s - 1 = d(x_0, x_{d+1}) - 1 \le \operatorname{diam}(G) - 1.$$

Finally, combining Proposition 5, we obtain the equality (3.8). □

4. Graphs with $\mathrm{QEC}(G) < -1/2$

The complete bipartite graph $K_{1,3}$ is called a *claw*. The complete tripartite graph $K_{1,1,2}$, which is also obtained by deleting an edge from the complete graph K_4, is called a *diamond*, see Figure 1. It is essential to note that

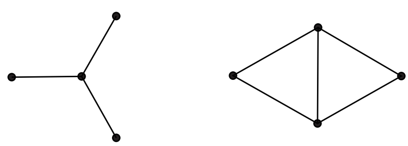

Figure 1. **Claw $K_{1,3}$ (left) and diamond $K_{1,1,2}$ (right)**

$$\mathrm{QEC}(K_{1,3}) = \mathrm{QEC}(K_{1,1,2}) = -\frac{1}{2}\,.$$

Since $\operatorname{diam}(K_{1,3}) = \operatorname{diam}(K_{1,1,2}) = 2$, we see from Propositions 2 and 3 that if a graph G contains $K_{1,3}$ or $K_{1,1,2}$ as an induced subgraph, we have $\mathrm{QEC}(G) \geq -1/2$. Thus, we come to the following criterion.

Proposition 7 ([4, Corollary 4.1]). *Any graph G with $\mathrm{QEC}(G) < -1/2$ does not contain a claw $K_{1,3}$ nor a diamond $K_{1,1,2}$ as an induced subgraph. In short, the claw and diamond are forbidden subgraphs for a graph with $\mathrm{QEC}(G) < -1/2$.*

Lemma 5. *Let G be a graph with $\mathrm{QEC}(G) < -1/2$. If H_1 and H_2 are maximal cliques of G with $H_1 \neq H_2$, then $H_1 \cap H_2 = \emptyset$ or $|H_1 \cap H_2| = 1$.*

Proof. In order to prove the assertion by contradiction, we suppose $|H_1 \cap H_2| \geq 2$ and take $x, y \in H_1 \cap H_2$ with $x \neq y$. By Lemma 1 there exist $a \in H_1 \backslash H_2$ and $b \in H_2 \backslash H_1$ such that $a \not\sim b$. Then $\langle x, y, a, b\rangle$ forms a diamond, which is a forbidden subgraph as stated in Proposition 7. □

Lemma 6. *Let G be a graph with $\mathrm{QEC}(G) < -1/2$. If H_1, H_2 and H_3 are mutually distinct maximal cliques of G, then $H_1 \cap H_2 \cap H_3 = \emptyset$.*

Proof. In order to prove the assertion by contradiction we suppose that $H_1 \cap H_2 \cap H_3 \neq \emptyset$. Then $H_1 \cap H_2 \neq \emptyset$ and by Lemma 5 we have $H_1 \cap H_2 = \{x\}$ for some $x \in V$. Hence $H_2 \cap H_3 = H_3 \cap H_1 = H_1 \cap H_2 \cap H_3 = \{x\}$. On the other hand, by Lemma 1 there exist $a \in H_1 \backslash H_2$ and $b \in H_2 \backslash H_1$ such that $a \not\sim b$. Note that there exists $c \in H_3 \backslash H_1$ such that $c \not\sim a$. In fact, if any $c \in H_3 \backslash H_1$ is adjacent to a, then $H_3 \cup \{a\}$ becomes a clique and we come to a contradiction. Thus, we have chosen four vertices x, a, b, c. There are two cases. In case of $b \not\sim c$, the induced subgraph $\langle x, a, b, c\rangle$ becomes a claw. In case of $b \sim c$ note that there exist $c' \in H_3 \backslash H_2$ such that $c' \not\sim b$. In fact, if any $c' \in H_3 \backslash H_2$ is adjacent to b, $H_3 \cup \{b\}$ becomes a clique and we come to a contradiction. Thus, taking $c' \in H_3 \backslash H_2$ such that $c' \not\sim b$, we see that $\langle x, b, c, c'\rangle$ becomes a diamond. In any case we obtain a forbidden subgraph as stated in Proposition 7 and arrive to a contradiction. □

