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Research Article

Optimization problems with nonconvex multiobjective generalized Nash equilibrium problem constraints

El-yahyaoui Youness*, Lafhim Lahoussine†

Laboratory LASMA, Department of mathematics, Sidi Mohamed Ben Abdellah University, Fez, Morocco
*youness.elyahyaoui@usmba.ac.ma
†wcshiu@associate.hkbu.edu.hk

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Abstract: This work discusses a category of optimization problems in which the lower-level problems include multiobjective generalized Nash equilibrium problems. Despite the fact that it has various possible applications, there has been little research into it in the literature. We provide a single-level reformulation for these types of problems and highlight their equivalence in terms of global and local minimizers. Our method consists of transforming our problem into a one-level optimization problem, utilizing the kth-objective weighted-constraint and optimal value reformulation. The Mordukhovich generalized differentiation calculus is then used to derive completely detailed first-order necessary optimality conditions in the smooth setting.

Keywords: mathematical programming, generalized Nash equilibrium problem, kth-objective weighted-constraint approach, optimal value function, optimality conditions.

AMS Subject classification: 90C26, 90C30, 90C31, 90C46

1. Introduction

Optimization problems with mathematical programs in the constraints are structured analogously to hierarchical programming problems; see [2–4, 7–10, 16] for an introduction to hierarchical optimization and some results, with the exception that the objective mapping of the lower-level (or follower's) problem is a vector function. This problem is known as bilevel. Given that the usual definition of a minimizer does not apply to multiobjective programs, the lower-level decision maker must, for instance, compute the set of efficient or weakly efficient points for each fixed value of the upper-level (or leader's) variable in general. For more information, see [11, 14] for an introduction to multicriteria optimization.

* *Corresponding Author*

Recently, multi-objective optimization programs have received attention in the literature, motivated by the fact that decision makers often have several conflicting objectives that should be optimized simultaneously when faced with a real-world problem. Mathematical programming is applicable to real-world problems related to electricity markets, as seen in [1] and the references therein, as well as the modeling of inverse multicriteria optimization problems. [3, 5, 15, 17, 26] explored the essential optimality conditions for finite-dimensional mathematical optimization problems, while [12] addressed this type of problem in the convex case. The goal of this paper is twofold: first, we generalized the existing [12] for the generalized Nash equilibrium problem to a nonconvex case, and second, we employed a new scalarization technique called weighted sum to extend the existing result for bilevel to two players; see [9, 15, 26].

In fact, at the lower level of a bilevel optimization problem, there may be several followers who play in a noncooperative generalized Nash game (see Section 3). In this paper, we name such kinds of problems "the nonconvex mathematical program with multiobjective generalized Nash equilibrium problems". The Nash equilibrium is a fundamental concept in many fields. Its importance lies not only in its theoretical foundations within game theory but also in its practical applications spanning economics, political science, biology, computer science, and many other fields; see [23, 24].

Considering that the lower-level decision maker must solve a multiobjective generalized Nash equilibrium problem for each fixed value of the upper-level variable, the overall weak Pareto front can be computed using the kth-objective weighted-constraint approach, and the original mathematical programming problem is transformed into several standard programming problems. This topic was first examined in [15], where the author emphasized the relationship of the surrogate problems to the original mathematical programming problem.

In what follows, we develop two approaches, with the first being to study the mathematical programming problem with a multiobjective generalized Nash equilibrium problem at the lower level. Our method consists of several steps: First, we apply the kth-objective weighted constraint approach for tranforming the scalarization parameters into new upper-level variables. After that, the original version of the mathematical programming form for the generalized Nash equilibrium problem is turned into several typical programming problems, with the Nash equilibrium problem at the lower level. The fundamental advantage of this scalarization is that the scalarization parameters are unique and strictly positive, and this scalarization is applicable to nonconvex problems. We reformulate the optimization problem into a single-level optimization problem that is globally (locally) equivalent to the initial problem using the optimal value function of the obtained lower-level generalized Nash equilibrium problems.

The second objective of the paper is to present optimality conditions in terms of the limiting subdifferentials and the limiting normal cones using the weak basic CQ condition suitable for mathematical programming for generalized Nash equilibrium optimization problems. The generalized differentiation calculus of Mordukhovich is then used to derive completely detailed first-order necessary optimality conditions in a smooth setting.

The remainder of the paper is organized as follows: In Section 2, we review the fundamental concepts and the results of variational analysis and generalized differentiation. In Section 3, we present a concept of equilibrium for generalized Nash equilibrium problems with many goals and provide its scalarization formulation. In Section 4, we first introduce the mathematical programming problem of interest as well as its associated surrogate standard optimization program. We study the equivalence in the sense of global minimizers and local minimizers. We give optimality conditions in terms of the limiting subdifferentials and the limiting normal cones using the weak basic constraint qualification appropriate for mathematical optimization problems. The smooth situation is then used to derive completely detailed first-order necessary optimality conditions using Mordukhovich's generalized differentiation calculus.

2. Preliminaries

In this section, we present various definitions, notations, and results which will be used in the sequel. Let A be a subset of $\mathbb{R}^n$, co A and clA signify the convex hull and the closure of A, respectively, and $\|.\|$ signifies an arbitrary norm in $\mathbb{R}^n$.
Following that, we give some variational analysis material that will be useful in our research. Allow Ω to be a locally closed subset of $\mathbb{R}^n$ centered on $\bar{x} \in \Omega$.

Definition 1. [19] Let $\Omega \subset \mathbb{R}^n$ be locally closed around $\bar{x} \in \Omega$. Then the Fréchet normal cone $\hat{N}(\bar{x};\Omega)$ and the Mordukhovich normal cone (limiting normal cone) $N(\bar{x};\Omega)$ to Ω at $\bar{x}$ are defined, respectively, by

$$\hat{N}(\bar{x};\Omega) := \left\{ x^* \in \mathbb{R}^n : \limsup_{x \to \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\}$$

$$N(\bar{x};\Omega) := \limsup_{x \xrightarrow{\Omega} \bar{x}} \hat{N}(x;\Omega),$$

where $x \xrightarrow{\Omega} \bar{x}$ stands for $x \to \bar{x}$ with $x \in \Omega$.

Definition 2. [18] Let $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous around $\bar{x}$.

$(\mathcal{D}_1)$: The Fréchet subdifferential of ϕ at $\bar{x}$ is

$$\hat{\partial}\phi(\bar{x}) := \left\{ x^* \in \mathbb{R}^n : \liminf_{x \to \bar{x}} \frac{\phi(x) - \phi(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}.$$

$(\mathcal{D}_2)$: The following formula represents the Mordukhovich (limited) subdifferential of ϕ at $\bar{x}$.

$$\partial\phi(\bar{x}) := \limsup_{x \to \bar{x}} \widehat{\partial}\phi(x),$$

where $x \xrightarrow{\phi} \bar{x}$ means that $x \to \bar{x}$ with $\phi(x) \to \phi(\bar{x})$.

$\partial\phi(\overline{x})$ is nonempty and compact for a local Lipschitz continuous function. Furthermore, its convex hull is the Clarke subdifferential, implying that the Clarke subdifferential of ϕ at $\overline{x}$ can be defined by

$$\partial_C\phi(\overline{x}) := \text{co } \partial\phi(\overline{x}). \tag{2.1}$$

We have the subsequent convex hull property:

$$\text{co}\,\partial(-\phi)(\overline{x}) := -\,\text{co } \partial\phi(\overline{x}). \tag{2.2}$$

The concept of semismoothness can be applied to sets by means of the Euclidean distance function d. A set $\Omega \subseteq \mathbb{R}^n$ is called semismooth at $\overline{x} \in cl\,\Omega$ if for any sequence $x_k \longrightarrow \overline{x}$ with $x_k \in \Omega$ and $\|x_k - \overline{x}\|^{-1}(x_k - \overline{x}) \longrightarrow d$ it holds that $\langle x_k^*, d\rangle \longrightarrow d$ for all selections $x_k^* \in \partial_C d_\Omega(x_k)$.
For a set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, the sets $\text{dom}\,\Phi := \{x \in \mathbb{R}^n \mid \Phi(x) \neq \emptyset\}$, and $\text{gph}\,\Phi := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in \Phi(x)\}$, denote the domain and graph of Φ, respectively. The expression

$$\limsup_{x\to\bar{x}} \Phi(x) := \{x^* \in \mathbb{R}^m : \ \exists x_k \to \bar{x},\ x_k^* \to x^* \text{ with } x_k^* \in \Phi(x_k) \text{ as } k \to \infty\}.$$

signifies the sequential Painlevé–Kuratowski upper/outer limit of Φ. Finally, the coderivative for the set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $(\bar{x},\bar{y}) \in \text{gph}\,\Phi$ is the set-valued mapping $D^*\Phi(\bar{x},\bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, given by

$$D^*\Phi(\bar{x},\bar{y})(v) := \{u \in \mathbb{R}^n \mid (u,-v) \in N_{\text{gph}\,\Phi}(\bar{x},\bar{y})\} \text{ for } \quad v \in \mathbb{R}^m.$$

Furthermore, we have the representation of the function $h : \mathbb{R}^n \to \mathbb{R}^m$ that is strictly differentiable at $\bar{x}$.

$$D^*h(\bar{x})(y^*) = \left\{\nabla h(\bar{x})^\top y^*\right\} \quad \text{for all } y^* \in \mathbb{R}^m.$$

- The set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be inner semicompact at $\bar{x}$, $\Phi(\overline{x}) \neq \emptyset$, if and only if, for every sequence $x_k \longrightarrow \bar{x}$ with $\Phi(x_k) \neq \emptyset$ there is a sequence of $y_k \in \Phi(x_k)$ that contains a convergent subsequence as $k \longrightarrow +\infty$.
- The set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be inner semicontinuous at $(\bar{x},\bar{y})$ if for any sequence $x_k \to \bar{x}$ there exists a sequence $y_k \in \Phi(x_k)$ which converges to $\bar{y}$ as $k \to \infty$.

Remark 1. [10] The inner semicompactness of the set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ holds whenever Φ is uniformly bounded and has nonempty values around $\bar{x}$, i.e., there exists a neighborhood U of $\bar{x}$ and a bounded set $V \subset \mathbb{R}^m$ such that $\Phi(x) \subset V$, for all $x \in U$.

3. The weighted-constraint scalarization to weakly generalized Nash equilibrium

Consider a multiobjective generalized Nash equilibrium problem $(MGNEP)$ composed by N players such that for any $\upsilon = 1, \ldots, N$, and $y^{-\upsilon}$, the υth player chooses y^{υ} that solves

$$\begin{cases} \min\limits_{y^{\upsilon}} \ f^{\upsilon}\left(y^{\upsilon}, y^{-\upsilon}\right) = \left(f_1^{\upsilon}\left(y^{\upsilon}, y^{-\upsilon}\right), \ldots, f_{s_{\upsilon}}^{\upsilon}\left(y^{\upsilon}, y^{-\upsilon}\right)\right) \\ \text{s.t } y^{\upsilon} \in Y^{\upsilon}\left(y^{-\upsilon}\right), \end{cases} \qquad (P^{\upsilon}[y^{-\upsilon}])$$

by controling his or her own strategy $y^{\upsilon} \in \mathbb{R}^{m_{\upsilon}}$ while taking into account the choices made by opponents, which are represented by the vector $y^{-\upsilon} \in \mathbb{R}^{m-m_{\upsilon}}$, where $m = m_1 + \ldots + m_N$ represents the total number of players' decision variables. Sometimes we use the notation $(y^{\upsilon}, y^{-\upsilon})$ instead of just y to emphasize the variables of player within the vector y. The block components are not implied to be rearranged in any manner by this notation, and $y = (y^{\upsilon}, y^{-\upsilon})$ is always the case. Furthermore, each player has an objective $f^{\upsilon} : \mathbb{R}^{m_{\upsilon}} \to \mathbb{R}^{s_{\upsilon}}$, called utility function or payoff function, which depends on his own vaiables y^{υ} as well as the variables $y^{-\upsilon}$ of the other players. Moreover, each player's strategies belong to a set $Y^{\upsilon}\left(y^{-\upsilon}\right)$ that depends on the rival player's strategie. This continuous function's components are $f_1^{\upsilon}, \ldots, f_{s_{\upsilon}}^{\upsilon} : \mathbb{R}^{m_{\upsilon}} \to \mathbb{R}$. Here after, for each υ we denote by $\Psi_{weff}^{\upsilon}\left(y^{-\upsilon}\right)$ the weakly efficient solution set of problem ($P^{\upsilon}[y^{-\upsilon}]$).
Assuming that the functions $f_k^{\upsilon}\left(., y^{-\upsilon}\right)$ have a lower bound on the constraint set $Y^{\upsilon}\left(y^{-\upsilon}\right)$ and that this lower bound is known, it is reasonable to assert that there is no loss of generality in imposing the condition below.

$$\min_{k=1,\cdots,s_{\upsilon}} \left\{ \min_{y^{\upsilon} \in Y^{\upsilon}(y^{-\upsilon})} \ f_k^{\upsilon}\left(y^{\upsilon}, y^{-\upsilon}\right) \right\} > 0, \quad \forall \upsilon = 1, \ldots, N.$$

