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***Research Article***

# Strength based domination in graphs

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**Abstract:** Let $G = (V, E)$ be a connected graph. Let $A \subseteq V$ and $v \in V - A$. The dominating strength of $A$ on $v$ is defined by $s(v, A) = \sum_{u \in A} \frac{1}{d(u,v)}$. A subset $D$ of $V$ is called a strength based dominating set if for every vertex $v \notin D$, there exists a subset $A$ of $D$ such that $s(v, A) \geq 1$. The $sb$-domination number $\gamma_{sb}(G)$ is the minimum cardinality of a strength based dominating set of $G$. In this paper we initiate a study of this parameter and indicate directions for further research.

**Keywords:** distance, domination, dominating strength, $sb$-domination.

**AMS Subject classification:** 05C69.

## 1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected, connected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to the book [1]. A subset $D$ of $V$ is called a dominating set of $G$ if every vertex $v$ in $V - D$ is adjacent to a vertex $u$ in $D$. The minimum cardinality of a dominating set of $G$ is called the domination number of $G$ and is denoted by $\gamma(G)$. The concept of domination in graphs and its several variants have been extensively investigated. For fundamentals of domination in graphs we refer to [5].

Different types of dominating sets have been formulated by putting restrictions on the induced subgraph $G[D]$. Connected domination, total domination, independent

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domination and paired domination are some of the domination parameters under this category. For a detailed study of total domination in graphs we refer to the book [6]. Another type of generalization is by putting restrictions on $N(v) \cap D$ and examples of such a type are weak domination, strong domination, $k$-domination and perfect domination. For further details of various types of domination models we refer to the Appendix in [5].

The distance $d(u, v)$ between two vertices $u$ and $v$ in a graph is the length of a shortest $u$-$v$ path in $G$. The eccentricity of a vertex $v$ is defined by $ecc(v) = \max\{d(u, v) : u \in V\}$. The radius and diameter of $G$ are defined by $\text{rad}(G) = \min\{ecc(v) : v \in V\}$ and diam $(G) = \max\{ecc(v) : v \in V\}$.

In several real life situations such as social networks, communication networks and biological networks, the influence of a vertex extends beyond its neighborhood but decreases with distance. To address this problem Dankelmann et al. [2] introduced the concept of exponential domination and exponential domination number of a graph, in which the dominating power of a vertex is decreasing exponentially by the factor $\frac{1}{2}$ with distance.

Goddard et al. [4] introduced the concept of disjunctive domination number of a graph. This concept reconsider in [3].

**Definition 1.** Let $G = (V, E)$ be a connected graph. A subset $D$ of $V$ is called a disjunctive dominating set of $G$, if every vertex $v \notin D$ is adjacent to a vertex in $D$ or has at least two distinct vertices at a distance two from $v$. The minimum cardinality of a disjunctive dominating set of $G$ is called disjunctive domination number of $G$ and is denoted by $\gamma_2^d(G)$.

**Theorem 1.** *([4]) Let $G$ be any graph. Then $\gamma_2^d(G) \leq \gamma(G)$.*

**Theorem 2.** *([4]) For any positive integer $n$, $\gamma_2^d(P_n) = \left\lceil \frac{n+1}{4} \right\rceil$.*

**Theorem 3.** *([4]) For any positive integer $n \geq 3$,*

$$\gamma_2^d(C_n) = \begin{cases} 2 & \text{if } n = 4 \\ \lceil \frac{n}{4} \rceil & \text{if } n \neq 4 \end{cases}$$

We need the following definitions and theorem.

**Definition 2.** Let $G_1$ and $G_2$ be two graphs. The corona $G_1 \circ G_2$ is the graph obtained from one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$ by joining the $i^{th}$ vertex of $G_1$ to all the vertices in the $i^{th}$ copy of $G_2$.

**Definition 3.** The Cartesian product $G = G_1 \square G_2$ of two graphs $G_1$ and $G_2$ has $V(G) = V(G_1) \times V(G_2)$ and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjcent in $G$ if either $u_1 = u_2$ and $v_1 v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G_1)$.

**Theorem 4.** *([5], Page 56) For any connected graph $G$, $\left\lceil \frac{diam(G)+1}{3} \right\rceil \leq \gamma(G)$.*

In this paper we introduce the concept of strength based domination in graphs which is a variant of the exponential model considered by Dankelmann et al. [2]. We present several basic results on strength based domination number and indicate directions for further research.