Proposition 8. *For a graph $G = (V, E)$ with $\mathrm{QEC}(G) < -1/2$ the clique graph $\Gamma(G)$ is a tree.*

Proof. Suppose that the clique graph $\Gamma(G)$ is not a tree and take a smallest cycle, say,

$$H_1 \sim H_2 \sim \cdots \sim H_k \sim H_1, \qquad k \geq 3, \tag{4.1}$$

where H_i is a maximal clique of G and $H_i \cap H_{i+1} \neq \emptyset$ for $1 \le i \le k$ (understanding $H_{k+1} = H_1$). It follows from Lemma 5 that there exists a unique vertex x_i such that $H_i \cap H_{i+1} = \{x_i\}$ for $1 \le i \le k$. Then, obviously

$$x_1 \sim x_2 \sim \cdots \sim x_k \sim x_1.$$

(Case 1) $k = 3$. In that case x_1, x_2, x_3 are mutually distinct. We note that $x_1, x_3 \in H_1$ and $x_2 \notin H_1$. If $H_1 = \{x_1, x_3\}$, namely $H_1 \backslash \{x_1, x_3\} = \emptyset$, then $H_1 \cup \{x_2\}$ becomes a clique containing H_1 properly and we come to a contradiction. Hence $H_1 \backslash \{x_1, x_3\} \neq \emptyset$. If any $y \in H_1 \backslash \{x_1, x_3\}$ is adjacent to x_2, then $H_1 \cup \{x_2\}$ becomes a clique containing H_1 properly and we come to a contradiction again. Therefore, there exists $y \in H_1 \backslash \{x_1, x_3\}$ such that $y \not\sim x_2$. Thus, $\langle x_1, x_2, x_3, y \rangle$ becomes a diamond, which is a forbidden subgraph by Proposition 7. Consequently, $\Gamma(G)$ does not contain a cycle (4.1) with $k = 3$.

(Case 2) $k \ge 4$. Using the assumption that (4.1) is a smallest cycle, one can show easily that $x_1, \dots, x_k$ are mutually distinct.

We first prove that the induced subgraph $C = \langle x_1, x_2, \dots, x_k \rangle$ becomes a cycle C_k. In fact, if not, there exist $1 \le i, j \le k$ such that $i + 1 < j$ and $x_i \sim x_j$. Let J be a maximal clique containing $\{x_i, x_j\}$. Then $x_i \in H_i \cap H_{i+1} \cap J$ and $x_j \in H_j \cap H_{j+1} \cap J$. In view of Lemma 6 we obtain $J = H_i$ or $J = H_{i+1}$ from the former condition, and similarly $J = H_j$ or $J = H_{j+1}$ from the latter. In any case we come to a contradiction against that (4.1) is a smallest cycle.

We next show that the cycle $C = \langle x_1, x_2, \dots, x_k \rangle \cong C_k$ is isometrically embedded in G. Suppose otherwise. Then there exist $1 \le i, j \le k$ such that

$$d_G(x_i, x_j) < d_{C_k}(x_i, x_j), \tag{4.2}$$

where the right-hand side is the distance in the cycle C_k. Without loss of generality, we may assume that $1 \le i < j \le k$. Then, (4.2) becomes

$$d_G(x_i, x_j) < \min\{j - i, k - (j - i)\}. \tag{4.3}$$

For $j = i + 1$ we have $x_i \sim x_j$ and (4.3) does not hold. For $j = i + 2$ it follows from (4.3) that $d_G(x_i, x_j) = 1$ and come to a contradiction against the argument in the previous paragraph. Thus, it is sufficient to derive a contradiction from (4.3) for some $j \ge i + 3$. Now take a shortest path

$$x_i \sim y_0 \sim y_1 \sim \cdots \sim y_s = x_j, \qquad s = d_G(x_i, x_j).$$

For $1 \le i \le s$ take a maximal clique J_i such that $\{y_{i-1}, y_i\} \subset J_i$. Since $x_i = y_0 \in H_i \cap H_{i+1} \cap J_1$, by Lemma 6 we have $J_1 = H_i$ or $J_1 = H_{i+1}$. Similarly, we see from $x_j = y_s \in H_j \cap H_{j+1} \cap J_s$ that $J_s = H_j$ or $J_s = H_{j+1}$. Thus, the path $J_1 \sim J_2 \sim \cdots \sim J_s$, which is a shortest path by Lemma 3, gives rise to an alternative

