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Research Article

Two techniques to reduce the Pareto optimal solutions in multi-objective optimization problems

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Abstract: In this study, for a decomposed multi-objective optimization problem, we propose the direct sum of the preference matrices of the subproblems provided by the decision maker (DM). Then, using this matrix, we present a new generalization of the rational efficiency concept for solving the multi-objective optimization problem (MOP). A problem that sometimes occurs in multi-objective optimization is the existence of a large set of Pareto optimal solutions. Hence, decision making based on selecting a unique preferred solution becomes difficult. Considering models with the concept of generalized rational efficiency can relieve some of the burden from the DM by shrinking the solution set. This paper discusses both theoretical and practical aspects of rationally efficient solutions related to this concept. Moreover, we present two techniques to reduce the Pareto optimal solutions using. The first technique involves using the powers of the preference matrix, while the second technique involves creating a new preference matrix by modifying the decomposition of the MOP.

Keywords: pareto, nondominated, A_P-efficiency, multi-objective programming.

AMS Subject classification: 90C29, 91B08, 90B50

1. Introduction

Because of the conflicting nature of the objective functions, a MOP usually does not have a single optimal solution for all objectives, but a set of Pareto optimal solutions exists. Several multi-objective optimization approaches exist that generate subsets of Pareto optimal solutions, which can be overwhelming to the DM in the task of selecting the most appropriate solution to implement. More recently, these

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approaches have categorized into a priori, a posteriori, interactive, and pruning methods by Petchrompo et al. [12].
In the a priori method, a MOP is converted into a single objective problem before obtaining a solution. A priori methods use DM preferences to bias the search of optimal solutions towards a preferred region, for example by changing the definition of dominance [7, 9, 19], by giving different weights to the objectives [17], by assigning reference values (goals) and priority levels to the objectives [18], by assuming a utility function describing the DM behaviour and interest in the alternative solutions [8]. The a posteriori method generates a representative set of Pareto optimal solutions and the DM chooses the best one among them, [16, 20]. Interactive methods allow the DM to guide the search by alternating optimization, and preference articulation iteratively, [8, 15]. Pruning methods are applied before, during, or after the optimization process to reduce the number of Pareto optimal solutions, [12, 13].
One approach to reduce the size of the Pareto optimal set is to combine multiple objectives, i.e. by summing them, before employing tools to solve the resulting MOP, [2]. This method also reduces the objective space dimension using DM's preference matrix. In particular, it can be used to discard certain unwanted solutions, especially the extreme points which are found by minimizing just one of the objectives given in the classical sense while disregarding the rest. Unfortunately, the problem of combining several objectives does not seem to have attracted too much attention in the literature so far.
Berman and Naumov [1] were perhaps the first to construct a matrix of a cone to represent DM's preferences by interval trade-offs. Noghin [10, 11] investigated the relative importance of objectives and provided a definition of weights (which he called coefficients) for objective functions as well as for groups of objective functions. Podinowski [14] extended the concept of "one criterion is more important than another" to "‘one criterion is n-times more important than another" by applying pairwise coefficients of relative importance obtained from the DM. Also, using these coefficients, he constructed a matrix that represents DM's preferences. The polyhedral cone and the partial order characterized by it are discussed by Engau and Wiecek [3]. They discussed the relationship between the optimal solutions with respect to a polyhedral cone represented by a matrix A and the optimal solutions of a multi-objective optimization with respect to the natural order after linear transformation of the objective functions by a matrix A. Hunt et al. [5] developed a convex polyhedral cone-based preference modeling framework for decision making with multiple criteria, which extends the classical notion of Pareto optimality and accounts for the relative importance of criteria. Dempe et al. [2] investigated the impact of using linear combinations of objectives and which solutions are eliminated by doing so. They showed how the strategy of combining objectives linearly influences the efficient set.
In this paper, we do not focus on constructing a preference matrix. Instead, we investigate two pruning techniques to reduce the number of Pareto optimal solutions, by applying the strategy of linear combination of objectives. The present study is an extension of some results obtained by [6, 7].
The paper is organized as follows. In Section 2, we give some basic and preliminary

concepts. In Section 3, we introduce the concept of rational A_P-efficiency, and we define rational A_P-efficiency in terms of vector inequalities, in order to make it practical. In Section 4, the concept of A_P^r-efficiency is examined to generate a subset of Pareto optimal solutions, for $r = 1, 2, \ldots$. Moreover, the A_P-efficient solutions are reduced by constructing a new preference matrix which based on P and A_P, in Section 5. Also, an algorithm is presented to generate a subset of A_P-efficient solutions. In addition, the numerical examples are provided to confirm the efficiency of these methods. Finally, Section 6 concludes the paper.

2. Terminology

In this article, the following notations will be used. Let $\mathbb{R}^m$ be the Euclidean vector space and $y', y'' \in \mathbb{R}^m$. $y' \leqq y''$ denotes $y'_i \leq y''_i$ for all $i = 1, \ldots, m$. $y' < y''$ denotes $y'_i < y''_i$ for all $i = 1, \ldots, m$. $y' \leq y''$ denotes $y' \leqq y''$ but $y' \neq y''$.
Consider a decision problem defined as an optimization problem with m objective functions. For simplification, we assume, without losing generality, that the objective functions should be minimized. The problem can be defined as follows:

$$\begin{aligned} &\min\,(f_1(x), \ldots, f_m(x)) \\ &\text{subject to } x \in X, \end{aligned} \tag{2.1}$$

where x denotes a vector of decision variables selected from the feasible set X and $f(x) = (f_1(x), \ldots, f_m(x))$ is a vector function that maps the feasible set X into the objective (criterion) space $\mathbb{R}^m$. We refer to the elements of the objective space as outcome vectors. An outcome vector y is attainable if it expresses the outcomes of a feasible solution, i.e., $y = f(x)$ for some $x \in X$. The set of all attainable outcome vectors will be denoted by $Y = f(X)$.
In single objective minimization problems, we compare the objective values at different feasible decisions to select the best decision. Decisions are ranked according to the objective values at those decisions and the decision resulting in the least smallest objective value is the most preferred decision. Similarly, to make the multi-objective optimization model operational, one needs to assume some solution concept specifying what it means to minimize multi-objective functions. The solution concepts are defined by the properties of the corresponding preference model. We assume that solution concepts depend only on the evaluation of the outcome vectors while not taking into account any other solution properties not represented within the outcome vectors. Thus, we can limit our considerations to the preference model in the objective space Y. In the following, some basic concepts and definitions of preference relations are reviewed from [6].