## 2. On *sb*-domination

In a social network a member $v$ is very often influenced by another member $u$ who is not in its neighborhood $N(v)$. In fact there is a possibility that the member $v$ is influenced by a group of members who are not in $N(v)$. We propose the concept of dominating strength and the associated parameters to address the above situation.

**Definition 4.** Let $G = (V, E)$ be a connected graph and let $u, v \in V$. The dominating strength $s(u, v)$ between $u$ and $v$ is defined as $s(u, v) = \frac{1}{d(u,v)}$. The dominating strength $\text{ds}(v)$ of $v$ is defined as $\text{ds}(v) = \sum_{u \neq v} s(u, v) = \sum_{u \neq v} \frac{1}{d(u,v)}$. The sequence $\Pi = (\text{ds}(v_1), \text{ds}(v_2), \ldots, \text{ds}(v_n))$ where $\text{ds}(v_1) \geq \text{ds}(v_2) \geq \cdots \geq \text{ds}(v_n)$ is called the dominating strength sequence or simply the ds-sequence of $G$.

**Example 1.** For the graph $G$ given in Figure 1,

$$\text{ds}(v_i) = \begin{cases} \frac{157}{60} & \text{if } i = 1 \text{ or } 6 \\ \frac{43}{12} & \text{if } i = 2 \text{ or } 5 \\ \frac{13}{3} & \text{if } i = 3 \text{ or } 4 \\ \frac{11}{3} & \text{if } i = 7. \end{cases}$$

Hence the ds-sequence $\Pi$ is given by

$$\begin{aligned} \Pi &= \left(\frac{13}{3}, \frac{13}{3}, \frac{11}{3}, \frac{43}{12}, \frac{43}{12}, \frac{157}{60}, \frac{157}{60}\right) \\ &= (\text{ds}(v_3), \text{ds}(v_4), \text{ds}(v_7), \text{ds}(v_2), \text{ds}(v_5), \text{ds}(v_1), \text{ds}(v_6)). \end{aligned}$$

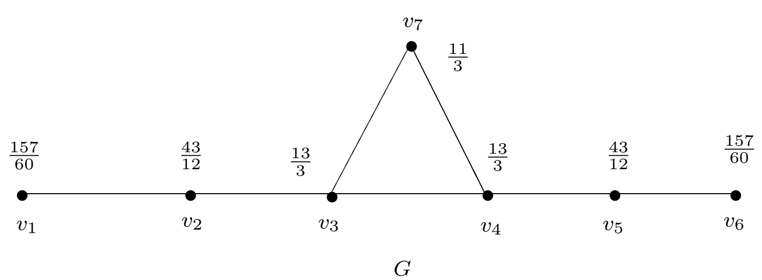


**Figure 1.** **A graph and its ds-sequence**

**Observation 5.** Let $G$ be a connected graph of order $n$ and let $v \in V$. Then $deg(v) \leq \text{ds}(v)$ and equality holds if and only if $deg(v) = n-1$.

**Definition 5.** Let $G$ be a connected graph of order $n$. Let $A \subseteq V$ and $v \in V - A$. Then the dominating strength of $A$ on $v$ is defined by $\text{ds}(A, v) = \sum_{u \in A} \frac{1}{d(v,u)}$.

**Example 2.** A graph $G$ with its ds-sequence is given in Figure 1. In this graph

$$\text{ds}(\{v_1, v_2\}, v_6) = \frac{1}{d(v_6, v_1)} + \frac{1}{d(v_6, v_2)} = \frac{1}{5} + \frac{1}{4} = 0.45.$$

If $D$ is a dominating set of $G$, then any vertex $v \notin D$ is dominated by a vertex in $D$. In the case of disjunctive domination, $v$ is dominated by a single vertex in $D$ or is dominated by a set of two vertices in $D$ each at distance 2 from $v$. We now introduce the concept of strength based domination in which $v$ is dominated by a subset $D_1$ of $D$ and this is a generalization of domination and disjunctive domination.

**Definition 6.** Let $G = (V, E)$ be a connected graph. A subset $D$ of $V$ is called a strength based dominating set or a *sb*-dominating set of $G$ if for every $v \in V - D$, there exists a subset $D_1$ of $D$ such that $\text{ds}(D_1, v) \geq 1$. The minimum cardinality of a *sb*-dominating set of $G$ is called the *sb*-domination number of $G$ and is denoted by $\gamma_{sb}(G)$. Also *sb*-dominating set of cardinality $\gamma_{sb}$ is called a $\gamma_{sb}$-set of $G$.