path connecting two vertices of C, and hence two cyclic walks. We will consider these two cyclic walks in details.
Consider the case where

$$J_1 = H_i, \quad \text{and} \quad J_s = H_j. \tag{4.4}$$

We obtain two cyclic walks:

$$H_1 \sim \cdots \sim H_i = J_1 \sim J_2 \sim \cdots \sim J_s = H_j \sim H_{j+1} \sim \cdots \sim H_k \sim H_1. \tag{4.5}$$

and

$$H_i = J_1 \sim J_2 \sim \cdots \sim J_s = H_j \sim H_{j-1} \sim \cdots \sim H_{i+1} \sim H_i\,. \tag{4.6}$$

The length of these walks are $(i-1)+(s-1)+(k-j)+1 = k+s+i-j$ and $(s-1)+(j-i) = s+j-i-1$, respectively. In view of (4.3) we consider two cases. First, in case of $s < \min\{j-i, k-(j-i)\} = j-i$ we have $k+s+i-j < k$, that is, the length of (4.5) is less than k. This contradicts to the choice of k. Second, in case of $s < \min\{j-i, k-(j-i)\} = k-(j-i)$ we have $s+j-i-1 < k-1$, that is, the length of (4.5) is less than $k-1$. This contradicts to the choice of k. Thus (4.4) does not occur.
Other than (4.4) there are three more cases. In each of these cases, in a similar manner as in the previous case, we may find a cycle in $\Gamma(G)$ which is smaller than (4.1), and come to a contradiction. As a result, the cycle $C = \langle x_1, x_2, \dots, x_k\rangle \cong C_k$ is isometrically embedded in G.
We now recall that $\mathrm{QEC}(C_k) > -1/2$ for any $k \geq 4$. In fact, the exact value of $\mathrm{QEC}(C_k)$ is known [22]. As a consequence of (Case 2), we obtain $\mathrm{QEC}(G) \geq \mathrm{QEC}(C_k) > -1/2$ and come to a contradiction. Hence, $\Gamma(G)$ does not contain a smallest cycle (4.1) with $k \geq 4$. This completes the proof. □

Summing up the above results, we state the following

Theorem 1. *Let $G = (V, E)$ be a graph with $\mathrm{QEC}(G) < -1/2$. Then the clique graph $\Gamma(G)$ is a tree. Any pair of adjacent maximal cliques H_1 and H_2 intersect with a single vertex, i.e., $|H_1 \cap H_2| = 1$. Moreover, mutually distinct three maximal cliques H_1, H_2 and H_3 do not intersect, i.e., $H_1 \cap H_2 \cap H_3 = \emptyset$.*

Thus, we say naturally that a graph $G = (V, E)$ with $\mathrm{QEC}(G) < -1/2$ is a block graph which admits "cactus-like" structure, see Figure 2. On the other hand, a *cactus* is defined to be a connected graph in which no edge lies on more than one cycle. This definition traces back to [8], though there is ambiguity in the usage in literature. Note that such a cactus is different from our "cactus-like" graph.

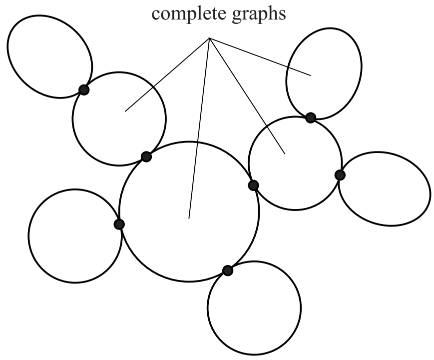

Figure 2. **"Cactus-like" graph**

5. Graphs Consisting of Two Maximal Cliques

For natural numbers $l \geq 1$, $m > l$ and $n > l$, let $V = \{1, 2, \dots, m+n-l\}$ and consider its two subsets:

$$H_1 = \{1, 2, \dots, m\}, \qquad H_2 = \{m-l+1, m-l+2, \dots, m-l+n\}. \tag{5.1}$$

Note that $V = H_1 \cup H_2$. Let E be the set of two-element subsets $\{x, y\} \subset V$ satisfying $x, y \in H_1$ or $x, y \in H_2$. Then $G = (V, E)$ becomes a graph which is denoted by $G = K_m \cup_l K_n$, see Figure 3. Obviously, $G = K_m \cup_l K_n$ has exactly two maximal cliques H_1 and H_2. We will prove that any graph (recall that we always assume that a graph is connected) with exactly two maximal cliques is of this form.