Definition 1. Let $y', y'' \in \mathbb{R}^m$ and let $\preceq$ be a relation of weak preference defined on $\mathbb{R}^m \times \mathbb{R}^m$. The corresponding relations of strict preference $\prec$ and indifference $\simeq$ are defined as follows:

$$y' \prec y'' \;\Leftrightarrow\; (y' \preceq y'' \textit{and not } y'' \preceq y'), \tag{2.2}$$
$$y' \simeq y'' \;\Leftrightarrow\; (y' \preceq y'' \textit{and } y'' \preceq y'). \tag{2.3}$$

Definition 2. Preference relations satisfying the following axioms are called rational preference relations:
1. Reflexivity: for all $y \in \mathbb{R}^m$: $y \preceq y$.
2. Transitivity: for all $y', y'', y''' \in \mathbb{R}^m$: $y' \preceq y''$ and $y'' \preceq y''' \Rightarrow y' \preceq y'''$.
3. Strict monotonicity: for all $y \in \mathbb{R}^m$: $y - \epsilon e_i \prec y$ for $\epsilon > 0$ where e_i denotes the i^{th} unit vector in $\mathbb{R}^m$.

The rational preference relations allow us to formalize the Pareto optimal solution concept with the following definitions.

Definition 3. Let $y', y'' \in Y$. We say that y' rationally dominates y'', and denote by $y' \prec_r y''$ iff $y' \prec y''$ for all rational preference relations $\preceq$. An outcome vector y is rationally nondominated if and only if there does not exist another outcome vector y' such that $y' \prec_r y$. Analogously, a feasible solution $x \in X$ is a rationally efficient solution of the MOP (2.1) if and only if $y = f(x)$ is rationally nondominated.

A relationship between the weak rational preference relation $\preceq_r$ and the Pareto relation has been established in [6]. As the following proposition shows finding nondominated points with respect to the relation $\preceq_r$ can be done by means of Pareto preference.

Proposition 1. *([6], Proposition 1.1) For any two vectors $y', y'' \in Y$, we have*

$$y' \preceq_r y'' \;\Leftrightarrow\; y' \leqq y'',$$
$$y' \prec_r y'' \;\Leftrightarrow\; y' \leq y''.$$

The set of all rationally efficient solutions $x \in X$ is denoted by X_E and called the efficient set. The set of all rationally nondominated points $y = f(x) \in Y$, where $x \in X_E$, is denoted by Y_N and called the nondominated set.

3. The concept of rational A_P-efficiency

In this section, we will introduce a dominance relation to generate solutions that are rationally A_P-efficient. The following definitions are necessary for the concepts of interest in this paper.

Definition 4. Let $M = \{1, \ldots, m\}$ be the index set of objective functions $f = (f_1, \ldots, f_m)$ and n be a positive integer such that $n \leqslant m$. A collection $P = \{P_k \subseteq M : k = 1, \ldots, n\}$ is called a decomposition of M, and also it is said a partition of M if $\bigcup_{k=1}^{n} P_k = M$, and

$P_i \cap P_j = \emptyset$ for all $i \neq j$, where $i, j \in \{1, \ldots, n\}$ and P_k is index set of objective functions in class k. The multi-objective problem

$$\begin{aligned} &\min(f_j(x))_{j \in P_k} \qquad (k = 1, \ldots, n), \\ &\text{subject to } x \in X, \end{aligned} \tag{3.1}$$

is called a subproblem of the multi-objective problem (2.1) and the collection of all these subproblems is called a decomposition of the multi-objective problem (2.1).

To simplify the notations, we put $f^k = (f_j)_{j \in P_k}$ and $f^k(x) = (f_j(x))_{j \in P_k}$. Also, we will use the notation $y = (y_{P_k})_{k=1}^n$, where $y_{P_k} = (y_i)_{i \in P_k}$, for any vector $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$. It is noteworthy that the functions f^k maps the feasible set X into the space $\mathbb{R}^{|P_k|}$, where $|P_k|$ denotes the cardinality of the set P_k for $k = 1, \ldots, n$, and $m = |P_1| + \cdots + |P_n|$.
The preference information identifies the relative importance of different objectives in the optimization problem. The DM can select the preference matrix A_k that displays the relative importance of different objectives in the subproblem (3.1) as follows. Let $i, j \in P_k$, $i \neq j$ and criterion j be relatively more important than criterion i, so that an improvement of the former is desired even when the latter decays. Let $a_{ij}^k \geqslant 0$ represent the minimum desired improvement in the j-th objective function for a unit loss in the i-th objective function. If criteria i and j are equally important than there is no specific preference relation between them and the corresponding coefficient $a_{ij}^k = 0$. For $i = j$, let $a_{ij}^k = 1$. The elements a_{ij}^k form a $|P_k| \times |P_k|$ matrix $A_k = [a_{ij}^k]$. For the case of Pareto domination, the matrix A_k is an identity matrix with $a_{ij}^k = 1$ for $i = j$ and 0 otherwise. In general, the preference matrix represents a polyhedral cone that plays the role of a domination cone for the used preference model, which is based on the trade-off between objectives. For more details, the reader can refer to [1–3, 5]. In this paper, we do not focus on how to construct a preference matrix that is provided by a DM. Instead, we suggest two pruning methods that are based on preference matrix. The idea behind these is that the DM classifies the objective functions in different classes and determines a partition $P = \{P_k \subseteq M : k = 1, \ldots, n\}$ of $\{1, \ldots, m\}$ according to the importance of objective functions. The DM should provide the preference matrix A_k for the relative importance of the objectives of subproblem (3.1) for $k = 1, \ldots, n$. Thus, we introduce the matrix $A_P = A_1 \oplus \ldots \oplus A_n$, which is the direct sum of the matrices $A_1, \ldots A_n$, i.e.

$$A_P = \begin{bmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_n \end{bmatrix}.$$

We have $A_P(y) = (A_1(y_{P_1}), \ldots, A_n(y_{P_n}))$ for $y \in Y$, where $y_{P_k} = (y_j)_{j \in P_k}$ for $k = 1, \ldots, n$.