**Observation 6.** Let $D$ be a *sb*-dominating set of $G$ and let $v \in V - D$. Obviously $\text{ds}(D, v) \geq 1$ if and only if there exists a subset $D_1$ of $D$ such that $\text{ds}(D_1, v) \geq 1$. The concept of *sb*-domination has interesting applications in social networks and in a large network identifying a subset $D_1$ such that the members of $D_1$ can collectively influence a member $v \in V - D$ is a significant and relevant issue. Thus from application perspective the subset $D_1$ in the above definition plays a crucial role.

**Observation 7.** For any graph $G$, we have $\gamma(G) = \gamma_{sb}(G) = 1$ if and only if $\Delta = n - 1$.

**Observation 8.** Clearly any dominating set of $G$ and any disjunctive dominating set of $G$ are *sb*-dominating sets. Hence $\gamma_{sb}(G) \leq \gamma_2^d(G) \leq \gamma(G)$.

**Example 3.** For the Petersen graph $G$ given in Figure 2, $D = \{v_1, v_2\}$ is a *sb*-dominating set of $G$, since $d(u, v_1) = d(u, v_2) = 2$ for all vertices $u \in (V - D) \cup N(v_1) \cup N(v_2)$. Also $\Delta = 3 < n - 1$. Hence $\gamma_{sb}(G) = 2$.

**Example 4.** Let $G = K_n \circ K_1$. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and let $w_i$ be the pendent vertex adjacent to $v_i$. Clearly $D = \{v_1, v_2\}$ is a *sb*-dominating set of $G$ and hence $\gamma_{sb}(G) \leq 2$. Since $\Delta < |V(G)| - 1, \gamma_{sb}(G) \geq 2$. Hence $\gamma_{sb}(G) = 2$. Since $\gamma(G) = n$, it follows that the difference between $\gamma_{sb}(G)$ and $\gamma(G)$ can be arbitrarily large.

**Theorem 9.** *For any positive integer $k$, there exists a graph $G$ with $\gamma_{sb}(G) = k$.*

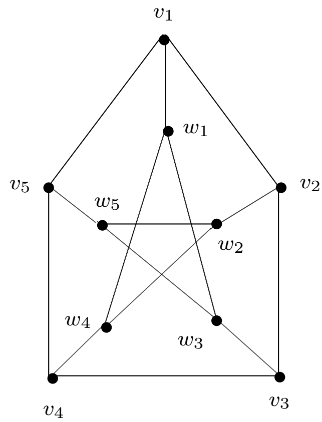


**Figure 2.** **Petersen graph** $G : \gamma_{sb}(G) = 2$.

*Proof.* For any graph $G$ with $\gamma(G) = 1$ or 2, we have $\gamma_{sb}(G) = \gamma(G)$.
Suppose $k \geq 3$. Let $G$ be the graph obtained from $K_{1,k}$ by sub-dividing each edge $(k-2)$ times. Let $V(K_{1,k}) = \{v_0, v_1, \ldots, v_k\}$ and $deg(v_0) = k$. Let $w_{i_1}, w_{i_2}, \ldots, w_{i_{(k-2)}})$ be the vertices sub-dividing the edge $v_0v_i$. Let $P_i = (v_0, w_{i_1}, w_{i_2}, \ldots, w_{i_{(k-2)}}, v_i)$. Let $D = \{w_{11}, w_{21}, \ldots, w_{k1}\}$. Now, $\text{ds}(D, v_i) = (k-1)\frac{1}{k} + \frac{1}{k-2} = 1 + \left(\frac{1}{k-2} - \frac{1}{k}\right) > 1$. Also, $\text{ds}(D, u) \geq \text{ds}(D, v_i)$ for all $u \in V - D$. Hence $D$ is a *sb*-dominating set of $G$ and therefore $\gamma_{sb}(G) \leq k$. Now let $S$ be any $\gamma_{sb}$-set of $G$. If $S \cap V(P_i) = \emptyset$ for some $i$, then $\text{ds}(v_i, S) \leq \frac{k-1}{k} + \frac{1}{k-1} < 1$. Hence $S \cap V(P_i) \neq \emptyset$ for all $i, 1 \leq i \leq k$ and so $|S| \geq k$. Thus $\gamma_{sb}(G) \geq k$ and hence $\gamma_{sb}(G) = k$. □

## 3. Bounds for $\gamma_{sb}$

The following theorems give an upper bound for $\gamma_{sb}(G)$ and a characterization of all extremal graphs which attain the bound.