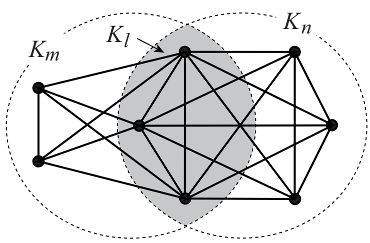

Figure 3. **A connected graph consisting of two maximal cliques**

Proposition 9. *Let $G = (V, E)$ be a (connected) graph with exactly two maximal cliques. Then there exist three natural numbers $l \geq 1$, $m > l$ and $n > l$ such that $G \cong K_m \cup_l K_n$.*

Although Proposition 9 pertains to an elementary understanding of graph theory, for later convenience we show an outline of the argument.

Lemma 7. *Let $G = (V, E)$ be a graph with exactly two maximal cliques H_1 and H_2.*

(1) *For any $a \in H_1 \backslash H_2$ and $b \in H_2 \backslash H_1$ we have $a \not\sim b$.*

(2) *$V = H_1 \cup H_2$.*

(3) *$H_1 \cap H_2 \neq \emptyset$.*

(4) *For any $a \in H_1 \backslash H_2$ and $b \in H_2 \backslash H_1$ we have $d_G(a, b) = 2$.*

Proof. (1) Suppose that $a \in H_1 \backslash H_2$ and $b \in H_2 \backslash H_1$ are adjacent. There exists a maximal clique containing $\{a, b\}$, which is different from H_1 and H_2. This implies that $a \not\sim b$ for any pair of $a \in H_1 \backslash H_2$ and $b \in H_2 \backslash H_1$.
(2) In order to prove by contradiction we suppose $V \neq H_1 \cup H_2$. Take $a \in V \backslash (H_1 \cup H_2)$. Since G is connected, there exists $b \in V$ such that $a \sim b$. Then a maximal clique containing $\{a, b\}$ exists and is different from H_1 and from H_2. We thus come to a contradiction.
(3) Suppose that $H_1 \cap H_2 = \emptyset$. Take $a \in H_1$ and $b \in H_2$ arbitrarily. Since G is connected, there exists a walk connecting a and b. Since this walk is kept in $H_1 \cup H_2$ by (2), we may find $a' \in H_1$ and $b' \in H_2$ such that $a' \sim b'$. This contradicts to the result of (1).
(4) By (1) we know that $d_G(a, b) \geq 2$. On the other hand, taking $x \in H_1 \cap H_2$ we obtain a walk $a \sim x \sim b$, which implies that $d_G(a, b) \leq 2$. □

Proof of Proposition 9. Let H_1 and H_2 be the two maximal cliques of G. We set $m = |H_1|$, $n = |H_2|$ and $l = |H_1 \cap H_2|$. By Lemma 7 we see that $H_1 \cong K_m$, $H_2 \cong K_n$ and $H_1 \cap H_2 \cong K_l$ with $l \geq 1$ and $m, n > l$. Moreover, there is no edge connecting vertices $a \in H_1 \backslash H_2$ and $b \in H_2 \backslash H_1$. We conclude that $G \cong K_m \cup_l K_n$. □

The distance matrix of $G = K_m \cup_l K_n$ is easily written down according to (5.1). In fact, taking Lemma 7 (4) into account, we obtain the distance matrix D in a block-matrix form as follows:

$$D = \begin{bmatrix} J - I & J & J \\ J & J - I & 2J \\ J & 2J & J - I \end{bmatrix}, \tag{5.2}$$

where I is the identity matrix and J the matrix whose entries are all one (the sizes of these matrices are understood in the context). Then $\mathrm{QEC}(G)$ is obtained by means of the basic formula in Proposition 1. The computation is just a routine and is deferred to the Appendix.