Definition 5. Let $y', y'' \in Y$ be two outcome vectors. We say that y' rationally A_P-dominates y'', and denote by $y' \prec_{rA_P} y''$ if and only if $A_P(y') \prec A_P(y'')$ for all rational preference relations $\preceq$. An outcome vector y is called rationally A_P-nondominated if and only if there is not another outcome vector y' such that $y' \prec_{rA_P} y$. Analogously, a feasible solution $x \in X$ is called a rationally A_P-efficient solution of the problem (2.1) if and only if $y = f(x)$ is rationally A_P-nondominated point.

The set of all rationally A_P-efficient solutions $x \in X$ is denoted by X_{A_PE} and is called the rationally A_P-efficient set. The set of all rationally A_P-nondominated points $y = f(x) \in Y$, where $x \in X_{A_PE}$, is denoted by Y_{A_PN} and is called the rationally A_P-nondominated set.
Let A be the preference matrix representing the relative importance of different objectives in the problem (2.1). It should be noted that $A_P = A$, when $n = 1$ or $P_1 = \{1, \dots, m\}$. Therefore, we can rewrite the above definition for the matrix A as follows.

Definition 6. Let $y', y'' \in Y$. We say that y' rationally A-dominates y'', and denote by $y' \prec_{rA} y''$ if and only if $A(y') \prec A(y'')$ for all rational preference relations $\preceq$. An outcome vector y is called rationally A-nondominated if and only if there is not another outcome vector y' such that $y' \prec_{rA} y$. Analogously, a feasible solution $x \in X$ is called a rationally A-efficient solution of the problem (2.1) if and only if $y = f(x)$ is rationally A-nondominated point.

It is worth noting that for $n = 1$ or $P_1 = \{1, \dots, m\}$, we have $X_{A_PE} = X_{AE}$ and $Y_{A_PN} = Y_{A_PN}$. Let I denote the identity matrix. If $P_k = \{k\}$ and $A_k = I$ for all $k = 1, \dots, m$, then the relation $\prec_{rA_P}$ becomes the rational relation $\prec_r$.
Similar to the relation $\prec_{rA_P}$, we can define the relation of rational A_P-indifference, $\simeq_{rA_P}$, and the relation of rational weak A_P-dominance, $\preceq_{rA_P}$. We say that $y' \simeq_{rA_P} y''$ if and only $A_P(y') \simeq A_P(y'')$ for all rational preference relations $\preceq$, and also $y' \preceq_{rA_P} y''$ if and only $A_P(y') \preceq A_P(y'')$ for all rational preference relations $\preceq$. The relations $\prec_{rA_P}$, $\simeq_{rA_P}$ and $\preceq_{rA_P}$ satisfy conditions (2.2-2.3).
It is clear that the preference relation $\preceq_{rA_P}$ satisfies the reflexivity and transitivity axioms. To continue, we express some conditions that guarantee the relation $\preceq_{rA_P}$ is a rational preference relation. Throughout this section, we assume that $e_i^k \in \mathbb{R}^k$ is the unit vector with the *i-th* component equal to one and the remaining ones equal to zero, where $k = 1, \dots$ and $i \in \{1, \dots, k\}$.

Theorem 1. *The strict monotonicity axiom for the preference $\preceq_{rA_P}$ is equivalent to the condition*

$$a_i^k \geq 0 \qquad (i = 1, \dots, |P_k|), \tag{3.2}$$

where a_i^k is the column i of the matrix A_k for $k = 1, \dots, n$.

Proof. We will prove that if matrix A_p satisfies condition (3.2), then the strict monotonicity axiom holds for preference $\preceq_{rA_P}$. For $y \in Y$, $i \in \{1, \ldots, m\}$, $\epsilon > 0$ and $y' = y - \epsilon e_i^m$, we show that $y' \prec_{rA_P} y$. There exists an index $k \in \{1, \ldots, n\}$ such that $i \in P_k$. It's obvious that $y'_{P_j} = y_{P_j}$ and $A_j(y'_{P_j}) = A_j(y_{P_j})$, when $j \in \{1, \ldots, n\} - \{k\}$. Since $y'_{P_k} = y_{P_k} - \epsilon e_i^{|P_k|}$, we have $y'_{P_k} \leq y_{P_k}$. According to condition (3.2), we obtain

$$\sum_{j=1}^{|P_k|} a_{ij}^k (y'_{P_k})_j \leqslant \sum_{j=1}^{|P_k|} a_{ij}^k (y_{P_k})_j \qquad (i = 1, \ldots, |P_k|),$$

where strict inequality holds at least once. So $A_k(y'_{P_k}) \leq A_k(y_{P_k})$ and the proof is complete by applying Proposition 1.

Conversely, assuming that the preference $\preceq_{rA_P}$ follows the strict monotonicity axiom. We have

$$e_j^m - e_j^m \prec_{eA_P} e_j^m, \qquad (j = 1, \ldots, m),$$

which, together with Proposition 1, imply that matrix A_P satisfies condition (3.2). □

To make it practical, rational A_P-efficiency is defined in terms of vector inequalities. To do that, we define a certain preference relation.