**Theorem 10.** *Let $G$ be a connected graph of order $n$. Let $\Delta_{sb} = \max\{\text{ds}(v) : v \in V\}$. Then $\gamma_{sb}(G) \leq n - \lfloor \Delta_{sb} \rfloor$.*

*Proof.* Let $v \in V$ and $\text{ds}(v) = \Delta_{sb}$. Let $n - i = \lfloor \Delta_{sb} \rfloor$. Hence $n - i \leq \Delta_{sb} < n - i + 1$. Therefore, $n - i \leq \text{ds}(v) < n - i + 1$. Also $\text{ds}(v) \leq deg(v) + \frac{n-1-deg(v)}{2} = \frac{deg(v)}{2} + \left(\frac{n-1}{2}\right)$ and hence $n - i \leq \frac{deg(v)}{2} + \left(\frac{n-1}{2}\right)$. Thus, $deg(v) \geq n - 2i + 1$.
Since $\lfloor \Delta_{sb} \rfloor = n - i$, we have $deg(v) \leq n - i$. Hence, $n - 2i + 1 \leq deg(v) \leq n - i$. Thus, $deg(v) = n - 2i + j$ where $1 \leq j \leq i$. Now let $G_1 = G[V - N[v]]$. Clearly, $|V(G_1)| = n - deg(v) - 1 = 2i - j - 1$. Let $A$ denote the set of all isolated vertices in $G_1$ and let $|A| = k$. Then $G_1 - A$ is a graph of order $2i - j - 1 - k$ and has no isolated

vertices. Let $D$ be a $\gamma$-set of $G_1 - A$. Therefore

$$|D| = \gamma(G_1 - A) \leq \left\lfloor \frac{2i - j - 1 - k}{2} \right\rfloor \leq \left\lfloor \frac{2i - 2 - k}{2} \right\rfloor = i - 1 - \left\lceil \frac{k}{2} \right\rceil.$$

Hence

$$|D| \leq i - 1 - \left\lceil \frac{k}{2} \right\rceil. \tag{3.1}$$

If $k = 0$, let $B = \{v\}$.
If $k = 1$ or 2, let $B$ be a subset of $N(v)$ of minimum order such that $B$ dominates $A$. Clearly $|B| = 1$ if $k = 1$ and $|B| = 1$ or 2 if $k = 2$.
If $k \geq 3$, let $B = \{v, u, w\}$ where $u, w \in A$. Let $D_1 = D \cup B$. It follows from (3.1) that

$$|D_1| \leq i \;\text{ if } k = 0, 3 \textit{ or } 4 \text{ or } k = 2 \text{ and } |B| = 2 \tag{3.2}$$

$$\text{and } |D_1| < i \;\text{ if } k = 2 \text{ and } |B| = 1 \text{ or } k \geq 5. \tag{3.3}$$

Since $D$ dominates $G_1 - A$ and $B$ $sb$-dominates $N[v] \cup A$, it follows that $D_1$ is a $sb$-dominating of $G$. Also $|D_1| \leq i = n - \lfloor \Delta_{sb} \rfloor$. Hence $\gamma_{sb} \leq n - \lfloor \Delta_{sb} \rfloor$. □

**Theorem 11.** *Let $G = (V, E)$ be a connected graph of order $n$. Then $\gamma_{sb}(G) = n - \lfloor \Delta_{sb} \rfloor$ if and only if the following conditions hold.*

*(i) There exists a vertex $v$ such that $\text{ds}(v) = \Delta_{sb}$ and $deg(v) = 2\lfloor \Delta_{sb} \rfloor + 1 - n$.*

*(ii) The number of isolated vertices $k$ in $G - N[v]$ is at most 4 and if $k = 2$, then the two isolated vertices have no common neighbor in $N(v)$.*

*Proof.* $\gamma_{sb} = n - \lfloor \Delta_{sb} \rfloor$ if and only if equality holds in (1) and (2) of Theorem 10. Also equality holds in (1) if and only if $j = 1$ and $\gamma(G - N[v]) = \left\lfloor \frac{|V(G) - N[v]|}{2} \right\rfloor + k$. Equality holds in (2) if and only if $k \leq 4$ and $|B| = 2$ when $k = 2$. Hence the result follows. □

We now proceed to obtain lower bounds for $\gamma_{sb}$.