Theorem 2. *For $l \geq 1$, $m > l$ and $n > l$ we have*

$$\mathrm{QEC}(K_m \cup_l K_n) = -1 + \frac{-(m - l)(n - l) + \sqrt{mn(m - l)(n - l)}}{m + n - l}. \tag{5.3}$$

Corollary 1 ([4, Proposition 4.4]). *Let $m \geq 2$ and $n \geq 2$. Then $K_m \cup_1 K_n$ is a graph obtained from K_m and K_n by concatenating a vertex, in other words, it is the star product $K_m \cup_1 K_n = K_m * K_n$, and we have*

$$\begin{aligned}\mathrm{QEC}(K_m \cup_1 K_n) &= \mathrm{QEC}(K_m * K_n)\\ &= \frac{-mn + \sqrt{mn(m-1)(n-1)}}{m+n-1}\\ &= -\left(1 + \sqrt{\left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{n}\right)}\right)^{-1}. \end{aligned} \tag{5.4}$$

Corollary 2. *Let $m \geq 3$ and $n \geq 3$. Then $K_m \cup_2 K_n$ is a graph obtained from K_m and K_n by concatenating an edge and we have*

$$\mathrm{QEC}(K_m \cup_2 K_n) = \frac{-mn + m + n - 2 + \sqrt{mn(m-2)(n-2)}}{m+n-2}. \tag{5.5}$$

Remark 1. By changing parameters we obtain an alternative form of (5.3) in Theorem 2. For $l, m, n \geq 1$ we have

$$\mathrm{QEC}(K_{m+l} \cup_l K_{n+l}) = -1 + \frac{l}{1 + \sqrt{\left(1 + \frac{l}{m}\right)\left(1 + \frac{l}{n}\right)}}.$$

This is useful to discuss estimates of $\mathrm{QEC}(K_{m+l} \cup_l K_{n+l})$.

6. Characterization of Graphs Along $QEC(P_d)$

Proposition 10. *Let $d \geq 3$. If $\mathrm{QEC}(G) < \mathrm{QEC}(P_d)$, we have $\mathrm{diam}\,(G) \leq d-2$ and $\mathrm{diam}\,(\Gamma(G)) \leq d-3$.*

Proof. Suppose that $\mathrm{diam}\,(G) > d-2$. Then $\mathrm{diam}\,(G) \geq d-1$ and P_d is isometrically embedded in G. By Proposition 3 we obtain $\mathrm{QEC}(P_d) \leq \mathrm{QEC}(G)$, which contradicts to the assumption. Therefore, if $\mathrm{QEC}(G) < \mathrm{QEC}(P_d)$, we have $\mathrm{diam}\,(G) \leq d-2$. In that case, since $\mathrm{QEC}(G) < \mathrm{QEC}(P_d) < -1/2$, it follows from Theorem 1 that $\Gamma(G)$ is a tree. We then see from Proposition 6 that $\mathrm{diam}\,(\Gamma(G)) = \mathrm{diam}\,(G) - 1 \leq d-3$. □

6.1. $\mathrm{QEC}(G) < \mathrm{QEC}(P_3)$

For a graph with $\mathrm{QEC}(G) < \mathrm{QEC}(P_3)$ we have $\mathrm{diam}\,(\Gamma(G)) = 0$, which means that G has just one maximal clique. Hence $G = K_n$ with $n \geq 2$. Since $\mathrm{QEC}(K_n) = -1$, we have the following assertions immediately.

Proposition 11 ([4]). *For a graph G we have $\mathrm{QEC}(G) = \mathrm{QEC}(P_2) = -1$ if and only if $G = K_n$ with $n \geq 2$.*

Proposition 12 ([4]). *There exists no graph G such that $\mathrm{QEC}(P_2) < \mathrm{QEC}(G) < \mathrm{QEC}(P_3) = -2/3$.*