Definition 7. Suppose that $y', y'' \in Y$ are two outcome vectors. We define the relation $\leq_{A_P}$ as follows:

$$y' \leq_{A_P} y'' \Leftrightarrow A_P(y') \leqq A_P(y''). \tag{3.3}$$

Also, we can define the relations $<_{A_P}$ and $=_{A_P}$ as follows:

$$\begin{aligned} y' <_{A_P} y'' &\Leftrightarrow (y' \leq_{A_P} y'' \textit{and not } y'' \leq_{A_P} y'), \\ y' =_{A_P} y'' &\Leftrightarrow (y' \leq_{A_P} y'' \textit{and } y'' \leq_{A_P} y'). \end{aligned}$$

It is clear that the preference relation $\leq_{A_P}$, satisfies reflexivity, transitivity and monotonicity. This means that, the relation (3.3) is a rational preference relation. Note that the relation $\leq_{A_P}$ becomes the Pareto relation, when $A = I$ and $P_k = \{k\}$ for all $k = 1, \ldots, m$.

In the following, we will discuss the relationship between two preferences $\preceq_{rA_P}$ and $\leq_{A_P}$.

Corollary 1. *Let $y', y'' \in Y$ be two outcome vectors. We have*

$$y' \preceq_{rA_P} y'' \Leftrightarrow y' \leq_{A_P} y'',$$

$$y' \prec_{rA_P} y'' \Leftrightarrow y' <_{A_P} y''.$$

Proof. Based on Definition 5, Proposition 1 and Definition 7, we can conclude that

$$\begin{aligned} y' \preceq_{rA_P} y'' &\Leftrightarrow A_P(y') \preceq_r A_P(y'') \\ &\Leftrightarrow A_P(y') \leqq A_P(y'') \\ &\Leftrightarrow y' \leq_{A_P} y''. \end{aligned}$$

□

The statement below is true according to Theorem 1 and Corollary 1.

Corollary 2. *The preference $\leq_{A_P}$ is a rational preference relation if and only if the matrix A_P satisfies condition (3.2).*

Note that Corollary 1 permits one to express rational A_P-efficiency for problem (2.1) in terms of the standard efficiency for the MOP with objectives $A_P(f(x))$,

$$\begin{aligned} &\min A_P(f(x)) \\ &\text{subject to } x \in X. \end{aligned} \tag{3.4}$$

Corollary 3. *A feasible solution $x \in X$ is a rationally A_P-efficient solution to the problem (2.1) if and only if it is an efficient solution to the problem (3.4).*

Remark 1. If $P_1 = \{1, \dots, m\}$ in Corollary 3, then any feasible solution $x \in X$ is a rationally A-efficient solution of the problem (2.1), if and only if it is an efficient solution of the problem

$$\begin{aligned} &\min A(f(x)) \\ &\text{subject to } x \in X. \end{aligned} \tag{3.5}$$

Also, if $A_k = I$ is identity matrix for all $k = 1, \dots, n$, then we have Corollary 3.2 from [7]. In addition, if $A_k = I$ and $P_k = \{k\}$ for all $k = 1, \dots, m$, we have Proposition 1.1 from [6].

Example 1. Let

$$X = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 \geqslant 1 \text{ and } x_1, x_2, x_3 \geqslant 0\right\},$$

and $f(x_1, x_2, x_3) = (x_1, x_2, x_3)$ and $P_1 = \{1, 2\}$ and $P_2 = \{3\}$. We get

$$X_E = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 = 1 \text{ and } x_1, x_2 \geqslant 0, x_3 = 0\right\}.$$

For $A_1 = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1 \end{bmatrix}$, we have $A_P(f(x)) = (x_1, \alpha x_1 + x_2, x_3)$, where $\alpha \geqslant 0$ and $x \in X$. The assumption $x_1 + x_2 = 1$ implies that $\alpha x_1 + x_2 = (\alpha - 1)x_1 + 1$. Hence, it is easy to obtain that $X_{A_P E} = X_E$, when $0 \leqslant \alpha < 1$.
Now, let $\alpha \geqslant 1$ and $\hat{x} \in X$ be a feasible solution with $\hat{x}_1 > 0$. For $0 < \epsilon < \hat{x}_1$, we put $x_1 = \hat{x}_1 - \epsilon$, $x_2 = \hat{x}_2 + \epsilon$ and $x_3 = \hat{x}_3$. After some calculations, we conclude that $A_P(f(x)) \leq A_P(f(\hat{x}))$, and thus $X_{A_P E} = \{(0, 1, 0)\}$.

Finally, we show that rational A_P-efficiency is scale invariant with respect to linear scaling with a positive factor.

Theorem 2. *Let $g : R \to R$ be a strictly increasing linear function. The feasible solution $x \in X$ is a rationally A_P-efficient solution of the problem (2.1), if and only if it is a rationally A_P-efficient solution of the problem*

$$\min_{x \in X} \left(g(f_1(x)), \dots, g(f_m(x))\right). \tag{3.6}$$

Proof. Let x be a rationally A_P-efficient solution of the problem (2.1). If x is not a rationally A_P-efficient solution of the problem (3.6). Then there is a feasible vector x' such that the outcome vectors $y = (g(f_1(x)), \dots, g(f_m(x)))$ and $y' = (g(f_1(x')), \dots, g(f_m(x')))$ satisfy

$$A_k(y'_{P_k}) \leqq A_k(y_{P_k}) \qquad (k = 1, \dots, n),$$

and $A_k(y'_{P_k}) \leq A_k(y_{P_k})$ for some $k \in \{1, \dots, n\}$. Since the function g is strictly increasing and linear, we obtain

$$A_k(f_{P_k}(x')) \leqq A_k(f_{P_k}(x)) \qquad (k = 1, \dots, n),$$

and $A_k(f_{P_k}(x')) \leq A_k(f_{P_k}(x))$ for some $k \in \{1, \dots, n\}$. Hence $f(x')$ rationally A_P-dominates $f(x)$, which contradicts the rationally A_P-efficient of x. The proof of converse part can be done in a similar way. □

4. The concept of A_P^r-efficiency

In this section, we investigate the relationship between A_P-efficient solutions and A_P^r-efficient solutions. Then, we introduce the concept of A_P^∞-efficient to generate a subset of efficient solutions, which aims to offer a limited number of representative solutions to the DM. At first, we study the relationship between A_P-efficient solutions and $(AB)_P$-efficient solutions. To do this, we require the following theorem.