Definition 3. [12, 25] A point $\bar{y} = \left(\bar{y}^1, \ldots, \bar{y}^N\right)$ is said to be a weakly efficient generalized Nash equilibrium of the multiobjective generalized Nash equilibrium problem $(MGNEP)$ if

$$\bar{y}^{\upsilon} \in \Psi_{\text{weff}}^{\upsilon}\left(\bar{y}^{-\upsilon}\right) \quad \forall \ \upsilon = 1, \cdots, N.$$

The set of weakly efficient generalized Nash equilibriums will be denoted by $\mathcal{H}_{\text{weff}}$. That is

$$\mathcal{H}_{\text{weff}} = \left\{ \bar{y} = \left(\bar{y}^1, \ldots, \bar{y}^N\right) \ \ \bar{y}^{\upsilon} \in \Psi_{\text{weff}}^{\upsilon}\left(\bar{y}^{-\upsilon}\right) \quad \forall \ \upsilon = 1, \cdots, N \right\}.$$

For $\upsilon = 1, \cdots, N$ set $S^{\upsilon} = \{1, \ldots, s_{\upsilon}\}$, and let

$$\mathbb{W}^{\upsilon} = \left\{ w \in \mathbb{R}^{s_{\upsilon}} : \ w_k \geqslant 0, \ \sum_{k=1}^{s_{\upsilon}} w_k = 1 \right\}, \tag{3.1}$$

and

$$\mathbb{W}_+^{\upsilon} = \left\{ w \in \mathbb{R}_+^{s_\upsilon} : \; w_k > 0, \; \sum_{k=1}^{s_\upsilon} w_k = 1 \right\}. \tag{3.2}$$

Our goal now is to transform the multiobjective generalized Nash equilibrium problem $(MGNEP)$ into a single-objective generalized Nash equilibrium problem $(GNEP)$. To do this, for each $\upsilon \in \{1, \ldots, N\}$, fix $k \in S^{\upsilon}$ and $w^{\upsilon} \in \mathbb{W}_+^{\upsilon}$, and then consider the following scalarized problem:

$$\begin{cases} \min\limits_{y^{\upsilon}} \; f_k^{\upsilon}\left(y^{\upsilon}, y^{-\upsilon}\right) \\ \text{s.t } y^{\upsilon} \in Y_k^{\upsilon}\left(y^{-\upsilon}, w^{\upsilon}\right), \end{cases} \tag{$P_k^{\upsilon}[y^{-\upsilon}, w^{\upsilon}]$}$$

where for any $y^{-\upsilon}$ and w^{υ} the feasible set and the solution set of the problem $(P_k^{\upsilon}[y^{-\upsilon}, w^{\upsilon}])$ is given by

$$Y_k^{\upsilon}\left(y^{-\upsilon}, w^{\upsilon}\right) = \left\{ y^{\upsilon} \in Y\left(y^{-\upsilon}\right) : w_r^{\upsilon} \; f_r^{\upsilon}\left(y^{\upsilon}, y^{-\upsilon}\right) \leq w_k^{\upsilon} \; f_k^{\upsilon}\left(y^{\upsilon}, y^{-\upsilon}\right), \forall \, r \in S^{\upsilon} \backslash \{k\} \right\}, \tag{3.3}$$

and $\Psi_k^{\upsilon}\left(y^{-\upsilon}, w^{\upsilon}\right)$.
For each fixed $w^{\upsilon} \in \mathbb{W}_+^{\upsilon}$, one can easily check that

$$Y^{\upsilon}\left(y^{-\upsilon}\right) = \bigcup_{k=1}^{s_\upsilon} Y_k^{\upsilon}\left(y^{-\upsilon}, w^{\upsilon}\right).$$

In the sequel, for $w \in \mathbb{W}_+ = \mathbb{W}_+^1 \times \ldots \times \mathbb{W}_+^N$ with $w = \left(w^1, \ldots, w^N\right)$, we need the following construction:

$$\mathbb{W}_+^{\upsilon}\left(y^{-\upsilon}\right) = \left\{ w^{\upsilon} \in \mathbb{W}_+^{\upsilon} : \; y^{\upsilon} \in \Psi_k^{\upsilon}\left(y^{-\upsilon}, w^{\upsilon}\right), \text{ for all } k = 1, \ldots, s_\upsilon \right\}, \quad \text{and} \tag{3.4}$$

$$\mathcal{H}(w) = \left\{ y \in \mathbb{R}^m \; : \; y^{\upsilon} \in \bigcap_{k=1}^{s_\upsilon} \Psi_k^{\upsilon}\left(y^{-\upsilon}, w^{\upsilon}\right), \quad \text{holds for each } \upsilon = 1, \cdots, N \right\}. \tag{3.5}$$

where, for each $\upsilon = 1, \ldots, N$: $\bigcap_{k=1}^{s_\upsilon} \Psi_k^{\upsilon}\left(y^{-\upsilon}, w_0^{\upsilon}\right)$ denote the commun solution.
The next result gives a relationship between the set of weakly efficient generalized Nash equilibrium $\mathcal{H}_{weff}$ and $\mathcal{H}\left(w\right)$. Before that, some comments are in order. For fixed $\upsilon = 1, \cdots, N$, and for all $w_0^{\upsilon} \in \mathbb{W}_+^{\upsilon}$ we can see from [6] that

$$\bigcap_{k=1}^{s_\upsilon} \Psi_k^{\upsilon}\left(y^{-\upsilon}, w_0^{\upsilon}\right) \subseteq \Psi_{\text{weff}}^{\upsilon}\left(y^{-\upsilon}\right) \subset \bigcup_{w \in \mathbb{W}_+^{\upsilon}} \left[\bigcap_{k=1}^{s_\upsilon} \Psi_k^{\upsilon}\left(y^{-\upsilon}, w^{\upsilon}\right) \right]. \tag{3.6}$$

We deduce from the last inclusion that if for some $w_0^\upsilon \in \mathbb{W}_+^\upsilon$ we have that $\bigcap_{k=1}^{s_\upsilon} \Psi_k^\upsilon(y^{-\upsilon}, w_0^\upsilon) \neq \emptyset$, then the left side of the above inclusion furnishes a method for computing weak efficient points. However, if $\bigcap_{k=1}^{s_\upsilon} \Psi_k^\upsilon(y^{-\upsilon}, w_0^\upsilon) = \emptyset$, we need the following assumptions to ensure the nonemptiness of $\mathbb{W}_+^\upsilon(y^{-\upsilon})$ and consequently the existence of a common solution of $(P_k^\upsilon[y^{-\upsilon}, w^\upsilon])$.

$(\mathcal{A}_1)$**:** There exists an element $w^\upsilon \in \mathbb{W}_+^\upsilon$ such that the sets $\Psi_r^\upsilon(y^{-\upsilon}, w^\upsilon) \neq \emptyset$, for all $r \in S^\upsilon$, and there exists $k \in S^\upsilon$ and $y_k^\upsilon \in \Psi_k^\upsilon(y^{-\upsilon}, w^\upsilon)$ such that $\forall r \neq k$, there exists an element $y_r^\upsilon \in \Psi_r^\upsilon(y^{-\upsilon}, w^\upsilon)$ satisfying the inequality

$$f_r^\upsilon\left(y_k^\upsilon, y_k^{-\upsilon}\right) \leq f_r^\upsilon\left(y_r^\upsilon, y_r^{-\upsilon}\right).$$

$(\mathcal{A}_2)$**:** There exists w^υ an element of $\mathbb{W}_+^\upsilon$ such that for any $k \in S^\upsilon$, we can find $\left(y_1^\upsilon, \ldots, y_{s_\upsilon}^\upsilon\right)$ an element of the cartesian product of $\Psi_1^\upsilon(y^{-\upsilon}, w^\upsilon) \times \ldots \times \Psi_{s_\upsilon}^\upsilon(y^{-\upsilon}, w^\upsilon)$ satisfying the following inequality

$$\forall r, k \in S^\upsilon, \quad f_r^\upsilon\left(\bar{y}_k^\upsilon, \bar{y}_k^{-\upsilon}\right) \leq f_r^\upsilon\left(y_r^\upsilon, y_r^{-\upsilon}\right).$$

Consequently, using [6, Proposition 3.3], under the hypothesis $(\mathcal{A}_1)$, we get that $y_k^\upsilon \in \Psi_{\text{weff}}^\upsilon(y_k^{-\upsilon})$. Under the hypothesis $(\mathcal{A}_2)$, using [6, Corollary 3.1] instead of [6, Proposition 3.3], we also find that $y_k^\upsilon \in \Psi_{\text{weff}}^\upsilon(y_k^{-\upsilon})$. Consequently, by [6, Proposition 3.2],

$$\emptyset \neq \mathbb{W}_+^\upsilon(y_k^\upsilon) = \{w_0^\upsilon\} \text{ and } y_k^\upsilon \in \bigcap_{r=1}^{s_\upsilon} \Psi_r^\upsilon(\bar{w}, y_k^{-\upsilon}).$$

Remark 2. As a result, if we suppose that for all $\upsilon = 1, \ldots, N$, $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold true, then the set $\mathcal{H}(w)$ is nonempty.

Now, we are really going to state the link between $\mathcal{H}_{weff}$ and $\mathcal{H}(w)$.

Theorem 1. *Let $w \in \mathbb{W}_+$. Suppose that $\mathbb{W}_+^\upsilon\left(y^{-\upsilon}\right) \neq \emptyset$ for any $\upsilon = 1, \ldots, N$. Then, we have*

$$\mathcal{H}_{weff} = \bigcup_{w \in \mathbb{W}_+} \mathcal{H}(w).$$

Proof. Let $\bar{y} \in \mathcal{H}_{weff}$, then $\bar{y}$ is a weakly efficient generalized Nash equilibrium. Then, from the definition 3, we have $\bar{y}^\upsilon \in \Psi_{\text{weff}}^\upsilon(\bar{y}^{-\upsilon})$ for all $\upsilon = 1, \ldots, N$. Moreover, from the right inclusion of (3.6), for each $\upsilon = 1, \ldots, N$ it follows that there exists $w^\upsilon \in \mathbb{W}_+^\upsilon$ such that $\bar{y}^\upsilon \in \bigcap_{k=1}^{s_\upsilon} \Psi_k^\upsilon(\bar{y}^{-\upsilon}, w^\upsilon)$, which implies that $\bar{y}^\upsilon \in \mathcal{H}(w)$.

Conversely, fix an arbitrary $w \in \mathbb{W}_+$ satisfying $w^\upsilon \in \mathbb{W}_+^\upsilon$ for all $\upsilon = 1, \ldots, N$. Let $\bar{y} \in \mathcal{H}(w)$. Then, from the definition, we get $\bar{y}^\upsilon \in \bigcap_{k=1}^{s_\upsilon} \Psi_k^\upsilon(\bar{y}^{-\upsilon}, w^\upsilon)$ for all $\upsilon = 1, \ldots, N$. Using the left inclusion of (3.6), it follows that $\bar{y}^\upsilon \in \Psi_{\text{weff}}^\upsilon(\bar{y}^{-\upsilon})$ for all $\upsilon = 1, \ldots, N$. This implies that $\bar{y} \in \mathcal{H}_{weff}$. □

4. Necessary optimality conditions applying weighted-constraint scalarization

This study deals with the mathematical program with a multiobjective generalized Nash equilibrium problem in the constraint:

$$\min_{x,y} F(x,y): \; x \in X, \quad y \in \mathcal{H}_{\text{weff}}(x). \tag{P}$$

The leader's decision variables are represented by $x \in X = \{x \in \mathbb{R}^n : G_i(x) \leq 0, \quad i \in 1, \ldots, p = I\}$, the follower's decision variables are represented by y, and the function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ denotes the leader's objective function, while $\mathcal{H}_{\text{weff}}(x)$ is the set of weakly efficient generalized Nash equilibrium of the multiobjective generalized Nash equilibrium problem parameterized of x composed of N players, where each player $\upsilon = 1, \ldots, N$ is assumed to minimize

$$\begin{cases} \min\limits_{y^\upsilon} f^\upsilon(x, y^\upsilon, y^{-\upsilon}) = \left(f_1^\upsilon(x, y^\upsilon, y^{-\upsilon}), \ldots, f_{s_\upsilon}^\upsilon(x, y^\upsilon, y^{-\upsilon})\right) \\ \text{s.t } y^\upsilon \in Y^\upsilon(x, y^{-\upsilon}), \end{cases} \tag{$P^\upsilon[x, y^{-\upsilon}]$}$$

where $m = m_1 + \ldots + m_N$. We use the notation $(y^\upsilon, y^{-\upsilon})$ instead of simply y. We assume that each follower player has a vector-valued objective function $f^\upsilon : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{s_\upsilon}$, and for any $\upsilon = 1, \ldots, N$ and $y^{-\upsilon}$ we denote by $\Psi_{weff}^\upsilon(x, y^{-\upsilon})$ the weakly efficient solution of problem ($P^\upsilon[x, y^{-\upsilon}]$), with

$$Y^\upsilon\left(x, y^{-\upsilon}\right) = \left\{y^\upsilon \in \mathbb{R}^{m_\upsilon} : \; g_j^\upsilon\left(x, y^\upsilon, y^{-\upsilon}\right) \leq 0, \; j \in J^\upsilon = \{1, \ldots, q_\upsilon\}\right\}.$$

where, the mapping $g^\upsilon(x, ., y^{-\upsilon}) : \mathbb{R}^n \to \mathbb{R}^{q_\upsilon}$, and $q = q_1 + \ldots + q_N$.
Assuming that the functions f_k^υ have a lower bound on the constraint set $Y^\upsilon(x, y^{-\upsilon})$ and that those lower bounds are known, it is reasonable to assert that there is no loss of generality in imposing the condition below.