6.2. QEC(G) < QEC(P_4)

Let G be a graph satisfying QEC(G) < QEC(P_4). It follows from Proposition 10 that diam ($\Gamma(G)$) ≤ 1, that is, diam ($\Gamma(G)$) $= 0$ or diam ($\Gamma(G)$) $= 1$. The case of diam ($\Gamma(G)$) $= 0$ is discussed already in Subsection 6.1.
In the case of diam ($\Gamma(G)$) $= 1$, the clique graph $\Gamma(G)$ consists of two vertices, which means that G has exactly two maximal cliques. By Proposition 9 we obtain $G = K_m \cup_l K_n$ with $l \ge 1$, $m > l$ and $n > l$. On the other hand, since QEC(G) $< -1/2$, we see from Lemma 5 that $l = 1$. Thus, G is necessarily of the form $G = K_m \cup_1 K_n = K_m * K_n$ with $m \ge n \ge 2$.
With the help of the formula in Corollary 1 we may easily determine $m \ge n \ge 2$ such that QEC($K_m * K_n$) < QEC(P_4) $= -(2 - \sqrt{2})$. As a result we obtain the following assertions.

Proposition 13 ([4]). *For a graph G we have* QEC(G) = QEC(P_3) $= -2/3$ *if and only if* $G = P_3 = K_2 * K_2$.

Proposition 14 ([4]). *For a graph G we have* QEC(P_3) < QEC(G) < QEC(P_4) $= -(2 - \sqrt{2})$ *if and only if* $G = K_m * K_2$ *with* $m \ge 3$ *or* $G = K_3 * K_3$.

6.3. QEC(G) < QEC(P_5)

If a graph G satisfies QEC(G) < QEC(P_5) $= -(5-\sqrt{5})/5$, then diam ($\Gamma(G)$) ≤ 2. The case of diam ($\Gamma(G)$) $= 0$ is already discussed in Subsection 6.1. If diam ($\Gamma(G)$) $= 1$, we have $G = K_m * K_n$ with $m \ge n \ge 2$. Then, as is discussed in Subsection 6.2, we may employ the explicit formula for QEC($K_m * K_n$) in Corollary 1. The result will be stated in Propositions 15 and 16.
Consider the case of diam ($\Gamma(G)$) $= 2$. Since $\Gamma(G)$ is a tree, it is necessarily a star $\Gamma(G) = K_{1,s}$ with $s \ge 2$. Then G is a graph obtained as follows: Let $n \ge s$ and $m_1 \ge m_2 \ge \cdots \ge m_s \ge 2$. We choose s vertices from K_n and to each of the s vertices we make a star product with $K_{m_1}, \dots, K_{m_s}$, see Figure 4. Such a graph is denoted by $G = K_n * (K_{m_1}, \dots, K_{m_s})$.

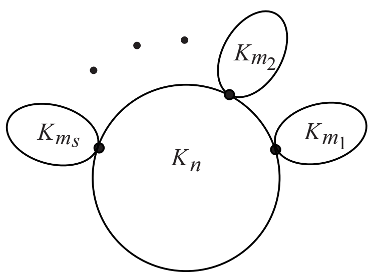

Figure 4. $K_n * (K_{m_1}, \dots, K_{m_s})$

Lemma 8. *Consider two graphs* $G = K_n * (K_{m_1}, \dots, K_{m_s})$ *with* $n \geq s$, $m_1 \geq m_2 \geq \dots \geq m_s \geq 2$, *and* $G' = K_{n'} * (K_{m'_1}, \dots, K_{m'_r})$ *with* $n' \geq r$ *and* $m'_1 \geq m'_2 \geq \dots \geq m'_r \geq 2$. *If* $n' \leq n$, $r \leq s$, $m'_1 \leq m_1, \dots, m'_r \leq m_r$, *then* G' *is isometrically embedded in* G. *Hence* $\mathrm{QEC}(G') \leq \mathrm{QEC}(G)$.

Proof. Obvious. □

Lemma 9. *Let* $n \geq s \geq 2$ *and* $m_1 \geq m_2 \geq \dots \geq m_s \geq 2$. *If* $m_1 \geq 3$, *we have*

$$\mathrm{QEC}(P_4) < \mathrm{QEC}(K_n * (K_{m_1}, \dots, K_{m_s})).$$

Proof. It is known that

$$\mathrm{QEC}(K_2 * (K_3, K_2)) = -\frac{2(6 - \sqrt{21})}{5} \approx -0.5669,$$