Theorem 3. *Let $A = (a_1, \dots, a_m)$ and $B = (b_1, \dots, b_m)$ be two $m \times m$ matrices, where a_j and b_j are column j of the matrices A and B, respectively.*

(i) Let $AB = C = (c_1, \dots, c_m)$, where c_j is column j of the matrix C. If $a_j \geq 0$ and $b_j \geq 0$ for all $j = 1, \dots, m$, then $c_j \geq 0$ for all $j = 1, \dots, m$.

(ii) If $a_j \geq 0$ for all $j = 1, \dots, m$ and y', y'' are two outcome vectors in R^m, then

$$\begin{aligned} y' <_B y'' &\implies y' <_{AB} y'', \\ y' \leq_B y'' &\implies y' \leq_{AB} y''. \end{aligned}$$

Hence $Y_{(AB)N} \subset Y_{BN}$. This implies that $X_{(AB)E} \subset X_{BE}$. In particular, we have $X_{AE} \subset X_E$, when $B = I$.

Proof. (*i*) We have

$$c_j = \left(\sum_{k=1}^{m} a_{ik} b_{kj} \right)_{i=1}^{m} .$$

The condition $b_j \geq 0$ implies that $b_{kj} \geq 0$ for all $k = 1, \ldots, m$ and $b_{k'j} > 0$ for some $k' \in \{1, \ldots, m\}$. Also $a_{k'} \geq 0$ conclude that $a_{ik'} \geq 0$ for any $i = 1, \ldots, m$ and $a_{i'k'} > 0$ for some $i' \in \{1, \ldots, m\}$. Thus $a_{ik} b_{kj} \geq 0$ for any $i, k = 1, \ldots, m$ and $a_{i'k'} b_{k'j} > 0$, which means that $c_j \geq 0$.
(*ii*) Let $y', y'' \in R^m_{\geqq}$ and $y' <_B y''$, hence

$$\sum_{j=1}^{m} b_{kj} y'_j \leq \sum_{j=1}^{m} b_{kj} y''_j \qquad (k = 1, \ldots, m),$$

and $\sum_{j=1}^{m} b_{k'j} y'_j < \sum_{j=1}^{m} b_{k'j} y''_j$ for some $k' \in \{1, \ldots, m\}$. Now according to assumption, we have $a_{ik} \geq 0$ for all $i, k = 1, \ldots, m$, and also there exists $i' \in \{1, \ldots, m\}$ such that $a_{i'k'} > 0$. This implies that

$$\sum_{k=1}^{m} a_{ik} \sum_{j=1}^{m} b_{kj} y'_j \leq \sum_{k=1}^{m} a_{ik} \sum_{j=1}^{m} b_{kj} y''_j \qquad (i = 1, \ldots, m),$$
$$\Longrightarrow \sum_{j=1}^{m} \left(\sum_{k=1}^{m} a_{ik} b_{kj} \right) y'_j \leq \sum_{j=1}^{m} \left(\sum_{k=1}^{m} a_{ik} b_{kj} \right) y''_j \qquad (i = 1, \ldots, m),$$

where the inequality is strict for $i = i'$. Thus $ABy' \leq ABy''$, which means that $y' <_{AB} y''$.

□

Theorem 4. *Let $A_P = A_1 \oplus \ldots \oplus A_n$ and $B_P = B_1 \oplus \ldots \oplus B_n$ be two $m \times m$ matrices and $a_j \geq 0$ for $j = 1, \ldots, m$, where a_j is column j of the matrix A_P. If y', y'' are two outcome vectors in R^m with $y' <_{B_P} y''$, then $y' <_{(AB)_P} y''$, hence $Y_{(AB)_P N} \subset Y_{B_P N}$. This concludes that $X_{(AB)_P E} \subset X_{B_P E}$.*

Proof. Let $y', y'' \in Y$ and $y' <_{B_P} y''$. Hence

$$B_k(y'_{P_k}) \leqq B_k(y''_{P_k}) \qquad (k = 1, \ldots, n),$$

and $B_k(y'_{P_k}) \leq B_k(y''_{P_k})$ for some $k \in \{1, \ldots, n\}$. Let a^k_j be column j of the matrix A_k for $k = 1, \ldots, n$. Since $a^k_j \geq 0$, by using Theorem 3, we have

$$A_k B_k(y'_{P_k}) \leqq A_k B_k(y''_{P_k}) \qquad (k = 1, \ldots, n),$$

and $A_k B_k(y'_{P_k}) \leq A_k B_k(y''_{P_k})$ for some $k \in \{1, \ldots, n\}$. This implies that $y' <_{(AB)_P} y''$. □

The following statement states the relationship among A_P-efficient solutions, A-efficient solutions and efficient solution of problem (2.1).

Corollary 4. *Let $A_P = A_1 \oplus \ldots \oplus A_n$ be an $m \times m$ matrix and $a_j \geq 0$ for $j = 1, \ldots, m$, where a_j is column j of the matrix A_P. If $y', y'' \in Y$ and $y' \leq y''$, then $y' <_{A_P} y''$, hence $Y_{A_P N} \subset Y_N$. This concludes that $X_{A_P E} \subset X_E$. Thus if $x \in X$ is an efficient solution of problem (3.4), then it is an efficient solution of problem (2.1). In particular, If $n = 1$, then $Y_{AN} \subset Y_N$ and $X_{AE} \subset X_E$. Thus, if $x \in X$ is an efficient solution of problem (3.5), then it is an efficient solution of problem (2.1).*

Proof. Setting $B_P = I$ in the above theorem gives the desired result. □

For a square matrix A and a positive integer r, A^r represents the product of A multiplied by itself r times. We also define $A^0 = I$.