$$\min_{k=1,\cdots,s_\upsilon} \left\{ \min_{y^\upsilon \in Y^\upsilon(x, y^{-\upsilon})} f_k^\upsilon\left(x, y^\upsilon, y^{-\upsilon}\right) \right\} > 0, \quad \forall \upsilon = 1, \ldots, N, \; \forall x \in X.$$

The feasible region of problem (P) is represented by

$$\Omega = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : \; x \in X, \; y \in \mathcal{H}_{\text{wef}}(x)\}.$$

4.1. Single-level reformulation

One option is to deal with hierarchical problems by transforming the optimization problem (P) into a single-level optimization problem. To accomplish this, we apply the weighted-sum technique introduced in section 3 for reformulating each lower-level problem ($P^{\upsilon}[x, y^{-\upsilon}]$) of player υ as s_{υ} standard one-level optimization problem. We begin by applying the following scalarization technique to the problem ($P^{\upsilon}[x, y^{-\upsilon}]$) for each $\upsilon \in 1, \dots, N$ and $w^{\upsilon} \in \mathbb{W}^{\upsilon}$, $\mathbb{W}^{\upsilon}$ is the sets stated in (3.1)

$$\begin{cases} \min\limits_{y^{\upsilon}} \; f_k^{\upsilon}\left(x, y^{\upsilon}, y^{-\upsilon}\right) \\ \text{s.t } y^{\upsilon} \in Y_k^{\upsilon}\left(x, y^{-\upsilon}, w^{\upsilon}\right), \end{cases} \qquad (P_k^{\upsilon}[x, y^{-\upsilon}, w^{\upsilon}])$$

where for any $y^{-\upsilon}$ and $w^{\upsilon} \in \mathbb{W}^{\upsilon}$, the feasible set and solution set of problem ($P_k^{\upsilon}[x, y^{-\upsilon}, w^{\upsilon}]$) are given by

$$Y_k^{\upsilon}\left(x, y^{-\upsilon}, w^{\upsilon}\right) = \left\{y^{\upsilon} \in Y^{\upsilon}\left(x, y^{-\upsilon}\right) : w_r^{\upsilon} \; f_r^{\upsilon}\left(x, y^{\upsilon}, y^{-\upsilon}\right) \leq w_k^{\upsilon} \; f_k^{\upsilon}\left(x, y^{\upsilon}, y^{-\upsilon}\right), \forall \; r \in S^{\upsilon} \backslash \{k\}\right\}, \tag{4.1}$$

$$\Psi_k^{\upsilon}\left(x, y^{-\upsilon}, w^{\upsilon}\right) := \left\{y^{\upsilon} \in Y_k^{\upsilon}\left(x, y^{-\upsilon}, w^{\upsilon}\right) : \; f_k^{\upsilon}\left(x, y^{\upsilon}, y^{-\upsilon}\right) - \varphi_k^{\upsilon}\left(x, y^{-\upsilon}, w^{\upsilon}\right) \leq 0, \quad k \in S^{\upsilon}\right\},$$

with

$$\varphi_k^{\upsilon}\left(x, y^{-\upsilon}, w^{\upsilon}\right) = \min_{y^{\upsilon}} \left\{f_k^{\upsilon}\left(x, y^{\upsilon}, y^{-\upsilon}\right) : y^{\upsilon} \in Y_k^{\upsilon}\left(x, y^{-\upsilon}, w^{\upsilon}\right), \quad k \in S^{\upsilon}\right\}, \text{ and } S^{\upsilon} = \{1, \dots, N\}.$$

In the sets stated above (3.4) and (3.5), we give the following sets parametrized by x. For $\upsilon = 1, \cdots, N$, let $w = \left(w^1, \dots, w^N\right) \in \mathbb{W}_+ = \mathbb{W}_+^1 \times \dots \times \mathbb{W}_+^N$, and set

$$\mathcal{H}(x, w) = \left\{y : \; y^{\upsilon} \in \bigcap_{k=1}^{s_{\upsilon}} \Psi_k^{\upsilon}\left(x, y^{-\upsilon}, w^{\upsilon}\right), \text{ for each } \upsilon = 1, \dots, N\right\},$$

$$\mathbb{W}_+^{\upsilon}\left(x, y^{\upsilon}\right) = \left\{w^{\upsilon} \in \mathbb{W}_+^{\upsilon} : \; y^{\upsilon} \in \Psi_k^{\upsilon}\left(x, y^{-\upsilon}, w^{\upsilon}\right)\right), \text{ for all } k = 1, \dots, s_{\upsilon}\} \quad \text{and}$$

$$\mathbb{W}_+(x, y) = \prod_{\upsilon=1}^{N} \mathbb{W}_+^{\upsilon}\left(x, y^{\upsilon}\right).$$

Remark 3. Let $k \in S^{\upsilon}$ and $w \in \mathbb{W}^{\upsilon}$. In case $w_k^{\upsilon} = 0$, we get $Y_k^{\upsilon}(x, y^{-\upsilon}, w^{\upsilon}) = \emptyset$.

Let $x \in X$, $y^{\upsilon} \in Y\left(x, y^{-\upsilon}\right)$ and $w_0^{\upsilon} \in \mathbb{W}_+^{\upsilon}$. Similarly to section 3, we have

$$\bigcap_{k=1}^{s_{\upsilon}} \Psi_k^{\upsilon}\left(x, y^{-\upsilon}, w_0^{\upsilon}\right) \subseteq \Psi_{\text{weff}}^{\upsilon} \; \left(x, y^{-\upsilon}\right) \subset \bigcup_{w \in \mathbb{W}_+^{\upsilon}} \left[\bigcap_{k=1}^{s_{\upsilon}} \Psi_k^{\upsilon}\left(x, y^{-\upsilon}, w^{\upsilon}\right)\right]. \tag{4.2}$$

Under the assumptions

$(\mathcal{B}_1)$: There exists $w^\upsilon \in \mathbb{W}_+^\upsilon$ such that the sets $\Psi_r^\upsilon\left(x, y^{-\upsilon}, w^\upsilon\right) \neq \emptyset$, for all $r \in S^\upsilon$ and there exists $k \in S^\upsilon$ and $y_k^\upsilon \in \Psi_k^\upsilon\left(x, y^{-\upsilon}, w^\upsilon\right)$ such that $\forall r \neq k$, there exists $y_r^\upsilon \in \Psi_r^\upsilon\left(x, y^{-\upsilon}, w^\upsilon\right)$ satisfying the inequality

$$f_r^\upsilon\left(\bar{x}, \bar{y}_k^\upsilon, \bar{y}_k^{-\upsilon}\right) \leq f_r^\upsilon\left(\bar{x}, \bar{y}_r^\upsilon, \bar{y}_r^{-\upsilon}\right).$$

$(\mathcal{B}_2)$: There exists $w^\upsilon \in \mathbb{W}_+^\upsilon$ such that $\left(y_1^\upsilon, \ldots, y_{s_\upsilon}^\upsilon\right) \in \Psi_1^\upsilon\left(x, y^{-\upsilon}, w^\upsilon\right) \times \ldots \times \Psi_{s_\upsilon}^\upsilon\left(x, y^{-\upsilon}, w^\upsilon\right)$ satisfying the following inequality

$$\forall r, k \in S^\upsilon, \quad f_r^\upsilon\left(\bar{x}, \bar{y}_k^\upsilon, \bar{y}_k^{-\upsilon}\right) \leq f_r^\upsilon\left(\bar{x}, \bar{y}_r^\upsilon, \bar{y}_r^{-\upsilon}\right),$$

we have $\mathbb{W}_+^\upsilon(x, y) \neq \emptyset$ and consequently the existence of a common solution for $(P_k^\upsilon[x, y^{-\upsilon}, w^\upsilon])$ is guaranteed.

Remark 4. Considering that, $(\mathcal{B}_1)$ and $(\mathcal{B}_2)$ are true for all $\upsilon = 1, \ldots, N$, the set $\mathcal{H}(x, w)$ is nonempty.

Remark 5. By [6, Proposition 3.3], under the hypothesis $(\mathcal{B}_1)$, we get that $y_k^\upsilon \in \Psi_{\text{weff}}^\upsilon\left(x, y_k^{-\upsilon}\right)$. Under the hypothesis $(\mathcal{B}_2)$, using [6, Corollary 3.1] instead of [6, Proposition 3.3], we find also that $y_k^\upsilon \in \Psi_{\text{weff}}^\upsilon\left(x, y_k^{-\upsilon}\right)$. Consequently, by [6, Proposition 3.2],

$$\emptyset \neq \mathbb{W}_+^\upsilon\left(x, y_k^\upsilon\right) = \{w_0^\upsilon\} \text{ and } y_k^\upsilon \in \bigcap_{r=1}^{s_\upsilon} \Psi_r(x, y_k^{-\upsilon}, w_0^\upsilon).$$

Now, we are really going to state the link between $\mathcal{H}_{weff}(x)$ and $\mathcal{H}(x, w)$.

Theorem 2. *Let $x \in X$ and $w \in \mathbb{W}_+$. Suppose that $\mathbb{W}_+^\upsilon\left(x, y^\upsilon\right) \neq \emptyset$ for any $\upsilon = 1, \ldots, N$. Then, we have*

$$\mathcal{H}_{weff}(x) = \bigcup_{w \in \mathbb{W}_+} \mathcal{H}(x, w).$$

Proof. It follows the path of that of Theorem 1. □

Hence, the optimization problem (P) with a multiobjective Nash equilibruim problem at the lower level can be replaced by the following optimization problem where each follower has a single objective

$$\begin{cases} \min\limits_{x,y,w} \; F(x, y) \\ \text{s.t} \;\; x \in X, \;\; w \in \mathbb{W}, \\ \qquad y \in \mathcal{H}(x, w). \end{cases} \tag{4.3}$$

The following result shows the equivalence between problem (P) and problem (4.3).

Proposition 1. *Consider problems (P) and (4.3).*

(1) Let $(\bar{x}, \bar{y})$ be a local (resp. global) optimal solution of problem (P). Then, for any $\bar{w} \in \mathbb{W}_+(\bar{x}, \bar{y})$, the point $(\bar{x}, \bar{y}, \bar{w})$ is a local (resp. global) optimal solution of problem (4.3).

(2) *Let $\bar{w} \in \mathbb{W}_+(\bar{x}, \bar{y})$. Assume that $(\bar{x}, \bar{y}, \bar{w})$ is a global optimal solution of problem (4.3). Then, the point $(\bar{x}, \bar{y})$ is a global optimal solution of problem (P).*

(3) *Assume that $(\bar{x}, \bar{y}, \bar{w})$ is a local optimal solution of (4.3) for all $\bar{w} \in \mathbb{W}_+(\bar{x}, \bar{y})$ and suppose that the set-valued mappings $\mathcal{H}$, is closed at all points from $\{\bar{x}\} \times \mathbb{W}_+$. Then, $(\bar{x}, \bar{y})$ is a local optimal solution of problem (P).*

Proof. (1) Suppose that there exists $\widehat{w} \in \mathbb{W}_+(\bar{x}, \bar{y})$ such that $(\bar{x}, \bar{y}, \widehat{w})$ is not a local solution to problem (4.3). Then, there exists a sequence $(x_\varepsilon, y_\varepsilon, w_\varepsilon)$ with $x_\varepsilon \longrightarrow \bar{x}$, $y_\varepsilon \longrightarrow \bar{y}$, and $\widehat{w}_\varepsilon \longrightarrow \widehat{w}$ such that

$$F(x_\varepsilon, y_\varepsilon) < F(\bar{x}, \bar{y}), \ x_\varepsilon \in X, \ \widehat{w}_\varepsilon \in \mathbb{W} = \mathbb{W}^1 \times \ldots \times \mathbb{W}^N, \ y_\varepsilon \in \mathcal{H}(x_\varepsilon, \widehat{w}_\varepsilon), \text{ for all } \varepsilon \in \mathbb{N}. \tag{4.4}$$

We claim that $\widehat{w}_\varepsilon \in \mathbb{W}_+$. Indeed, by the contrary, suppose that there exists $\upsilon = 1, \ldots, N$ such that $\widehat{w}_\varepsilon^\upsilon \notin \mathbb{W}_+^\upsilon$. ie there exists $k_\upsilon \in S^\upsilon$ such that $\widehat{w}_\varepsilon^{k_\upsilon} = 0$. Letting $\varepsilon \to +\infty$, we obtain $\widehat{w}^{k_\upsilon} = 0$ a contradiction with $\widehat{w}^\upsilon \in \mathbb{W}_+^\upsilon(\bar{x}, \bar{y})$.
Now, by (4.4), we get $\widehat{w}_\varepsilon \in \mathbb{W}_+$, $y_\varepsilon \in \mathcal{H}(x_\varepsilon, \widehat{w}_\varepsilon), \forall \varepsilon \in \mathbb{N}$. Then, $\widehat{w}_\varepsilon \in \mathbb{W}_+(x_\varepsilon, y_\varepsilon)$, $\forall \varepsilon \in \mathbb{N}$, which ensure that $\mathbb{W}_+(x_\varepsilon, y_\varepsilon) \neq \emptyset$. From Theorem 2 we get $y_\varepsilon \in \mathcal{H}_{\text{weff}}(x_\varepsilon)$ for all $\varepsilon \in \mathbb{N}$. As a conclusion, there exists a sequence $(x_\varepsilon, y_\varepsilon)$ converging to $(\bar{x}, \bar{y})$ with $x_\varepsilon \in X$, $y_\varepsilon \in \mathcal{H}_{\text{weff}}(x_\varepsilon)$ such that

$$F(x_\varepsilon, y_\varepsilon) < F(\bar{x}, \bar{y}), \text{ for all } \varepsilon \in \mathbb{N},$$

which is contrary to the fact that $(\bar{x}, \bar{y})$ is a local optimal solution of problem (P).