**Theorem 12.** *Let $G$ be a connected graph of order $n$. Then $\gamma_{sb}(G) \geq \left\lceil \frac{n}{1+\Delta_{sb}} \right\rceil$ and the bound is sharp.*

*Proof.* Let $D$ be a $\gamma_{sb}$-set of $G$. Since $\text{ds}(v) \leq \Delta_{sb}$ for all $v \in D$, we have $\sum_{v \in D} \text{ds}(v) \leq |D|\Delta_{sb}$. Also $\text{ds}(D, w) \geq 1$ for all $w \in V - D$ and hence $\sum_{v \in D} \text{ds}(v) \geq n - |D|$. Thus $n - |D| \leq \sum_{v \in D} \text{ds}(v) \leq |D|\Delta_{sb}$. Hence $n \leq |D|(\Delta_{sb} + 1)$ and so $\gamma_{sb} \geq \left\lceil \frac{n}{1+\Delta_{sb}} \right\rceil$. Also for the graph $G$ given in Figure 3, $\Delta_{sb} = 5.75$ and $D = \{v, w\}$ is a $\gamma_{sb}$-set of $G$. Hence $\gamma_{sb}(G) = 2 = \left\lceil \frac{n}{1+\Delta_{sb}} \right\rceil$. Thus the bound is sharp. □

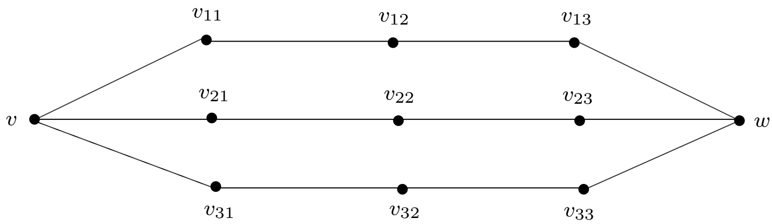


**Figure 3.** **A graph $G$ with $\gamma_{sb}(G) = \left\lceil \frac{n}{1+\Delta_{sb}} \right\rceil$.**

**Theorem 13.** *Let $\Pi = (\mathrm{ds}(v_1), \mathrm{ds}(v_2), \ldots, \mathrm{ds}(v_n))$ be the* ds*-sequence of a graph and let $\mathrm{ds}(v_1) \geq \mathrm{ds}(v_2) \geq \cdots \geq \mathrm{ds}(v_n)$. Let $t = \min\{k : k + \mathrm{ds}(v_1) + \mathrm{ds}(v_2) + \cdots + \mathrm{ds}(v_k) \geq n\}$. Then $\gamma_{sb}(G) \geq t$ and the bound is sharp.*

*Proof.* Let $S$ be any subset of $V$ with $|S| = r < t$. Hence $r + \mathrm{ds}(v_1) + \mathrm{ds}(v_2) + \cdots + \mathrm{ds}(v_t) < n$. Also $\sum\limits_{v \in S} \mathrm{ds}(v) \leq \mathrm{ds}(v_1) + \mathrm{ds}(v_2) + \cdots + \mathrm{ds}(v_t)$. Hence $|S| + \sum\limits_{v \in S} \mathrm{ds}(v) < n$. Thus

$$|S| + \sum_{v \in S} \left( \sum_{u \neq v} \frac{1}{d(v,u)} \right) < n. \tag{3.4}$$

Now, suppose $S$ is a $sb$-dominating of $G$.
Then $\sum\limits_{v \in S} \frac{1}{d(u,v)} \geq 1$ for all $u \in V - S$. Hence $\sum\limits_{u \in V-S} \left( \sum\limits_{v \in S} \frac{1}{d(u,v)} \right) \geq n - |S|$. Therefore $|S| + \sum\limits_{u \in V-S} \left( \sum\limits_{v \in S} \frac{1}{d(u,v)} \right) \geq n$ which contradicts (4). Hence $S$ is not a $sb$-dominating set of $G$. So $\gamma_{sb}(G) \geq t$. □

## 4. $sb$-domination and diameter

In this section we present several basic results and bounds for $\gamma_{sb}$ in terms of the diameter of a graph.