Corollary 5. *Suppose that the matrix A_P satisfies the condition $a_j \geq 0$ for $j = 1, \ldots, m$. If r is a non-negative integer, then $Y_{A_P^{r+1} N} \subset Y_{A_P^r N} \subset Y_N$.*

Proof. The result follows by replacing A^r instead of B, in Theorem 4. □

Let $\mathbb{R}^m_{\geqq} = \{d \in \mathbb{R}^m \colon d \geqq 0\}$ be non-negative orthant of $\mathbb{R}^m$. By Corollary 3, one can easily verify that the feasible solution $\hat{x} \in X$ is a rationally A_P^r-efficient solution if and only if

$$(A_P^r(f(\hat{x})) - \mathbb{R}^m_{\geqq}) \bigcap A_P^r(f(X)) = \{A_P^r(f(\hat{x}))\}. \tag{4.1}$$

The condition $a_j \geq 0$ for $j = 1, \ldots, m$, in the above results are necessary. To investigate this fact, we give the following example.

Example 2. Let

$$X = Y = \left\{(y_1, y_2) \colon y_1^2 + y_2^2 \leqslant 1\right\},$$

and $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $P_1 = \{1, 2\}$. Using the relation (4.1) for $r = 0$ and $r = 1$, we have

$$Y_N = \left\{(y_1, y_2) \colon y_1^2 + y_2^2 = 1,\ -1 \leqslant y_1, y_2 \leqslant 0\right\},$$

and

$$Y_{AN} = \left\{(y_1, y_2) \colon y_1^2 + y_2^2 = 1,\ -1 \leqslant y_1 \leqslant 0,\ 0 \leqslant y_2 \leqslant 1\right\},$$

respectively. Since $A^{2r} = I$ and $A^{2r+1} = A$ for $r = 0, 1, \ldots$, we get $Y_{A^{2r} N} = Y_N$ and $Y_{A^{2r+1} N} = Y_{AN}$.

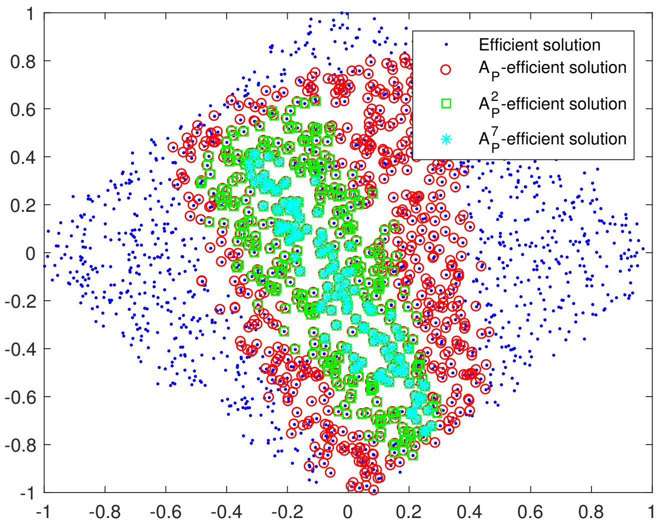

Figure 1. **The A_P^r-efficient solutions of the test problem (4.2) for $r = 0, 1, 2, 7$.**

In the following example, we investigate Corollary 5, and show that the set of A_P^r-efficient solutions are reducing when r is increasing. For this purpose, a large number of random solutions are generated for scalable test function. From this large set of solutions, efficient solutions and A_P^r-efficient are calculated for $r = 1, 2, 7$.

Example 3. Let us consider the following test problem from [4],

$$\begin{aligned} \min_{x \in R^2} \quad & y = (f_1(x), f_2(x), f_3(x), f_4(x), f_5(x), f_6(x)) \\ & f_1(x) = x_1^2 + (x_2 + 1)^2 \\ & f_2(x) = (x_1 - 0.5)^2 + (x_2 + 0.5)^2 \\ & f_3(x) = (x_1 - 1)^2 + x_2^2 \\ & f_4(x) = (x_1 + 1)^2 + x_2^2 \\ & f_5(x) = (x_1 - 0.5)^2 + (x_2 - 0.5)^2 \\ & f_6(x) = x_1^2 + (x_2 - 1)^2 \\ & x_1, x_2 \in [-1, 1]. \end{aligned} \tag{4.2}$$

To solve this problem, we use MATLAB software environment. We select 3000 random solutions $(x_1, x_2) \in [-1, 1] \times [-1, 1]$, and compare the values of the objective functions at these solutions. Among these solutions, there are 1741 efficient solutions, which are indicated by the blue points in Figure 1. Let $P_1 = \{1, 2, 3\}$, $P_2 = \{4, 5, 6\}$ and

$$A_1 = \begin{bmatrix} 1 & 0.5 & 0 \\ 0.9 & 0.8 & 0.5 \\ 1 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 0.7 & 0 \\ 1 & 0 & 0.8 \\ 0 & 0.5 & 0.8 \end{bmatrix},$$

be given by the DM. By comparing the values of $A_P^r(f(x))$ at the efficient solutions for $r = 1, 2, 7$, we obtain 888 A_P-efficient solutions, 420 A_P^2-efficient solutions and 129 A_P^7-efficient solutions, which are shown by red circles, green square and cyan star, respectively, in Figure 1.

Using the results above, we can define infinite order dominance as follows:

$$\leq_{A_P^\infty} = \bigcup_{r\in\mathbb{N}} \leq_{A_P^r},$$

where $\mathbb{N} = \{0, 1, \ldots\}$. This means that,

$$y' \leq_{A_P^\infty} y'' \Leftrightarrow y' \leq_{A_P^r} y'' \quad (for\ some\ r \in \mathbb{N}).$$

Definition 8. The outcome vector y is A_P^∞-nondominated if and only if there does not exist another outcome vector y' such that $y' \leq_{A_P^\infty} y$. Analogously, a feasible solution x is called an A_P^∞-efficient solution of the problem (2.1) if and only if $y = f(x)$ is A_P^∞-nondominated.

By using Definition 8, we can conclude that $Y_{A_P^\infty N} = \bigcap_{r\in\mathbb{N}} Y_{A_P^r N}$.