(2) Let $\bar{w} \in \mathbb{W}_+$, and let $(\bar{x}, \bar{y}, \bar{w})$ be a global optimal solution of problem (4.3). Assume that $(\bar{x}, \bar{y})$ is not a global optimal solution of problem (P). Then, we can find (x, y) with $x \in X$, and $y \in \mathcal{H}_{weff}(x)$ such that

$$F(x, y) < F(\bar{x}, \bar{y}).$$

From Theorem 2, there exists $w \in \mathbb{W}_+$ such that $y \in \mathcal{H}(x, w)$. Consequently (x, y, w) is a feasible point of problem (4.3) such that

$$F(x, y) < F(\bar{x}, \bar{y}),$$

which contrary to the fact that $(\bar{x}, \bar{y}, \bar{w})$ is a global optimal solution of problem (4.3).

(3) Let $\bar{w} \in \mathbb{W}_+(\bar{x}, \bar{y})$ such that $(\bar{x}, \bar{y}, \bar{w})$ is a local optimal solution of (4.3). Suppose that $(\bar{x}, \bar{y})$ is not a local optimal solution of problem (P). Then, there exists a sequence $(x_\varepsilon, y_\varepsilon)$ with $x_\varepsilon \longrightarrow \bar{x}$, and $y_\varepsilon \longrightarrow \bar{y}$, with $x_\varepsilon \in X$, and $y_\varepsilon \in \mathcal{H}_{\text{weff}}(x_\varepsilon)$ such that

$$F(x_\varepsilon, y_\varepsilon) < F(\bar{x}, \bar{y}), \text{ for all } \varepsilon \in \mathbb{N}.$$

For any (x, y), we have $\mathbb{W}_+(x, y) \subset \mathbb{B}(0, 1)$. Hence, $\mathbb{W}_+$ is uniformly bounded, then $\mathbb{W}_+$ inner semicompact at $(\bar{x}, \bar{y})$ (see Remark 1).

Since $y_\varepsilon \in \mathcal{H}_{\text{weff}}(x_\varepsilon)$, using Theorem 2, there exists $w_\varepsilon \in \mathbb{W}_+$ such that $y_\varepsilon \in \mathcal{H}(x_\varepsilon, w_\varepsilon)$. Consequently, $w_\varepsilon \in \mathbb{W}_+(x_\varepsilon, y_\varepsilon)$, and $\mathbb{W}_+(x_\varepsilon, y_\varepsilon) \neq \emptyset$. By the inner semicompactness of $\mathbb{W}_+$ at $(\bar{x}, \bar{y})$, we can find a sequence $w_\varepsilon \in \mathbb{W}_+(x_\varepsilon, y_\varepsilon)$ which has an accumulation point $w_0 \in \mathbb{W}$. Then, $w_\varepsilon \in \mathbb{W}_+$, $y_\varepsilon \in \mathcal{H}(x_\varepsilon, w_\varepsilon)$, for all $\varepsilon \in \mathbb{N}$.

Since the mappings $\mathcal{H}$ is closed at $(\bar{x}, w_0)$, we have $\bar{y} \in \mathcal{H}(\bar{x}, w_0)$.

We claim that $w_0 \in \mathbb{W}_+$. Indeed, suppose that there exists $\upsilon = 1, \ldots, N$ such that $w_0^\upsilon \notin \mathbb{W}_+^\upsilon$. By the opposite, if there are $k_\upsilon \in S^\upsilon$ such that $w_0^{k_\upsilon} = 0$, we get $\mathcal{H}(\bar{x}, w_0) = Y_k^\upsilon(\bar{x}, \bar{y}^{-\upsilon}, w_0) = \emptyset$. A contradiction to $\bar{y} \in \mathcal{H}(\bar{x}, w_0)$.

Then, $(\bar{x}, \bar{y}, w_0)$ is a feasible point of problem (4.3). Therefore, there exists $(x_\varepsilon, y_\varepsilon, w_\varepsilon)$ converging to $(\bar{x}, \bar{y}, w_0)$ such that

$$F(x_\varepsilon, y_\varepsilon) < F(\bar{x}, \bar{y}), \ x_\varepsilon \in X, \ w_\varepsilon \in \mathbb{W}_+, \ y_\varepsilon \in \mathcal{H}(x_\varepsilon, w_\varepsilon),$$

which contradicts the fact that $(\bar{x}, \bar{y}, \bar{w})$ is a local optimal solution of problem (4.3).

□

For $\upsilon = 1, \ldots, N$, $k \in S^\upsilon$, setting

$$\begin{aligned} \psi_k^\upsilon(x, y, w^\upsilon) &= f_k^\upsilon\left(x, y^\upsilon, y^{-\upsilon}\right) - \varphi_k^\upsilon\left(x, y^{-\upsilon}, w^\upsilon\right), \text{and} \\ h_{k,r}^\upsilon(x, y, w^\upsilon) &= w_r^\upsilon f_r^\upsilon(x, y) - w_k^\upsilon f_k^\upsilon(x, y), \ r \in S^\upsilon \setminus \{k\}. \end{aligned}$$

and using the optimal value reformulation, (4.3) is equivalent to the one-level optimization problem.

$$\begin{cases} \min\limits_{x,y,w} F(x, y) \\ x \in X, \ w^\upsilon \in \mathbb{W}^\upsilon, \upsilon = 1, \ldots, N, \\ g^\upsilon(x, y) \leq 0, \ \forall \upsilon = 1, \ldots, N, \\ h_{k,r}^\upsilon(x, y, w^\upsilon) \leq 0, \ \forall \upsilon = 1, \ldots, N, \ \forall r \in S^\upsilon \setminus \{k\}, \ \forall k \in S^\upsilon, \\ f_k^\upsilon\left(x, y^\upsilon, y^{-\upsilon}\right) - \varphi_k^\upsilon\left(x, y^{-\upsilon}, w^\upsilon\right) \leq 0, \ \forall \upsilon = 1, \ldots, N, \ \forall k \in S^\upsilon. \end{cases} \tag{4.5}$$

4.2. Necessary optimality conditions

In the second part of this section, we employ the results obtained above to derive new forms of necessary optimality conditions for the original mathematical program problem (P). In order to make the following as short as possible, we introduce the following notations: for any $\upsilon \in \{1, \cdots, N\}$, $k \in \mathbb{T} = \{1, \ldots, t\}$ with $t = \max\limits_{\upsilon=1}^{N} s_\upsilon$ and $y^{-\upsilon}$, we define

$$\Theta_k = \Big\{(x, y, w) : \ x \in X, \ w \in \mathbb{W}, \ y = \left(y^1, \ldots, y^N\right) \in Y_k^1\left(x, y^{-1}, w^1\right) \times \ldots \times Y_k^N\left(x, y^{-N}, w^N\right)\Big\}.$$

$$\Omega_k = \Big\{(x, y, w) \in \Theta_k : \ \psi_k(x, y, w) = \big(\psi_k^1(x, y, w), \ldots, \psi_k^N(x, y, w)\big) \leq 0\Big\}.$$

$$\Xi = \Big\{(x, y, w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{s_1 + \ldots + s_N} : \ x \in X, \ w \in \mathbb{W}, \ y \in Y^1\left(x, y^{-1}\right) \times \ldots \times Y^N\left(x, y^{-N}\right)\Big\},$$

$$\Pi_k = \Big\{(x, y, w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{s_1 + \ldots + s_N} : \Upsilon_k(x, y, w) \in \mathbb{R}_-^{s_1 + \ldots + s_N}\Big\},$$

where

$$\Upsilon_k^\upsilon(x, y, w^\upsilon) = \Big(h_{k,1}^\upsilon(x, y, w^\upsilon), \cdots, h_{k,(k-1)}^\upsilon(x, y, w^\upsilon), \psi_k^\upsilon(x, y, w^\upsilon), h_{k,(k+1)}^\upsilon(x, y, w^\upsilon), \cdots, h_{k,s_\upsilon}^\upsilon(x, y, w^\upsilon)\Big),$$

$$\Upsilon_k(x, y, w) = \Big(\Upsilon_k^1\left(x, y, w^1\right), \ldots, \Upsilon_k^N\left(x, y, w^N\right)\Big).$$

Remark 6. 1) Under the above notations, we have

$$\Omega = \bigcap_{k=1}^{t} \Omega_k \text{ and } \Omega_k = \Xi \cap \Pi_k.$$

2) Note that for $\upsilon = 1, \dots, N$, when $k \notin S^\upsilon$ we have $Y_k^\upsilon\left(x, y^{-\upsilon}, w^\upsilon\right) = \mathbb{R}^{m_\upsilon}$ and $f^\upsilon = 0$. In this case, the sets Ω_k, Θ_k are well defined.

To provide necessary optimalty condition for (P), we will make some hypotheses throughout the paper, as follows:

($\mathcal{R}_1$): The set Ξ is regular and semismooth at $(\bar{x}, \bar{y}, \bar{w})$ such that

$$D^*\Upsilon_k\left(\bar{x}, \bar{y}, \bar{w}\right)(\sigma) \ \cap \ (-\text{bd } N_\Xi\left(\bar{x}, \bar{y}, \bar{w}\right) = \emptyset, \text{ for all } \sigma \in N\left(\Upsilon_k\left(\bar{x}, \bar{y}, \bar{w}\right), \mathbb{R}^s_-\right) \setminus \{0\}.$$

($\mathcal{R}_2$): Suppose that

$$\left[x_1^* + \dots + x_t^* = 0, x_k^* \in N_{\Omega_k}\left(\bar{x}, \bar{y}, \bar{w}\right)\right] \Longrightarrow x_k^* = 0, \ k \in \mathcal{T}.$$

($\mathcal{R}_3$): The upper-level regularity of (P) at $\bar{x} \in X$ is given by

$$\left.\begin{array}{r} \sum\limits_{i=1}^{p} \alpha_i \nabla G_i(\bar{x}) = 0 \\ \alpha_i \geq 0, \quad \alpha_i G_i(\bar{x}) = 0, \quad i \in I, \end{array}\right\} \Longrightarrow \left[\alpha_i = 0, i \in I\right].$$

($\mathcal{R}_4$): For $\upsilon = 1, \dots, N$, the lower-level regularity of $(P^\upsilon[x, y^{-\upsilon}])$ at $(\bar{x}, \bar{y}^\upsilon) \in X \times Y^\upsilon\left(\bar{x}, \bar{y}^{-\upsilon}\right)$

$$\left.\begin{array}{r} \sum\limits_{j=1}^{q_\upsilon} \beta_j^\upsilon \nabla_{y^\upsilon} g_j^\upsilon(\bar{x}, \bar{y}) = 0, \\ \beta_j^\upsilon \geq 0, \quad \beta_j^\upsilon g_j^\upsilon(\bar{x}, \bar{y})) = 0, \quad j \in J^\upsilon, \end{array}\right\} \Longrightarrow \left[\beta_j^\upsilon = 0, j \in J^\upsilon.\right]$$