see [22], where $K_2 * (K_3, K_2)$ is referred to as No.5-7. Then we have

$$\mathrm{QEC}(P_4) < \mathrm{QEC}(K_2 * (K_3, K_2)) < \mathrm{QEC}(P_5).$$

On the other hand, for $n \geq s \geq 2$ and $m_1 \geq 3$, $K_n * (K_{m_1}, \dots, K_{m_s})$ contains $K_2 * (K_3, K_2)$ as an isometrically embedded subgraph. Hence

$$\mathrm{QEC}(P_4) < \mathrm{QEC}(K_2 * (K_3, K_2)) \leq \mathrm{QEC}(K_n * (K_{m_1}, \dots, K_{m_s})),$$

as desired. □

Lemma 10 ([4, Theorem 4.3]). *For* $n \geq s \geq 2$ *we have*

$$\mathrm{QEC}(K_n * (\overbrace{K_2, \dots, K_2}^{s \text{ times}})) = -(2 - \sqrt{2}) = \mathrm{QEC}(P_4).$$

Proposition 15. *For a graph* G *we have* $\mathrm{QEC}(G) = \mathrm{QEC}(P_4) = -(2 - \sqrt{2})$ *if and only if* $G = K_4 * K_3$ *or* $G = K_n * (K_2, \dots, K_2)$ *(*K_2 *appears* s *times) with* $n \geq s \geq 2$.

Proof. As is discussed in Subsection 6.2, a graph G with $\mathrm{diam}\,(\Gamma(G)) = 1$ is of the form $K_m * K_n$ with $m \geq n \geq 2$. Then, using the explicit formula for $\mathrm{QEC}(K_m * K_n)$ in Corollary 1, we see easily that $\mathrm{QEC}(K_m * K_n) = \mathrm{QEC}(P_4)$ if and only if $m = 4$ and $n = 3$.
A graph G with $\mathrm{diam}\,(\Gamma(G)) = 2$ is of the form $G = K_n * (K_{m_1}, \dots, K_{m_s})$ with $n \geq s \geq 2$ and $m_1 \geq m_2 \geq \dots \geq m_s \geq 2$. By Lemma 9, $\mathrm{QEC}(G) \leq \mathrm{QEC}(P_4)$ may occur only when $m_1 = m_2 = \dots = m_s = 2$. On the other hand, in that case, the equality $\mathrm{QEC}(G) = \mathrm{QEC}(P_4)$ holds by Lemma 10. □

Proposition 16. *For $m \geq n \geq 2$ we have* $\mathrm{QEC}(P_4) < \mathrm{QEC}(K_m * K_n) < \mathrm{QEC}(P_5)$ *if and only if*

(i) $n = 3$ *and* $5 \leq m \leq 54$;

(ii) $n = 4$ *and* $4 \leq m \leq 7$;

(iii) $n = m = 5$.

Proof. Straightforward by the explicit formula for $\mathrm{QEC}(K_m * K_n)$. □

By Proposition 16 all graphs G such that $\mathrm{QEC}(P_4) < \mathrm{QEC}(G) < \mathrm{QEC}(P_5)$ with $\mathrm{diam}\,(\Gamma(G)) = 1$ are determined. The case of $\mathrm{diam}\,(\Gamma(G)) = 2$, i.e., $G = K_n * (K_{m_1}, \dots, K_{m_s})$ remains to be checked. The work in this line is in progress.

Appendix: Calculating $\mathrm{QEC}(K_m \cup_l K_n)$

Let D be the distance matrix of $G = K_m \cup_l K_n$, where $l \geq 1$ and $m, n > l$. Using the block-matrix form of D as in (5.2), we will calculate $\mathrm{QEC}(G)$ explicitly.
For $f \in \mathbb{R}^l$, $g \in \mathbb{R}^{m-l}$ and $h \in \mathbb{R}^{n-l}$ we set

$$\begin{aligned}\psi(f,g,h) &= \left\langle \begin{bmatrix} f \\ g \\ h \end{bmatrix}, D \begin{bmatrix} f \\ g \\ h \end{bmatrix} \right\rangle \\ &= \langle 1,f\rangle^2 + \langle 1,g\rangle^2 + \langle 1,h\rangle^2 - \langle f,f\rangle^2 - \langle g,g\rangle^2 - \langle h,h\rangle^2 \\ &\quad + 2\langle 1,f\rangle\langle 1,g\rangle + 2\langle 1,f\rangle\langle 1,h\rangle + 4\langle 1,g\rangle\langle 1,h\rangle\end{aligned}$$

and

$$\begin{aligned}\varphi(f,g,h,\lambda,\mu) = \psi(f,g,h) &- \lambda(\langle f,f\rangle + \langle g,g\rangle + \langle h,h\rangle - 1) \\ &- \mu(\langle 1,f\rangle + \langle 1,g\rangle + \langle 1,h\rangle).\end{aligned} \tag{A.1}$$