Corollary 6. *If $x \in X$ is an A_P^∞-efficient solution for problem (2.1) and A_P is a matrix with the condition $a_j \geq 0$ for $j = 1, \ldots, m$. Then, x is an efficient solution for problem (2.1).*

Proof. By applying the definition of A_P^∞-efficiency and Corollary 5, the proof is trivial. □

Corollary 6 indicates that to reduce Pareto optimal solutions, we can use A_P^∞-efficient solutions.

5. The reduction of A_P-efficient solutions

In this section, our focus is on reducing the set of A_P-efficient by creating a new preference matrix. This is done by modifying the decomposition of the MOP. As in the previous sections, assume that the partition P and the preferences matrices $A_1^P, \ldots, A_n^P$ are provided by the DM and $A_P = A_1^P \oplus \ldots \oplus A_n^P$.

Definition 9. Let $P = \{P_1, \ldots, P_n\}$ be a partition of $\{1, \ldots, m\}$ and $F = \{F_1, \ldots, F_t\}$ be a partition of $\{1, \ldots, n\}$. The generated partition $E = \{E_1, \ldots, E_t\}$ by P and F of $\{1, \ldots, m\}$ is defined by $E_k = \bigcup_{j\in F_k} P_j (k = 1, \ldots, t)$, where F_k is index set classes in partition P should be integrated for class k in partition E.

Let $A_P = A_1^P \oplus \ldots \oplus A_n^P$. If $F_k = \{k_1, k_2, \ldots, k_{|F_k|}\}$ in Definition 9, we define

$$A_k^E = [A_{k_1}^{\prime P} \; A_{k_2}^{\prime P} \ldots A_{k_{|F_k|}}^{\prime P}],$$

where

$$A_{k_j}^{\prime P} = \begin{bmatrix} A_{k_j}^P \\ 0 \end{bmatrix}_{|E_k| \times |P_{k_j}|},$$

for $j = 1, \ldots, |F_k|$.

Example 4. Suppose that $P_1 = \{1,2\}$, $P_2 = \{3\}$, $P_3 = \{4\}$, $F_1 = \{1,2\}$ and $F_2 = \{3\}$. The generated partition by P and F is $E_1 = P_1 \cup P_2 = \{1,2,3\}$ and $E_2 = P_3 = \{4\}$. For example, if $A_1^P = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $A_2^P = [1]$ and $A_3^P = [4]$ then $A_1^{\prime P} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 0 & 0 \end{bmatrix}$, $A_2^{\prime P} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $A_1^E = [A_1^{\prime P} \; A_2^{\prime P}] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Also we have $A_2^E = A_3^{\prime P} = [4]$.

In the following, we will investigate the relationship between preference relations $\leq_{A_P}$ and $\leq_{A_E}$.

Theorem 5. *Let P, F, E and A_P, A_E be as in Definition 9. For any two outcome vectors $y', y'' \in Y$, we have*

$$y' <_{A_P} y'' \Rightarrow y' <_{A_E} y''.$$

Hence $Y_{A_E N} \subset Y_{A_P N}$. In particular, if $P = \{P_1, \ldots, P_n\}$ is an arbitrary partition of $\{1, \ldots, m\}$, then $Y_{AN} \subset Y_{A_P N}$, where $A = [A_1^{\prime P} \ldots A_n^{\prime P}]$ and

$$A_j^{\prime P} = \begin{bmatrix} A_j^P \\ 0 \end{bmatrix}_{m \times |P_j|}.$$

Proof. Let $y' <_{A_P} y''$. Using Corollary 1, we deduce that

$$A_k^P(y'_{P_k}) \leqq A_k^P(y''_{P_k}) \qquad (k = 1, \ldots, n),$$

and $A_k^P(y'_{P_k}) \leq A_k^P(y''_{P_k})$ for some $k \in \{1, \ldots, n\}$. Since $E_k = \bigcup_{j \in F_k} P_j$, for all $k = 1, \ldots, t$ and $A_k^E = [A_{k_1}^{\prime P} \ldots A_{k_{|F_k|}}^{\prime P}]$, we obtain $y' <_{A_E} y''$. Hence $Y_{A_E N} \subset Y_{A_P N}$. In particular, let $P = \{P_1, \ldots, P_n\}$ be an arbitrary partition of $\{1, \ldots, m\}$. For $F_1 = \{1, \ldots, n\}$, the generated partition by P and $F = \{F_1\}$ equal to $E = \{E_1\}$, where $E_1 = \bigcup_{j \in F_1} P_j = \{1, \ldots, m\}$. Now by applying the first part of the theorem, we obtain $Y_{AN} \subset Y_{A_P N}$. □

As shown in the example below, the converse of Theorem 5 is not always true.

Example 5. Let $P_1 = \{1\}$ and $P_2 = \{2\}$ and $F_1 = \{1, 2\}$. The generated partition by P and F is $E_1 = P_1 \cup P_2 = \{1, 2\}$. If $A_1^P = A_2^P = \begin{bmatrix}1\end{bmatrix}$, then $A_1^E = [A_1'^P \ A_2'^P] = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$. For $y' = (2, 1)$ and $y'' = (1, 3)$, we have $y' <_{A_E} y''$, while $y' \not<_{A_P} y''$.

In the following, we obtain the relationship between A_E-efficient solutions and A_P-efficient solutions by Theorem 5.

Corollary 7. *Let P, F and E be as in Definition 9 and let $x \in X$ be a feasible solution. If x is an A_E-efficient solution of problem (2.1), then it is an A_P-efficient solution of problem (2.1). Hence $X_{A_E E} \subseteq X_{A_P E}$. Therefore, if $x \in X$ is an efficient solution of the problem*

$$\begin{aligned} &\min A_E(f(x)) \\ &\textit{subject to } x \in X, \end{aligned} \tag{5.1}$$

then it is an efficient solution of the problem (3.4).

According to Theorem 5 and Corollary 7, an algorithm is offered to generate A_E-efficient solutions, whereby is reduced A_E-efficient solutions.

Algorithm 1.