($\mathcal{R}_5$): For $\upsilon = 1, \dots, N$, the lower-level regularity of $(P_k^\upsilon[x, y^{-\upsilon}, w^\upsilon])$,$k \in S^\upsilon$, at $(\bar{x}, \bar{y}^\upsilon, \bar{w}^\upsilon) \in X \times Y_k^\upsilon\left(\bar{x}, \bar{y}^{-\upsilon}, \bar{w}^\upsilon\right) \times \mathbb{W}^\upsilon$.
- In case $s_\upsilon \geq 2$, $\upsilon = 1, \dots, N$ it is given by

$$\left.\begin{array}{r} \sum\limits_{j=1}^{q_\upsilon} \beta_j^\upsilon \nabla_{y^\upsilon} g_j^\upsilon(\bar{x}, \bar{y}) + \sum\limits_{r \in S \setminus \{k\}} \gamma_r^k \nabla_{y^\upsilon} h_{k,r}^\upsilon(\bar{x}, \bar{y}, \bar{w}^\upsilon) = 0 \\ \beta_j^\upsilon \geq 0, \quad \beta_j^\upsilon g_j^\upsilon(\bar{x}, \bar{y}) = 0, \quad j \in J^\upsilon, \\ \gamma_r^k \geq 0, \gamma_r^k h_{k,r}^\upsilon\left(\bar{x}, \bar{y}, \bar{w}^\upsilon\right) = 0, r \in S^\upsilon \setminus \{k\} \end{array}\right\}$$
$$\Longrightarrow \left[\gamma_r^k = 0, \beta_j^\upsilon = 0, r \in S^\upsilon \setminus \{k\}, j \in J^\upsilon\right]$$

In case $s_\upsilon = 1$, $\upsilon = 1, \dots, N$ it is given by

$$\left.\begin{array}{r} \sum\limits_{j=1}^{q_\upsilon} \beta_j^\upsilon \nabla_{y^\upsilon} g_j^\upsilon(\bar{x}, \bar{y}) = 0, \\ \beta_j^\upsilon \geq 0, \quad \beta_j^\upsilon g_j^\upsilon(\bar{x}, \bar{y})) = 0, \quad j \in J^\upsilon, \end{array}\right\} \Longrightarrow \left[\beta_j^\upsilon = 0, j \in J^\upsilon.\right]$$

Theorem 3. *Let $(\bar{x},\bar{y},\bar{w})$ be a local optimal solution of problem (4.5). For $\upsilon=1,\dots,N$, assume that*

($\mathcal{T}_1$) *The functions f^υ and g^υ are strictly differentiable at $(\bar{x},y^\upsilon)$ for all $y^\upsilon \in \bigcap_{k=1}^{s} \Psi_k^\upsilon\left(\bar{x},\bar{y}^{-\upsilon},\bar{w}^\upsilon\right)$.*

($\mathcal{T}_2$) *The functions F and G are strictly differentiable at $(\bar{x},\bar{y})$ and $\bar{x}$, respectively.*

($\mathcal{T}_3$) *The solution set mappings $\Psi_k^\upsilon, k\in S^\upsilon$ are inner semicompact at $\left(\bar{x},\bar{y}^{-\upsilon},\bar{w}^\upsilon\right)$ and for all $y^\upsilon \in \bigcap_{k=1}^{s} \Psi_k^\upsilon\left(\bar{x},\bar{y}^{-\upsilon},\bar{w}^\upsilon\right)$ the point $(\bar{x},y^\upsilon,\bar{w}^\upsilon)$ is lower level regular.*

($\mathcal{T}_4$) *The hypotheses $(\mathcal{R}_1)$, $(\mathcal{R}_2)$, $(\mathcal{R}_3)$ and $(\mathcal{R}_4)$ are satisfied at $(\bar{x},\bar{y},\bar{w})$, $(\bar{x},\bar{y},\bar{w})$, $\bar{x}$ and $(\bar{x},\bar{y}^\upsilon)$ respectively.*

Then there are scalars $\sigma_k^\upsilon \in \mathbb{R}_+^{s_\upsilon}$, $\alpha_i \in \mathbb{R}_+^{p}$, $\beta_{k,l}^\upsilon, \beta_j^\upsilon \in \mathbb{R}_+^{q_\upsilon}$, $\pi_{k,l}^\upsilon \geq 0, \gamma_{k,l}^\upsilon \in \mathbb{R}_+^{s_\upsilon}$ and $y_{k,l}^\upsilon \in \Psi_k^\upsilon\left(\bar{x},y_{l,k}^{-\upsilon},\bar{w}^\upsilon\right)$, with $k\in S^\upsilon$ and $l=1,\dots,n+1$ such that $\sum_{l=1}^{n+1}\pi_{l,k}^\upsilon=1$ is satisfied with

$$
\begin{aligned}
0 \;=\;& \nabla_x F(\bar{x},\bar{y}) + \sum_{\upsilon=1}^{N}\left(\sum_{k=1}^{s_\upsilon}\left(\sum_{r\in S^\upsilon\setminus\{k\}} \sigma_{k,r}^\upsilon\left(\bar{w}_r^\upsilon\nabla_x f_r^\upsilon(\bar{x},\bar{y}) - \bar{w}_k^\upsilon\nabla_x f_k^\upsilon(\bar{x},\bar{y})\right)\right)\right)\\
&+\sum_{\upsilon=1}^{N}\left(\sum_{k=1}^{s_\upsilon}\sigma_k^\upsilon\left[\nabla_x f_k^\upsilon(\bar{x},\bar{y}) - \sum_{l=1}^{n+1}\pi_{k,l}^\upsilon\left(\nabla_x f_k^\upsilon\left(\bar{x},y_{k,l}^\upsilon,\bar{y}^{-\upsilon}\right) + \sum_{j=1}^{q_\upsilon}\beta_{j,k}^{\upsilon,l}\nabla_x g_j^\upsilon\left(\bar{x},y_{k,l}^\upsilon,\bar{y}^{-\upsilon}\right)\right.\right.\right.\\
&\left.\left.\left.+\sum_{r\in S^\upsilon\setminus\{k\}}\gamma_{k,r}^{\upsilon,l}\left(\bar{w}_r^\upsilon\nabla_x f_r^\upsilon\left(\bar{x},y_{k,l}^\upsilon,\bar{y}^{-\upsilon}\right) - \bar{w}_k^\upsilon\nabla_x f_k^\upsilon\left(\bar{x},y_{k,l}^\upsilon,\bar{y}^{-\upsilon}\right)\right)\right)\right] + \sum_{j=1}^{q_\upsilon}\beta_j^\upsilon\nabla_x g_j^\upsilon(\bar{x},\bar{y})\right)\\
&+\sum_{i=1}^{p}\alpha_i\nabla G_i(\bar{x}),
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& \nabla_y F(\bar{x},\bar{y}) + \sum_{\upsilon=1}^{N}\left[\sum_{k=1}^{s_\upsilon}\left(\sum_{r\in S^\upsilon\setminus\{k\}}\sigma_{k,r}^\upsilon\left(\bar{w}_r^\upsilon\nabla_{y^\upsilon} f_r^\upsilon(\bar{x},\bar{y}) - \bar{w}_k^\upsilon\nabla_{y^\upsilon} f_k^\upsilon(\bar{x},\bar{y})\right)\right)\right.\\
&\left.+\sum_{k=1}^{s_\upsilon}\sigma_k^\upsilon\;\nabla_{y^\upsilon} f_k^\upsilon(\bar{x},\bar{y}) + \sum_{j=1}^{q_\upsilon}\beta_j^\upsilon\nabla_{y^\upsilon} g_j^\upsilon(\bar{x},\bar{y})\right],
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& \nabla_{y^\upsilon} f_k^\upsilon\left(\bar{x},y_{k,l}^\upsilon,\bar{y}^{-\upsilon}\right) + \sum_{j=1}^{q_\upsilon}\beta_{j,k}^{\upsilon,l}\nabla_{y^\upsilon} g_j^\upsilon\left(\bar{x},y_{k,l}^\upsilon,\bar{y}^{-\upsilon}\right) + \sum_{r\in S^\upsilon\setminus\{k\}}\gamma_{k,r}^{\upsilon,l}\left(\bar{w}_r^\upsilon\nabla_{y^\upsilon} f_r^\upsilon\left(\bar{x},y_{k,l}^\upsilon,\bar{y}^{-\upsilon}\right)\right.\\
&\left.-\,\bar{w}_k^\upsilon\nabla_{y^\upsilon} f_k^\upsilon\left(\bar{x},y_{k,l}^\upsilon,\bar{y}^{-\upsilon}\right)\right), \qquad \upsilon=1,\dots,N, \forall k\in S^\upsilon, \forall l=1,\dots,n+s+1.
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& \gamma_{k,r}^{\upsilon,l}\left(\bar{w}_r^\upsilon f_r^\upsilon\left(\bar{x},y_{k,l}^\upsilon,\bar{y}^{-\upsilon}\right) - \bar{w}_k^\upsilon f_k^\upsilon\left(\bar{x},y_{k,l}^\upsilon,\bar{y}^{-\upsilon}\right)\right),\\
0 \;=\;& \beta_{j,k}^{\upsilon,l} g_j^\upsilon\left(\bar{x},y_{k,l}^\upsilon,\bar{y}^{-\upsilon}\right), \qquad \upsilon=1,\dots,N,\ \forall\, k\in S^\upsilon, \forall\, j\in J^\upsilon,\\
& \forall\, r\in S^\upsilon\setminus\{k\}, \forall\, l=1,\dots,n+s+1.
\end{aligned}
$$

$$
0 \;=\; \beta_j^\upsilon g_j^\upsilon(\bar{x},\bar{y}),\quad 0=\alpha_i G_i(\bar{x}),\qquad \upsilon=1,\dots,N,\ \forall\, j\in J^\upsilon, \forall\, i\in I.
$$

$$
0 \;=\; \gamma_{k,r}^{\upsilon}\left(\bar{w}_r^\upsilon f_r^\upsilon\left(\bar{x},y_k^\upsilon,\bar{y}^{-\upsilon}\right) - \bar{w}_k^\upsilon f_k^\upsilon\left(\bar{x},y_k^\upsilon,\bar{y}^{-\upsilon}\right)\right),\qquad \upsilon=1,\dots,N,\ \forall\, r\in S^\upsilon\setminus\{k\}, \forall\, k\in S^\upsilon.
$$

$$
0 \;=\; \sigma_{k,r}^{\upsilon}\left(\bar{w}_r^\upsilon f_r^\upsilon(\bar{x},\bar{y}) - \bar{w}_k^\upsilon f_k^\upsilon(\bar{x},\bar{y})\right),\qquad \upsilon=1,\dots,N,\ \forall\, r\in S^\upsilon\setminus\{k\}, \forall\, k\in S^\upsilon.
$$

Proof. Let $(\bar{x}, \bar{y}, \bar{w})$ be a local optimal solution of problem (4.5). Hence, from [19, Proposition 5.1], and using the strict differentiability assumption of the function F at $(\bar{x}, \bar{y})$, we have

$$0 \in \nabla F(\bar{x}, \bar{y}) + N_{\Omega}(\bar{x}, \bar{y}, \bar{w}).$$

According to the regularity hypothesis $(\mathcal{R}_2)$, using [18, Corollary 3.37], we get

$$0 \in \nabla F(\bar{x}, \bar{y}) + \sum_{k=1}^{t} N_{\Omega_k}(\bar{x}, \bar{y}, \bar{w}).$$

Using the inner semicompactness of $\Psi_k^{\upsilon}, \upsilon = 1, \ldots, N,\ k \in S^{\upsilon}$ at $(\bar{x}, \bar{y}^{-\upsilon}, \bar{w}^{\upsilon})$, and the lower level regularity $(\mathcal{R}_5)$ at $(\bar{x}, y^{\upsilon}, \bar{w})$ for all $y^{\upsilon} \in \Psi_k^{\upsilon}(\bar{x}, \bar{y}^{-\upsilon}, \bar{w}^{\upsilon})$, $\upsilon = 1, \ldots, N$, we deduce from [20, Theorem 5.2(ii)], that for $\upsilon = 1, \ldots, N$, φ_k^{υ} and Υ_k are Lipschitz continuity around $(\bar{x}, \bar{y}^{-\upsilon}, \bar{w}^{\upsilon})$ and $(\bar{x}, \bar{y}, \bar{w})$ respectively. Hence, applying [10, Lemma 3.3] there exist $\sigma_k \geqslant 0$ with $\sigma_k = (\sigma_k^1, \ldots, \sigma_k^N)$, $k \in \mathbb{T}$ where $(\sigma_k^1, \ldots, \sigma_k^N) = \left(\left(\sigma_{k,1}^1, \ldots, \sigma_{k,s_1}^1\right), \ldots, \left(\sigma_{k,1}^N, \ldots, \sigma_{k,s_N}^N\right)\right)$ such that

$$0 \in \nabla F(\bar{x}, \bar{y}) + \sum_{k=1}^{t} \partial\langle \sigma_k, \Upsilon_k\rangle(\bar{x}, \bar{y}, \bar{w}) + N_{\Xi}(\bar{x}, \bar{y}, \bar{w}),$$

while taking into account the regularity and semi-smoothness property of Ξ, the satisfaction of $(\mathcal{R}_1)$ at $(\bar{x}, \bar{y}, \bar{w})$ and the fact that $\Omega_k = \Xi \cap \Pi_k$,(see Remarks 6).