**Observation 14.** If $\mathrm{diam}(G) = 1$, then $G = K_n$ and $\gamma_{sb}(G) = diam(G) = 1$. If $\mathrm{diam}(G) = 2$, then $\gamma_{sb}(G) = \begin{cases} 1 & \text{if } \Delta = n-1 \\ 2 & \text{otherwise.} \end{cases}$

**Lemma 1.** *Let $G$ be a connected graph. Then $\gamma_{sb}(G) \leq diam(G)$ and the bound is sharp.*

*Proof.* Let $\mathrm{diam}(G) = d$ and let $D = \{v_1, v_2, \ldots, v_d\}$ be any subset of $V$ with $|D| = diam(G) = d$. Since $d(u,v) \leq d$ for all $u, v \in V$, it follows that $D$ is a $sb$-dominating set of $G$ and hence $\gamma_{sb}(G) \leq d$. By Observation 14, it follows that equality holds if $d = 1$ or $d = 2$ and $\Delta \neq n-1$. □

For any positive integer $t$, there exists a graph $G$ with $\gamma_{sb}(G) = diam(G) = t$, as shown in the following theorem.

**Theorem 15.** *For any positive integer $t$, there exists a graph $G$ with $\gamma_{sb}(G) = diam(G) = t$.*

*Proof.* Let $G = K_t \square K_t \square \cdots \square K_t$ be the Cartesian product of $t$ copies of $K_t$. Clearly, $\mathrm{diam}(G) = t$ and hence $\gamma_{sb}(G) \leq t$. Now let $D = \{v_1, v_2, \ldots, v_{t-1}\}$ be any subset of $V(G)$. Let $V_i = \{u_{i_1}, u_{i_2}, \ldots, u_{i_t}\}$ be the vertex set of the $i^{th}$ copy of $K_t$ in $G$. Let $v_j = (u_{1j_1}, u_{2j_2}, \ldots, u_{tj_t})$, where $1 \leq j \leq t-1$. Let $1 \leq r \leq t$. Since $|D| = t-1$, we can choose $u_{rk_r} \in V_r$ such that $u_{rk_r} \neq u_{rj_r}$ for all $j$ with $1 \leq j \leq t-1$. Let $u = (u_{1k_1}, u_{2k_2}, \ldots, u_{tk_t})$. Since $u$ differs from $v_j$ in all the $t$ coordinates $d(u, v_j) = t$. Hence $ds(D, u) = \frac{t-1}{t} < t$. Thus $D$ is not a $sb$-dominating set of $G$ and hence $\gamma_{sb}(G) \geq t$. Thus $\gamma_{sb}(G) = t = \mathrm{diam}(G)$. □

**Observation 16.** Let $r$ be the radius of $G$ and let $Z(G)$ denote the centre of $G$. If $|Z(G)| \geq r$, then $Z(G)$ is a $sb$-dominating set of $G$ and hence $\gamma_{sb}(G) \leq r$.

**Theorem 17.** *Let $G$ be a connected graph of order $n$ with $\gamma_{sb}(G) = 2$. Then $diam(G) \leq 6$.*

*Proof.* Let $D = \{u, v\}$ be a $sb$-dominating set of $G$. If $D$ is a dominating set of $G$, then it follows from Theorem 4 that $\mathrm{diam}(G) \leq 5$. Suppose $D$ is not a dominating set. Let $D_1 = N[u] \cup N[v]$ and $D_2 = V - D_1$. Clearly $D_2 \neq \emptyset$. Since $D$ is a $sb$-dominating set of $G$, it follows that $d(x, u) = d(x, v) = 2$ for all $x \in D_2$. Hence $d(x, y) \leq 4$ if $x, y \in D_2$ and $d(x, y) \leq 3$ if $x \in D_2$ and $y \in D_1$. Now, let $x, y \in D_1$. If $x, y \in N(u)$ or $x, y \in N(v)$ then $d(x, y) \leq 2$. If $x \in N(u)$ and $y \in N(u)$, let $w \in D_2$. Then $d(x, y) \leq d(x, u) + d(u, w) + d(w, v) + d(v, y) = 6$. Thus $d(x, y) \leq 6$ for all $x, y \in V$ and hence $\mathrm{diam}(G) \leq 6$. □

**Theorem 18.** *Let $k$ be a positive integer with $2 \leq k \leq 6$. Then there exists a graph $G$ such that $\gamma_{sb}(G) = 2, \gamma(G) > 2$ and $diam(G) = k$.*