Step 1: Determine a partition $P = \{P_1, \ldots, P_n\}$ of $\{1, \ldots, m\}$ and the matrix A_P, according to the DM.
Step 2: put $t = 1$.
Step 3: Consider the partition F, where $F_1 = \{1, \ldots, t\}$, $F_2 = \{t + 1\}$, ..., $F_{n-t+1} = \{n\}$.
Step 4: Calculate the partition E and the matrix A_E, where $E_k = \bigcup_{j \in F_k} P_j$ for $k = 1$, ..., $n - t + 1$, according to Definition 9.
Step 5: Solve the MOP (5.1).
Step 6: If the DM chooses the desired solution, stop.
Step 7: Otherwise put $t = t + 1$, if $t > n$ stop, the model does not answer.
Step 8: Otherwise, go to Step 3.

In the first iteration of Algorithm 1, we have $t = 1$. Thus $F_i = \{i\}$ for $i = 1, \ldots, n$, and hence $E_k = P_k$ for $k = 1, \ldots, n$. Therefore, the output of the first iteration of the algorithm is the generation of A_P-efficient solutions. In particular, if $P_j = \{j\}$ and $A_j^P = I$ for $j = 1, \ldots, m$, then Pareto optimal solutions are computed in the first iteration of this algorithm. For $t = 2$, we get $F_1 = \{1, 2\}$ and $F_i = \{i + 1\}$ for $i = 2, \ldots, n - 1$, and hence $E_1 = P_1 \cup P_2$ and $E_k = P_{k+1}$ for $k = 2, \ldots, n - 1$. Hence, in the second iteration of Algorithm 1, the A_E-efficient solutions are computed. Applying Corollary 7 for these two iterations, we conclude that $X_{A_E E} \subseteq X_{A_P E}$. By continuing this process, we observe that the set of solutions obtained in each iteration is reduced compared to the previous iteration, when t increases to a maximum of n. In the following example, we obtain the A_P-efficient solutions where the partition P of $\{1, \ldots, m\}$ and the matrix A_P are given by the DM. Then, we gradually reduce

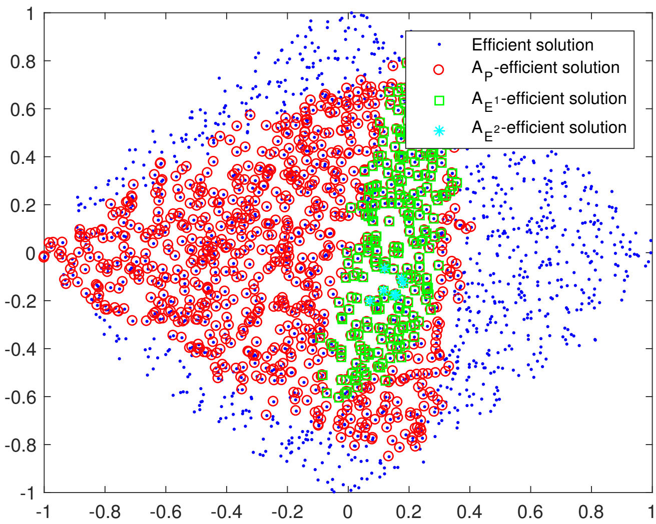

Figure 2. The Pareto optimal and A_P-efficient solutions with $P_1 = \{1, 2\}$, $P_2 = \{3, 4\}$, $P_3 = \{5, 6\}$ and A_E-efficient solutions generated by Algorithm 1.

these solutions by A_E-efficient solutions in the next iterations of the Algorithm 1. For this purpose, a large number of random solutions are generated for scalable test function. From this large set of solutions, efficient solutions, A_P-efficient solutions and A_E-efficient solutions are calculated.

Example 6. Let us consider again the test problem (4.2). In Figure 2 from 3000 random solutions, 1804 solutions (blue points) are efficient. Let $P_1 = \{1, 2\}$, $P_2 = \{3, 4\}$, $P_3 = \{5, 6\}$ and

$$A_1^P = \begin{bmatrix} 1 & 0.5 \\ 0.9 & 0.8 \end{bmatrix}, \qquad A_2^P = \begin{bmatrix} 1 & 0.7 \\ 0 & 0.5 \end{bmatrix}, \qquad A_3^P = \begin{bmatrix} 0.5 & 0.6 \\ 0.7 & 0.4 \end{bmatrix},$$

be given by the DM. In the first iteration of the Algorithm 1, 1042 solutions (red circles) are A_P-efficient. 271 solutions (green square) are A_{E^1}-efficient, which are obtained by assuming $E_1^1 = P_1 \cup P_2 = \{1, 2, 3, 4\}$ and $E_2^1 = P_3 = \{5, 6\}$, in the second iteration of the algorithm. Also in the third iteration of the Algorithm 1, 7 solutions (cyan star) are A_{E^2}-efficient, which are obtained by assuming $E_1^2 = P_1 \cup P_2 \cup P_3 = \{1, 2, 3, 4, 5, 6\}$.

6. Conclusion

The aim of this paper is introducing the new concept of rational A_P-efficiency for solving of the MOPs, where the preference matrix A_P is given by DM. This concept is obtained by rational preference relations on the certain cumulative vector $A_P(y)$ for $y \in Y$. We have investigated both the theoretical and practical aspects of rationally

A_P-efficient solutions. Using the powers of A_P, the concept of A_P^r-efficiency is expressed to generate a subset of Pareto optimal solutions for $r = 1, 2, \ldots$. Also, we proved that the A_P^r-efficient sets are decreasing with respect to r, and the intersection of these sets is the A_P^∞-efficient set. We investigated the relationship between A_P-efficient and A_E-efficient solutions, where A_E is derived from the preference matrix A_P. Moreover, two experiments were carried out on randomly generated solutions to better compare the efficient solutions with the A_P^r-efficient solutions and the A_P-efficient solutions with the A_E-efficient solutions. These experiments show that the A_P^r-efficient set is significantly smaller than the efficient set. Furthermore, the A_E-efficient set is considerably smaller than the A_P-efficient set.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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