Let us first calculate $A = \sum_{k=1}^{s} \partial\langle \sigma_k, \Upsilon_k\rangle(\bar{x}, \bar{y}, \bar{w})$. From [13, Theorem 4.1] one has

$$\partial\langle \sigma_k, \Upsilon_k\rangle(\bar{x}, \bar{y}, \bar{w}) = \sum_{\upsilon=1}^{N} \sum_{r \in S^{\upsilon}\setminus\{k\}} \sigma_{k,r}^{\upsilon} \nabla_{x, y^{\upsilon}, w^{\upsilon}} h_{k,r}^{\upsilon}(\bar{x}, \bar{y}, \bar{w}^{\upsilon}) + \sigma_k^{\upsilon} \partial_{x, y^{\upsilon}, w^{\upsilon}} \psi_{k,k}^{\upsilon}(\bar{x}, \bar{y}, \bar{w}^{\upsilon}).$$

By summing over k, we obtain

$$A = \sum_{k=1}^{t}\left[\sum_{\upsilon=1}^{N} \sum_{r \in S^{\upsilon}\setminus\{k\}} \sigma_{k,r}^{\upsilon} \nabla_{x, y^{\upsilon}, w^{\upsilon}} h_{k,r}^{\upsilon}(\bar{x}, \bar{y}, \bar{w}^{\upsilon}) + \sigma_k^{\upsilon} \partial_{x, y^{\upsilon}, w^{\upsilon}} \psi_{k,k}^{\upsilon}(\bar{x}, \bar{y}, \bar{w}^{\upsilon})\right].$$

Since $t = \max_{\upsilon=1}^{N} s_{\upsilon}$, reordering the summation, we get

$$A = \sum_{\upsilon=1}^{N}\left(\sum_{k=1}^{s_{\upsilon}}\left(\left(\sum_{r \in S^{\upsilon}\setminus\{k\}} \sigma_{k,r}^{\upsilon} \nabla_{x, y^{\upsilon}, w^{\upsilon}} h_{k,r}^{\upsilon}(\bar{x}, \bar{y}, \bar{w}^{\upsilon})\right) + \sigma_k^{\upsilon} \partial_{x, y^{\upsilon}, w^{\upsilon}} \psi_k^{\upsilon}(\bar{x}, \bar{y}, \bar{w}^{\upsilon})\right)\right).$$

Consequently

$$\begin{aligned} 0 \in \nabla F(\bar{x}, \bar{y}) \; + \; & \sum_{\upsilon=1}^{N}\left(\sum_{k=1}^{s_{\upsilon}}\left(\left(\sum_{r \in S^{\upsilon}\setminus\{k\}} \sigma_{k,r}^{\upsilon} \nabla_{x, y^{\upsilon}, w^{\upsilon}} h_{k,r}^{\upsilon}(\bar{x}, \bar{y}, \bar{w}^{\upsilon})\right)\right.\right. \\ & \left.\left. + \sigma_k^{\upsilon} \partial_{x, y^{\upsilon}, w^{\upsilon}} \psi_k^{\upsilon}(\bar{x}, \bar{y}, \bar{w}^{\upsilon})\right)\right) + N_{\Xi}(\bar{x}, \bar{y}, \bar{w}), \end{aligned} \tag{4.6}$$

and

$$\sigma_k^{\upsilon} \geq 0,\ \sigma_{r,k}^{\upsilon} \geq 0 \quad h_{k,r}^{\upsilon}\left(\bar{x},\bar{y},\bar{w}^{\upsilon}\right)=0, \qquad \upsilon=1,\ldots,N,\ \forall\ r\in S^{\upsilon}\backslash\{k\}.$$

For any $\upsilon=1,\ldots,N$, $k\in S^{\upsilon}$ and $r\in S^{\upsilon}\backslash\{k\}$ we have

$$\nabla h_{k,r}^{\upsilon}\left(\bar{x},\bar{y},\bar{w}^{\upsilon}\right)=\left\{\begin{array}{l}\bar{w}_r^{\upsilon}\nabla_x f_r^{\upsilon}\left(\bar{x},\bar{y}\right)-\bar{w}_k^{\upsilon}\nabla_x f_k^{\upsilon}\left(\bar{x},\bar{y}\right)\\ \bar{w}_r^{\upsilon}\nabla_{y^{\upsilon}} f_r^{\upsilon}\left(\bar{x},\bar{y}\right)-\bar{w}_k^{\upsilon}\nabla_{y^{\upsilon}} f_k^{\upsilon}\left(\bar{x},\bar{y}\right)\\ \nabla_{w^{\upsilon}} h_{k,r}^{\upsilon}\left(\bar{x},\bar{y},\bar{w}^{\upsilon}\right)\end{array}\right\}. \tag{4.7}$$

Now, let $\Phi(x,y,w)=[G_i(x),\ i\in I,\ g_j^{\upsilon}(x,y),\upsilon=1,\ldots,N,j\in J^{\upsilon},\ -w]^{\top}$, and $\phi(x,y,w)=w^{\top}e_{\mathbb{R}^s}-1$. Starting with $N_{\Xi}(\bar{x},\bar{y},\bar{w})$, let us note that Ξ can be rewritten as $\Xi=\{(x,y,w):\Phi(x,y,w)\leq 0,\phi(x,y,w)=0\}$. Now, the fulfillment of both the upper-level ($\mathcal{R}_3$) and lower-level ($\mathcal{R}_4$) regularity conditions implies the satisfaction of the following implication:

$$\left.\begin{array}{r}\langle u,\nabla\Phi(\bar{x},\bar{y},\bar{w})\rangle+\langle v,\nabla\phi(\bar{x},\bar{y},\bar{w})\rangle=0\\ u\geq 0,\langle u,\Phi(\bar{x},\bar{y},\bar{w})\rangle=0\end{array}\right\}\Longrightarrow[u=0,v=0].$$

Hence, from [22, Theorem 6.14], and for $\upsilon=1,\ldots,N$, we obtain

$$N_{\Xi}(\bar{x},\bar{y},\bar{w})\subseteq\left\{\begin{array}{c}\left(\begin{array}{c}\sum_{i=1}^{p}\alpha_i\nabla G_i(\bar{x})+\sum_{\upsilon=1}^{N}\sum_{j=1}^{q_{\upsilon}}\beta_j^{\upsilon}\nabla_x g_j^{\upsilon}(\bar{x},\bar{y})\\ \sum_{\upsilon=1}^{N}\sum_{j=1}^{q_{\upsilon}}\beta_j^{\upsilon}\nabla_{y^{\upsilon}} g_j^{\upsilon}(\bar{x},\bar{y})\\ -\nu+\iota e\end{array}\right):\\ \mu_r\geq 0,\ \mu_r\bar{w}_r=0,\quad r\in S^{\upsilon}\\ \alpha_i\geq 0,\quad \alpha_i G_i(\bar{x})=0,\quad i\in I,\\ \upsilon=1,\ldots,N,\ \beta_j^{\upsilon}\geq 0,\quad \beta_j^{\upsilon}g_j^{\upsilon}(\bar{x},\bar{y})=0,\quad j\in J^{\upsilon}.\end{array}\right\} \tag{4.8}$$

On the other hand, for $\upsilon=1,\ldots,N$, we have the following convex hull property:

$$\partial\left(-\varphi_k^{\upsilon}\left(\bar{x},\bar{y}^{-\upsilon},\bar{w}^{\upsilon}\right)\right)\subseteq co\left(-\partial\varphi_k^{\upsilon}\left(\bar{x},\bar{y}^{-\upsilon},\bar{w}^{\upsilon}\right)\right)\subseteq\ -co\ \partial\varphi_k^{\upsilon}\left(\bar{x},\bar{y}^{-\upsilon},\bar{w}^{\upsilon}\right).$$

Then

$$\partial\ \psi_k^{\upsilon}\left(\bar{x},\bar{y},\bar{w}^{\upsilon}\right)\subseteq\left\{\begin{array}{l}\nabla_x f_k^{\upsilon}(\bar{x},\bar{y})\\ \nabla_{y^{\upsilon}} f_k^{\upsilon}(\bar{x},\bar{y})\\ 0\end{array}\right\}-co\ \partial_{x,w^{\upsilon}}\varphi_k^{\upsilon}\left(\bar{x},\bar{y}^{-\upsilon},\bar{w}^{\upsilon}\right)\times\{0_{\mathbb{R}^{m_{\upsilon}}}\}.$$

According to [21, Theorem 7], because the solution set-valued mappings $\Psi_k^{\upsilon},k\in S^{\upsilon},\ \upsilon=1,\ldots,N$, are inner semicompact at $\left(\bar{x},\bar{y}^{-\upsilon},\bar{w}^{\upsilon}\right)$ and the point $\left(\bar{x},y_k^{\upsilon},\bar{w}^{\upsilon}\right)$ is lower-level regular for all $y_k^{\upsilon}\in\Psi_k^{\upsilon}\left(\bar{x},\bar{y}^{-\upsilon},\bar{w}^{\upsilon}\right),\ k\in S^{\upsilon}$, we get

$$\partial\varphi_k^{\upsilon}\left(\bar{x},\bar{y}^{-\upsilon},\bar{w}^{\upsilon}\right)\subset\bigcup_{y_k^{\upsilon}\in\Psi_k^{\upsilon}\left(\bar{x},\bar{y}^{-\upsilon},\bar{w}^{\upsilon}\right)}\ \bigcup_{\left(\beta_k^{\upsilon},\gamma_k^{\upsilon}\right)\in\Lambda_k^{\upsilon}\left(\bar{x},y_k^{\upsilon},\bar{y}^{-\upsilon},\bar{w}^{\upsilon}\right)} \tag{4.9}$$

$$\left\{\begin{array}{c}\left(\begin{array}{c}\nabla_x f_k^{\upsilon}\left(\bar{x},y_k^{\upsilon},\bar{y}^{-\upsilon}\right)+\sum_{j=1}^{q_{\upsilon}}\beta_{j,k}^{\upsilon}\nabla_x g_j^{\upsilon}(\bar{x},y_k^{\upsilon},\bar{y}^{-\upsilon})\\ \sum_{r\in S^{\upsilon}\backslash\{k\}}\gamma_{k,r}^{\upsilon}\nabla_{w^{\upsilon}}h_{k,r}^{\upsilon}\left(\bar{x},y_k^{\upsilon},\bar{y}^{-\upsilon},\bar{w}^{\upsilon}\right)\end{array}\right)+\\ \left[\begin{array}{c}\sum_{r\in S^{\upsilon}\backslash\{k\}}\gamma_{k,r}^{\upsilon}\left(\bar{w}_r^{\upsilon}\nabla_x f_r^{\upsilon}\left(\bar{x},y_k^{\upsilon},\bar{y}^{-\upsilon}\right)-\bar{w}_k^{\upsilon}\nabla_x f_k^{\upsilon}\left(\bar{x},y_k^{\upsilon},\bar{y}^{-\upsilon}\right)\right)\\ 0\end{array}\right]\end{array}\right\} \tag{4.10}$$

where $\Lambda_k^{\upsilon}\left(\bar{x}, y_k^{\upsilon}, y_k^{-\upsilon}, \bar{w}^{\upsilon}\right)$ is the collection of lower-level Lagrange multipliers defined by

$$\Lambda_k^{\upsilon}\left(\bar{x}, y_k^{\upsilon}, \bar{y}^{-\upsilon}, \bar{w}^{\upsilon}\right) = \left\{ \begin{array}{c} \left(\beta_k^{\upsilon}, \gamma_k^{\upsilon}\right) \in \mathbb{R}^{s_\upsilon + q_\upsilon} : \\ \beta_{j,k}^{\upsilon} \geq 0,\ \beta_{j,k}^{\upsilon} \nabla_{y^\upsilon} g_j^{\upsilon}(\bar{x}, y_k^{\upsilon}, \bar{y}^{-\upsilon}) = 0, \quad \upsilon = 1, \ldots, N, j \in J^{\upsilon}, \\ \gamma_{k,r}^{\upsilon} \geq 0,\ \gamma_{k,r}^{\upsilon}\ \left(\bar{w}_r^{\upsilon} \nabla_{y^\upsilon} f_r^{\upsilon}\left(\bar{x}, y_k^{\upsilon}, \bar{y}^{-\upsilon}\right) - \bar{w}_k^{\upsilon} \nabla_{y^\upsilon} f_k^{\upsilon}\left(\bar{x}, y_k^{\upsilon}, \bar{y}^{-\upsilon}\right)\right) = 0, \\ \upsilon = 1, \ldots, N,\ r \in S^{\upsilon} \backslash \{k\}, \\ \nabla_{y^\upsilon} f_k^{\upsilon}\left(x, y_k^{\upsilon}, \bar{y}^{-\upsilon}\right) + \sum_{j=1}^{q_\upsilon} \beta_j^{\upsilon,k} \nabla_{y^\upsilon} g_j^{\upsilon}(\bar{x}, y_k^{\upsilon}, \bar{y}^{-\upsilon}) \\ + \sum_{r \in S^{\upsilon} \backslash \{k\}} \gamma_{k,r}^{\upsilon}\left(\bar{w}_r^{\upsilon} \nabla_{y^\upsilon} f_r^{\upsilon}\left(\bar{x}, y_k^{\upsilon}, \bar{y}^{-\upsilon}\right) - \bar{w}_k^{\upsilon} \nabla_{y^\upsilon} f_k^{\upsilon}\left(\bar{x}, y_k^{\upsilon}, \bar{y}^{-\upsilon}\right)\right) = 0 \end{array} \right\} \tag{4.11}$$