*Proof.* If $k = 5$ then for the graph $G$ given in Figure 1, $\gamma_{sb}(G) = 2$, $\gamma(G) = 3$ and $\mathrm{diam}(G) = 5$. If $k = 6$, then for the path $G = P_7 = (v_1, v_2, v_3, v_4, v_5, v_6, v_7)$, we have $\gamma_{sb}(G) = 2, \gamma(G) = 3$ and $\mathrm{diam}(G) = 6$. If $k = 4$, then let $G = P_3 \circ K_1$ where $P_3 = v_1 v_2 v_3$ and let $w_i$ be the pendent vertex adjacent to $v_i$. Then $\{v_1, v_3\}$ is a $\gamma_{sb}$-set of $G, \{v_1, v_2, v_3\}$ is a $\gamma$-set of $G$ and $d(w_1, w_3) = 4$. Thus $\gamma_{sb}(G) = 2, \gamma(G) = 3$ and $\mathrm{diam}(G) = 4$.
Assume now that $k = 2$. For the graph $G$ given in Figure 4, $\mathrm{diam}(G) = 2$ and $\Delta(G) \leq n - 2$. Hence $\gamma_{sb}(G) = 2$. Also for any two vertices $x, y$ in $G$, $N(x) \cap N(y) \neq \emptyset, |N(x)| = |N(y)| = 4$ and hence $|N[x] \cup N[y]| \leq 9$. Thus $\{x, y\}$ is not a dominating set of $G$ and hence $\gamma(G) > 2$.

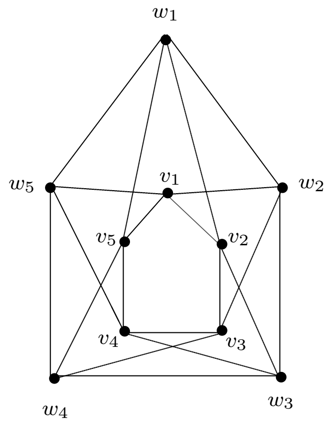


**Figure 4.** **A graph $G$ with $\gamma_{sb}(G) = diam(G) = 2$ and $\gamma(G) > 2$.**

Finally let $k = 3$. Let $G$ be the graph obtained from $K_4 \circ K_1$ by removing one pendent vertex. Let $V(K_4) = \{v_1, v_2, v_3, v_4\}$ and let $w_i$ be the pendent vertex adjacent to $v_i, 1 \leq i \leq 3$. Then $D = \{v_1, v_2\}$ is a $\gamma_{sb}$-set of $G$, $D_1 = \{v_1, v_2, v_3\}$ is a $\gamma$-set of $G$ and $d(w_1, w_3) = 3$. Thus $\gamma_{sb}(G) = 2, \gamma(G) = 3$ and $\text{diam}(G) = 3$. □

## 5. Conclusion and Scope

The dominating strength ds($v$) of a vertex is a topological index which is useful in identifying the most influential member in a social network. Also the concept of *sb*-domination is a natural generalization of domination and disjunctive domination in graphs. A study of total *sb*-domination, independent *sb*-domination and algorithmic aspects of *sb*-domination are a few promising directions for further research.

**Conflict of Interest:** The authors declare that they have no conflict of interest.

**Data Availability:** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

## References

[1] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & Digraphs, 6th ed.*, Chapman & Hall London, New York, 2015.

[2] P. Dankelmann, D. Day, D. Erwin, S. Mukwembi, and H. Swart, *Domination with exponential decay*, Discrete Math. **309** (2009), no. 19, 5877–5883. https://doi.org/10.1016/j.disc.2008.06.040.

[3] Z Gao, Y Shi, C Xi, and J Yue, *The extended dominating sets in graphs*, Asia-Pacific Journal of Operational Research **40** (2023), no. 5, Article ID: 2340015. https://doi.org/10.1142/S0217595923400158.

[4] W. Goddard, M.A. Henning, and C.A. McPillan, *The disjunctive domination number of a graph*, Quaest. Math. **37** (2014), no. 4, 547–561. https://doi.org/10.2989/16073606.2014.894688.

[5] T.W. Haynes, S. Hedetniemi, and P. Slater, *Fundamentals of Domination in Graphs*, CRC press, Boca Raton, 1998.

[6] M.A. Henning and A. Yeo, *Total Domination in Graphs*, Springer, New York, 2013.