It then follows from Proposition 1 that $\mathrm{QEC}(G)$ coincides with the maximum of $\lambda \in \mathbb{R}$ appearing in the stationary points of $\psi(f,g,h,\lambda,\mu)$. Since G is not complete, it is sufficient to explore stationary points of $\psi(f,g,h,\lambda,\mu)$ with $\lambda > -1$.
By direct computation together with condition

$$\langle 1,f\rangle + \langle 1,g\rangle + \langle 1,h\rangle = 0$$

we have

$$\frac{\partial\varphi}{\partial f_i} = -2(\lambda+1)f_i - \mu = 0. \tag{A.2}$$

Then we see that f_i is constant independent of $1 \le i \le l$, say $f_i = \xi$. Thus, (A.2) becomes

$$\xi = -\frac{1}{\lambda+1}\cdot\frac{\mu}{2}\,. \tag{A.3}$$

(From the beginning we may assume that $\lambda > -1$ as noted before.) Similarly, it follows from $\partial\varphi/\partial g_i = \partial\varphi/\partial h_i = 0$ that g_i and h_i are respectively constant. Setting $g_i = \eta$ and $h_i = \zeta$, we obtain

$$(\lambda+1)\eta - (n-l)\zeta = -\frac{\mu}{2}\,, \tag{A.4}$$

$$-(m-l)\eta + (\lambda+1)\zeta = -\frac{\mu}{2}\,, \tag{A.5}$$

and the constraints become

$$l\xi + (m-l)\eta + (n-l)\zeta = 0, \tag{A.6}$$

$$l\xi^2 + (m-l)\eta^2 + (n-l)\zeta^2 = 1. \tag{A.7}$$

Our task is to solve the system of equations (A.3)–(A.7).
In view of (A.4) and (A.5) we set

$$\Delta = \det\begin{bmatrix} \lambda+1 & -(n-l) \\ -(m-l) & \lambda+1 \end{bmatrix} = (\lambda+1)^2 - (m-l)(n-l). \tag{A.8}$$

(Case I) $\Delta \neq 0$. The equations (A.4) and (A.5) have a unique solution:

$$\eta = -\frac{\lambda+n+1-l}{\Delta}\cdot\frac{\mu}{2}\,, \qquad \zeta = -\frac{\lambda+m+1-l}{\Delta}\cdot\frac{\mu}{2}\,. \tag{A.9}$$

Inserting (A.3) and (A.9) into (A.6), we obtain

$$l\Delta + (\lambda+1)\{(m-l)(\lambda+n+1-l) + (n-l)(\lambda+m+1-l)\} = 0 \tag{A.10}$$

after simple calculation. The solutions are easily written down as

$$\lambda_\pm = -1 + \frac{-(m-l)(n-l) \pm \sqrt{mn(m-l)(n-l)}}{m+n-l}\,.$$

Checking that $\lambda_+ \neq -1$ and $\Delta \neq 0$ for $\lambda = \lambda_+$, we see that λ_+ is a candidate of $\mathrm{QEC}(G)$.
(Case II) $\Delta = 0$. From (A.4) and (A.5) we obtain

$$(\lambda+n-l+1)\frac{\mu}{2} = 0, \qquad (\lambda+m-l+1)\frac{\mu}{2} = 0.$$

If $m \neq n$, we obtain $\mu = 0$ and $\xi = \eta = \zeta = 0$, which do not fulfill (A.7). Hence there is no stationary points. Assume that $m = n$. Then $\lambda = -1 - (m - l)$ appears in the stationary points. However, since we are only interested in $\lambda > -1$, there exists no candidate for our QEC(G) in the case of $\Delta = 0$.
Finally, we conclude from (Case I) and (Case II) that $\lambda_+ = \mathrm{QEC}(G)$.

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Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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