For any pair $\upsilon = 1, \ldots, N$, $k \in S^{\upsilon}$ and $\left(a_k^{\upsilon}, b_k^{\upsilon}\right) \in \operatorname{co} \partial \varphi_k^{\upsilon}\left(\bar{x}, \bar{y}^{-\upsilon}, \bar{w}^{\upsilon}\right)$. Now, Carathéodory's theorem gives us from (4.9), and (4.11) $\pi_{k,l}^{\upsilon}$, $\gamma_{r,k}^{\upsilon,l}$, $r \in S^{\upsilon}$, $\beta_j^{\upsilon,l}$, $j \in J^{\upsilon}$, with $l = 1, \cdots n+1$, such that

$$\sum_{l=1}^{n+1} \pi_{k,l}^{\upsilon} = 1 \tag{4.12}$$

$$b_k^{\upsilon} = \sum_{l=1}^{n+1} \pi_{k,l}^{\upsilon} \sum_{r \in S^{\upsilon} \backslash \{k\}} \gamma_{k,r}^{\upsilon,l}\ \nabla_w h_{k,r}^{\upsilon}\left(\bar{x}, y_{k,l}^{\upsilon}, \bar{y}^{-\upsilon}, \bar{w}^{\upsilon}\right). \tag{4.13}$$

$$\begin{aligned} a_k^{\upsilon} &= \sum_{l=1}^{n+1} \pi_{k,l}^{\upsilon} \Bigg(\nabla_x f_k^{\upsilon}(\bar{x}, y_{k,l}^{\upsilon}, \bar{y}^{-\upsilon}) + \sum_{j=1}^{q_\upsilon} \beta_{j,k}^{\upsilon,l} \nabla_x g_j^{\upsilon}\left(\bar{x}, y_{k,l}^{\upsilon}, \bar{y}^{-\upsilon}\right) \\ &\quad + \sum_{r \in S^{\upsilon} \backslash \{k\}} \gamma_{k,r}^{\upsilon,l}\left(\bar{w}_r^{\upsilon} \nabla_x f_r^{\upsilon}\left(\bar{x}, y_{k,l}^{\upsilon}, y_{k,l}^{-\upsilon}\right) - \bar{w}_k^{\upsilon} \nabla_x f_k^{\upsilon}\left(\bar{x}, y_{k,l}^{\upsilon}, \bar{y}^{-\upsilon}\right)\right)\Bigg). \\ 0 &= \nabla_{y^\upsilon} f_k^{\upsilon}\left(\bar{x}, y_{k,l}^{\upsilon}, \bar{y}^{-\upsilon}\right) + \sum_{j=1}^{q_\upsilon} \beta_{j,k}^{\upsilon,l} \nabla_{y^\upsilon} g_j^{\upsilon}\left(\bar{x}, y_{k,l}^{\upsilon}, \bar{y}^{-\upsilon}\right) \\ &\quad + \sum_{r \in S^{\upsilon} \backslash \{k\}} \gamma_{k,r}^{\upsilon,l}\left(\bar{w}_r^{\upsilon} \nabla_{y^\upsilon} f_r^{\upsilon}\left(\bar{x}, y_{k,l}^{\upsilon}, \bar{y}^{-\upsilon}\right) - \bar{w}_k^{\upsilon} \nabla_{y^\upsilon} f_k^{\upsilon}\left(\bar{x}, y_{k,l}^{\upsilon}, \bar{y}^{-\upsilon}\right)\right) \\ 0 &= \beta_{j,k}^{\upsilon,l} \nabla_{y^\upsilon} g_j^{\upsilon}\left(\bar{x}, y_{k,l}^{\upsilon}, \bar{y}^{-\upsilon}\right),\ \beta_{j,k}^{\upsilon,l} \geqslant 0, \end{aligned} \tag{4.14}$$

$$0 = \gamma_{k,r}^{\upsilon,l}\left(\bar{w}_r^{\upsilon} \nabla_{y^\upsilon} f_r^{\upsilon}\left(\bar{x}, y_{k,l}^{\upsilon}, \bar{y}^{-\upsilon}\right) - \bar{w}_k^{\upsilon} \nabla_{y^\upsilon} f_k^{\upsilon}\left(\bar{x}, y_{k,l}^{\upsilon}, \bar{y}^{-\upsilon}\right)\right),\ \gamma_{k,r}^{\upsilon,l} \geqslant 0. \tag{4.15}$$

Combining (4.6),(4.7)-(4.8)-(4.9)-(4.13)-(4.15), we obtain the result. □

The upcoming example applies the preceding results to mathematical programs with multiobjective Nash equilibrium lower-level problems.

Example 1. Consider the mathematical program with multiobjective Nash equilibrium problem

$$\min_{x,y} F(x, y) = x^2 + \left(y^1\right)^2 :\ x \in X, \quad y \in \mathcal{H}_{\text{weff}}\ (x), \tag{P}$$

with $y = \left(y^1, y^2\right)$.

The leader's decision variables are represented by $x \in X = \{x \in \mathbb{R} : G_1(x) = -x \leq$

$0\} = \mathbb{R}_+$, and $\mathcal{H}_{\text{weff}}(x)$ is the set of weakly efficient generalized Nash equilibrium of the multiobjective generalized Nash equilibrium problem parameterized by x composed by two players represented below:

$$\begin{cases} \min\limits_{y^\upsilon} f^\upsilon\left(x, y^\upsilon, y^{-\upsilon}\right) \\ \text{s.t } y^\upsilon \in Y^\upsilon\left(x, y^{-\upsilon}\right). \end{cases} \qquad (P^\upsilon[x, y^{-\upsilon}])$$

For $\upsilon = 1$, we have

$$\begin{cases} \min\limits_{y^1} f^1\left(x, y^1; y^2\right) = \left(-y^1 - y^2, y^1 - y^2\right) \\ g^1\left(x, y^1, y^2\right) = \left(y^1 - 2; -y^1; -y^2 + 3\right) \leq 0, \end{cases} \qquad (P^1[x, y^2, w^1])$$

and for $\upsilon = 2$ we have

$$\begin{cases} \min\limits_{y^2} f^2\left(x, y^2, y^1\right) = \left(-2y^1 - y^2; -2y^1 + y^2\right), \\ g^2\left(x, y^1, y^2\right) = \left(y^2 - 3; -y^2; -y^1 + 2\right) \leq 0. \end{cases} \qquad (P^2[x, y^1, w^2])$$

Applying the weighted-sum technique, we get

$$\begin{cases} \min\limits_{y^1} f_1^1\left(x, y^1, y^2\right) = -y^1 - y^2, \\ y^1 \in Y_1^1\left(x, y^2, w_1^1, 1 - w_1^1\right), \\ y^2 \in [3, +\infty[. \end{cases} \quad \text{and} \quad \begin{cases} \min\limits_{y^1} f_2^1\left(x, y^1; y^2\right) = y^1 - y^2, \\ y^1 \in Y_2^1\left(x, y^2, w_1^1, 1 - w_1^1\right), \\ y^2 \in [3, +\infty[. \end{cases}$$

$$\begin{cases} \min\limits_{y^2} f_1^2\left(x, y^2, y^1\right) = -2y^1 - y^2, \\ y^2 \in Y_1^2\left(x, y^1, w_1^2, 1 - w_1^2\right), \\ y^1 \in [2, +\infty[. \end{cases} \quad \text{and} \quad \begin{cases} \min\limits_{y^2} f_2^2\left(x, y^2, y^1\right) = -2y^1 + y^2, \\ y^2 \in Y_1^2\left(x, y^1, w_1^2, 1 - w_1^2\right), \\ y^1 \in [2, +\infty[. \end{cases}$$

where

$$Y_1^1\left(x, y^2, w_1^1, 1 - w_1^1\right) = \begin{cases} [0, 2] & \text{if } 0 \leq w_1^1 < \dfrac{y^2 - 2}{2y^2} \\ [0, y^2(1 - 2w_1^1)] & \text{if } \dfrac{y^2 - 2}{2y^2} \leq w_1^1 \leq \dfrac{1}{2} \\ \emptyset & \text{if } \dfrac{1}{2} < w_1^1 \leq 1. \end{cases}$$

$$Y_2^1\left(x, y^2, w_1^1, 1 - w_1^1\right) = \begin{cases} \emptyset & \text{if } 0 \leq w_2^1 < \dfrac{y^2 - 2}{2y^2}, \\ [y^2(1 - 2w_1^1), 2] & \text{if } \dfrac{y^2 - 2}{2y^2} \leq w_1^1 \leq \dfrac{1}{2}, \\ [0, 1] & \text{if } \dfrac{1}{2} < w_1^1 \leq 1. \end{cases}$$

$$Y_1^2\left(x, y^1, w_1^2, 1 - w_1^2\right) = \begin{cases} [0, 3] & \text{if } 0 \leq w_1^2 < \dfrac{2y^1 - 3}{4y^1}, \\ [0, y^1(2 - 4w_1^2)] & \text{if } \dfrac{2y^1 - 3}{4y^1} \leq w_1^2 \leq \dfrac{1}{2}, \\ \emptyset & \text{if } \dfrac{1}{2} < w_1^2 \leq 1. \end{cases}$$

$$Y_2^2\left(x,y^1,w_1^2,1-w_1^2\right) = \begin{cases} \emptyset & \text{if } 0 \le w_2^2 < \dfrac{2y^1-3}{4y^1}, \\ [y^1(2-4w_1^2),3] & \text{if } \dfrac{2y^1-3}{4y^1} \le w_2^2 \le \dfrac{1}{2}, \\ [0,3] & \text{if } \dfrac{1}{2} < w_2^2 \le 1. \end{cases}$$

It is easy to check that the optimal values are, respectively:

$$\varphi_1^1\left(x,y^2,w_1^1,1-w_1^1\right) = \begin{cases} -2-y^2 & \text{if } 0 \le w_1^1 < \dfrac{y^2-2}{2y^2} \\ -y^2(1-2w_1^1)-y^2 & \text{if } \dfrac{y^2-2}{2y^2} \le w_1^1 \le \dfrac{1}{2} \\ +\infty & \text{if } \dfrac{1}{2} < w_1^1 \le 1. \end{cases}$$

$$\varphi_2^1\left(x,y^2,w_1^1,1-w_1^1\right) = \begin{cases} +\infty & \text{if } 0 \le w_1^1 < \dfrac{y^2-2}{2y^2}, \\ y^2(1-2w_1^1)-y^2 & \text{if } \dfrac{y^2-2}{2y^2} \le w_1^1 \le \dfrac{1}{2}, \\ -y^2 & \text{if } \dfrac{1}{2} < w_1^1 \le 1. \end{cases}$$

$$\varphi_1^2\left(x,y^1,w_1^2,1-w_1^2\right) = \begin{cases} -3-2y^1 & \text{if } 0 \le w_1^2 < \dfrac{2y^1-3}{4y^1}, \\ -y^1(2-4z_1^2)-2y^1 & \text{if } \dfrac{2y^1-3}{4y^1} \le w_1^2 \le \dfrac{1}{2}, \\ +\infty & \text{if } \dfrac{1}{2} < w_1^2 \le 1. \end{cases}$$

$$\varphi_2^2\left(x,y^1,w_1^2,1-w_1^2\right) = \begin{cases} +\infty & \text{if } 0 \le w_1^2 < \dfrac{2y^1-3}{4y^1}, \\ y^1(2-4w_1^2)-2y^1 & \text{if } \dfrac{2y^1-3}{4y^1} \le w_1^2 \le \dfrac{1}{2}, \\ -2y^1 & \text{if } \dfrac{1}{2} < w_1^2 \le 1, \end{cases}$$

supported by the solution sets provided by:

$$\Psi_1^1\left(x,y^2,w_1^1,1-w_1^1\right) = \begin{cases} 2 & \text{if } 0 \le w_1^1 < \dfrac{y^2-2}{2y^2} \\ y^2(1-2w_1^1) & \text{if } \dfrac{y^2-2}{2y^2} \le w_1^1 \le \dfrac{1}{2} \\ \emptyset & \text{if } \dfrac{1}{2} < w_1^1 \le 1. \end{cases}$$

$$\Psi_2^1\left(x,y^2,w_1^1,1-w_1^1\right) = \begin{cases} \emptyset & \text{if } 0 \le w_1^1 < \dfrac{y^2-2}{2y^2}, \\ y^2(1-2w_1^1) & \text{if } \dfrac{y^2-2}{2y^2} \le w_1^1 \le \dfrac{1}{2}, \\ 0 & \text{if } \dfrac{1}{2} < w_1^1 \le 1. \end{cases}$$

$$\Psi_1^2\left(x,y^1,w_1^2,1-w_1^2\right) = \begin{cases} 3 & \text{if } 0 \leq w_1^2 < \dfrac{2y^1-3}{4y^1}, \\ y^1(2-4w_1^2) & \text{if } \dfrac{2y^1-3}{4y^1} \leq w_1^2 \leq \dfrac{1}{2}, \\ \emptyset & \text{if } \dfrac{1}{2} < w_1^2 \leq 1. \end{cases}$$

$$\Psi_2^2\left(x,y^2,w_1^2,1-w_1^2\right) = \begin{cases} \emptyset & \text{if } 0 \leq w_1^2 < \dfrac{2y^1-3}{4y^1}, \\ y^1(2-4w_1^2) & \text{if } \dfrac{2y^1-3}{4y^1} \leq w_1^2 \leq \dfrac{1}{2}, \\ 0 & \text{if } \dfrac{1}{2} < w_1^2 \leq 1. \end{cases}$$

Hence, the optimization problem (P) with the multiobjective Nash equilibruim problem can be replaced by the following optimization problem of the form:

$$\begin{cases} \min\limits_{x,y,w} F(x,y) = x^2 + \left(y^1\right)^2 \\ \text{s.t} \quad x \in X,\ y \in \mathcal{H}(x,w). \end{cases} \tag{4.16}$$

with

$$\mathcal{H}(x,w) = \{y = (y^1,y^2) :\ y^1 \in \Psi_1^1\left(x,y^2,w_1^1\right) \cap \Psi_2^1\left(x,y^2,w_2^1\right)\ ,y^2 \in \Psi_1^2\left(x,y^1,w_1^2\right) \cap \Psi_2^2\left(x,y^2,w_2^2\right)\}$$

Using the optimal value reformulation, (4.16) is equivalent to the one-level optimization problem.

$$\begin{cases} \min\limits_{x,y,w} F(x,y) = x^2 + \left(y^1\right)^2 \\ (x,y,w_1^1,w_2^1,w_1^2,w_2^2) \in \Omega. \end{cases} \tag{4.17}$$

We remark that $\bar{u} = (\bar{x},\bar{y},\bar{w}_1^1,\bar{w}_2^1,\bar{w}_1^2,\bar{w}_2^2) = (0,2,3,\frac{1}{6},\frac{5}{6},\frac{1}{8},\frac{7}{8})$ is a local optimal solution to problem (4.17). Moreover, the set Ξ and the normal cone to Ξ at $\bar{u} \in \Xi$ are given by:

$$\Omega = \Xi = [0,+\infty[\times\{2\}\times\{3\}\times\left\{\frac{1}{6},\frac{5}{6},\frac{1}{8},\frac{7}{8}\right\}, \qquad N_{\Xi}(\bar{u}) =]-\infty,0]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$$

Let

$$\psi_1^1\left(x,y^1,y^2,w_1^1,1-w_1^1\right) = \begin{cases} -y^1+2 & \text{if } 0 \leq w_1^1 < \dfrac{y^2-2}{2y^2} \\ -y^1+y^2(1-2w_1^1) & \text{if } \dfrac{y^2-2}{2y^2} \leq w_1^1 \leq \dfrac{1}{2} \\ -\infty & \text{if } \dfrac{1}{2} < w_1^1 \leq 1. \end{cases}$$

$$\psi_2^1\left(x,y^1,y^2,w_1^1,1-w_1^1\right) = \begin{cases} -\infty & \text{if } 0 \leq w_1^1 < \dfrac{y^2-2}{2y^2}, \\ y^1-y^2(1-2w_1^1) & \text{if } \dfrac{y^2-2}{2y^2} \leq w_1^1 \leq \dfrac{1}{2}, \\ y^1 & \text{if } \dfrac{1}{2} < w_1^1 \leq 1. \end{cases}$$

$$\psi_1^2\left(x,y^1,y^2,w_1^2,1-w_1^2\right) = \begin{cases} -y^2+3 & \text{if } 0\le w_1^2<\dfrac{2y^1-3}{4y^1},\\ -y^2+y^1(2-4w_1^2) & \text{if } \dfrac{2y^1-3}{4y^1}\le w_1^2\le\dfrac{1}{2},\\ -\infty & \text{if } \dfrac{1}{2}<w_1^2\le 1.\end{cases}$$

$$\psi_2^2\left(x,y^1,y^2,w_1^2,1-w_1^2\right) = \begin{cases} -\infty & \text{if } 0\le w_1^2<\dfrac{2y^1-3}{4y^1},\\ y^2-y^1(2-4w_1^2) & \text{if } \dfrac{2y^1-3}{4y^1}\le w_1^2\le\dfrac{1}{2},\\ y^2 & \text{if } \dfrac{1}{2}<w_1^2\le 1.\end{cases}$$

we have

$$D^*\psi\left(\bar{u}\right)(\sigma)=\left\{(0,-\sigma^1+\sigma^2+\frac{3}{2}\sigma^3-\frac{3}{2}\sigma^4,\frac{1}{2}\sigma^1-\frac{1}{2}\sigma^2-\sigma^3+\sigma^4,-6\sigma^1+6\sigma^2,0,-8\sigma^3+8\sigma^4,0)\right\}$$

Consequently

$$D^*\psi\left(\bar{u}\right)(\sigma)\cap\left(-bd\ N_{\Xi}\left(\bar{x},\bar{y}\right)\right)=\emptyset$$

where $\sigma=(\sigma^1,\sigma^2,\sigma^3,\sigma^4)$ and

$$\psi(x,y^1,y^2,w_1^1,1-w_1^1,w_1^2,1-w_1^2)=\Big(\psi_1^1\left(x,y^2,w_1^1,1-w_1^1\right),\psi_2^1\left(x,y^2,w_1^1,1-w_1^1\right),$$
$$\psi_1^2\left(x,y^1,y^2,w_1^2,1-w_1^2\right),\psi_2^2\left(x,y^1,y^2,w_1^2,1-w_1^2\right)\Big)$$

The set valued mappings Ψ_1^1,Ψ_2^1, Ψ_1^2,Ψ_2^2 are inner semicompact at $\left(0,2,3,\frac{1}{6},\frac{5}{6}\right)$, $\left(0,2,3,\frac{1}{6},\frac{5}{6}\right)$, $\left(0,2,3,\frac{1}{8},\frac{7}{8}\right)$ and $\left(0,2,3,\frac{1}{8},\frac{7}{8}\right)$ respectively, and the hypotheses $(\mathcal{R}_3)$, $(\mathcal{R}_4)$ and $(\mathcal{R}_5)$ hold true.
Then, there exist $(\sigma_1^1,\sigma_2^1)=(\frac{1}{2},\frac{1}{2})$, $(\sigma_1^2,\sigma_2^2)=(\frac{1}{4},\frac{1}{4})$, $(\sigma_{1,2}^1,\sigma_{2,1}^1)=(\frac{1}{5},\frac{1}{5})$, $(\sigma_{1,2}^2,\sigma_{2,1}^2)=(1,1)$, $\alpha_1=0$, $(\beta_1^1,\beta_2^l)=(\frac{4}{3},\frac{1}{3})$, $(\beta_1^2,\beta_2^2)=(\frac{5}{3},\frac{1}{3})$, $(\gamma_{1,2}^{1,l},\gamma_{2,1}^{1,l})=(\frac{1}{3},\frac{1}{3})$, $(\gamma_{1,2}^{2,l},\gamma_{2,1}^{2,l})=(1,1)$, $(\beta_1^{1,l},\beta_2^{1,l})=(3,3)$, $(\beta_1^{2,l},\beta_2^{2,l})=(2,2)$ and $y_l^{1,l}=y_2^{1,l}=y_l^{2,l}=y_2^{2,l}=\{0\}$, with $l=1\ldots,4$, $\sum_{l=1}^{4}\pi_l^1=\sum_{l=1}^{4}\pi_l^2=1$ such that

$$\begin{aligned}0 &= \nabla_x F(0,0)+\sum_{\upsilon=1}^{2}\left(\sum_{k=1}^{2}\left(\sum_{r\in\{1,2\}\setminus\{k\}}\sigma_{k,r}^{\upsilon}\left(\bar{w}_r^{\upsilon}\nabla_x f_r^{\upsilon}(0,0)-\bar{w}_k^{\upsilon}\nabla_x f_k^{\upsilon}(0,0)\right)\right)\right)\\ &+\sum_{\upsilon=1}^{2}\left(\sum_{k=1}^{2}\sigma_k^{\upsilon}\left[\nabla_x f_k^{\upsilon}(0,0)-\sum_{l=1}^{2}\pi_{k,l}^{\upsilon}\left(\nabla_x f_k^{\upsilon}\left(0,0,0\right)+\sum_{j=1}^{2}\beta_{j,k}^{\upsilon,l}\nabla_x g_j^{\upsilon}\left(0,0,0\right)\right.\right.\right.\\ &\left.\left.\left.+\sum_{r\in\{1,2\}\setminus\{k\}}\gamma_{k,r}^{\upsilon,l}\ \left(\bar{w}_r^{\upsilon}\nabla_x f_r^{\upsilon}\left(0,0,0\right)-\bar{w}_k^{\upsilon}\nabla_x f_k^{\upsilon}\left(0,0,0\right)\right)\right)\right]+\sum_{j=1}^{2}\beta_j^{\upsilon}\nabla_x g_j^{\upsilon}(0,0)\right)\\ &+\alpha_1\nabla G_1(0),\end{aligned}$$

$$
\begin{aligned}
0 &= \nabla_y F(0,0) + \sum_{\upsilon=1}^{2}\left[\sum_{k=1}^{2}\left(\sum_{r\in\{1,2\}\setminus\{k\}} \sigma_{k,r}^{\upsilon}\left(\bar{w}_r^{\upsilon}\nabla_{y^{\upsilon}} f_r^{\upsilon}(0,0) - \bar{w}_k^{\upsilon}\nabla_{y^{\upsilon}} f_k^{\upsilon}(0,0)\right)\right)\right.\\
&\quad \left. + \sum_{k=1}^{2} \sigma_k^{\upsilon}\ \nabla_{y^{\upsilon}} f_k^{\upsilon}(0,0) + \sum_{j=1}^{2} \beta_j^{\upsilon}\nabla_{y^{\upsilon}} g_j^{\upsilon}(0,0)\right],
\end{aligned}
$$

$$
\begin{aligned}
0 &= \nabla_{y^{\upsilon}} f_k^{\upsilon}\left(0,0,0\right) + \sum_{j=1}^{2}\beta_{j,k}^{\upsilon,l}\nabla_{y^{\upsilon}} g_j^{\upsilon}\left(0,0,0\right) + \sum_{r\in\{1,2\}\setminus\{k\}} \gamma_{k,r}^{\upsilon,l}\left(\bar{w}_r^{\upsilon}\nabla_{y^{\upsilon}} f_r^{\upsilon}\left(0,0,0\right)\right.\\
&\quad \left. - \bar{w}_k^{\upsilon}\nabla_{y^{\upsilon}} f_k^{\upsilon}\left(0,0,0\right)\right), \qquad \upsilon = 1,2, \forall k\in\{1,2\}, \forall l = 1,\ldots,4.
\end{aligned}
$$

$$
\begin{aligned}
0 &= \gamma_{k,r}^{\upsilon,l}\left(\bar{w}_r^{\upsilon} f_r^{\upsilon}\left(0,0,0\right) - \bar{w}_k^{\upsilon} f_k^{\upsilon}\left(0,0,0\right)\right),\\
0 &= \beta_{j,k}^{\upsilon,l} g_j^{\upsilon}\left(0,0,0\right), \qquad \upsilon = 1,2,\ \forall\ k\in\{1,2\}, \forall\ j = 1,2, \forall\ r\in\{1,2\}\setminus\{k\}, \forall\ l = 1,\ldots,4.
\end{aligned}
$$

$$
0 = \beta_j^{\upsilon} g_j^{\upsilon}\left(0,0\right),\quad 0 = \alpha_1 G_1\left(0\right), \qquad \upsilon = 1,2,\ \forall\ j = 1,2.
$$

$$
0 = \gamma_{k,r}^{\upsilon}\left(\bar{w}_r^{\upsilon} f_r^{\upsilon}\left(0,0,0\right) - \bar{w}_k^{\upsilon} f_k^{\upsilon}\left(0,0,0\right)\right), \qquad \upsilon = 1,2,\ \forall\ r\in\{1,2\}\setminus\{k\}, \forall\ k\in\{1,2\}.
$$

$$
0 = \sigma_{k,r}^{\upsilon}\left(\bar{w}_r^{\upsilon} f_r^{\upsilon}\left(0,0\right) - \bar{w}_k^{\upsilon} f_k^{\upsilon}\left(0,0\right)\right), \qquad \upsilon = 1,2,\ \forall\ r\in\{1,2\}\setminus\{k\}, \forall\ k\in\{1,2\}.
$$

5. Conclusions

We are concerned in this article with a mathematical programming problem (P). We presented a concept of equilibrium for generalized Nash equilibrium problems with many goals and provide its scalarization formulation. Based on this, we looked at the relationship between problem (P) and a related scalar programming problem (4.3), which is established by applying the kth-objective weighted-constraint technique to problem (P)'s multiobjective lower level problem. It has been proven that this relationship is nonhazardous when globally optimal solutions are considered; however, the analysis of locally optimal solutions is more sensitive. We exploited the equivalence of (P) and (4.3) in order to derive new necessary optimality conditions. Finally, we showed our discovery using an example.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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