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Short Note

A note on graphs with integer Sombor index

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Abstract: For a graph G, the Sombor index of G is defined as $SO(G) = \sum_{uv\in E(G)} \sqrt{\deg(u)^2 + \deg(v)^2}$, where $\deg(u)$ is referring to the degree of vertex u in G. In this paper, we present a construction, namely R_k-construction which produce infinitely many families of graphs whose Sombor indices are integers.

Keywords: topological index, Sombor index, integer.

AMS Subject classification: 05C69, 05C78

1. Introduction

Throughout this paper, $G = (V(G), E(G))$ is a simple graph with $V(G)$ as its vertex set and $E(G)$ as its edge set. The open *neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$ (or just $N(v)$) is the set $\{u : uv \in E(G)\}$. The *degree* of a vertex v, $\deg_G(v)$ (or just $\deg(v)$) is the number of neighbors of v in G, that is, $\deg(v) = |N_G(v)|$.

A graph invariant is a numerical quantity that remains the same under graph isomorphism. In chemical graph theory, graph invariants are usually referred to as the topological indices, which are computed from the molecular graph of a chemical compound. One of the widely studied topological indices recently is the Sombor index which was introduced by Gutman [3]. The Sombor index of a graph G, denoted by $SO(G)$, is defined as

$$SO(G) = \sum_{uv\in E(G)} \sqrt{\deg(v)^2 + \deg(u)^2}.$$

Many papers have been published related to Sombor index, see for instance [1, 2, 4, 5, 9, 10]. One of the main focus related to Sombor index is studying the graphs in which their Sombor indices are integers. The graphs with integer Sombor index have been studied in recent years, see for example, [6–8, 11, 12]. It was claimed in [2] that, the Sombor index of a connected bipartite graph G is an integer if and only if G is bipartite semi-regular and its two degrees δ and Δ appear as non-maximal elements in a Pythagorean triple. It was later shown in [7] that the 'only if' part of the claim is not true. The author constructed infinite number of connected bipartite graphs such that there are three or four distinct numbers in their degree sequences. In this paper, we continue the study of graphs with integer Sombor index. We provide a construction to produce from a given graph G with integer Sombor index infinitely many families of graphs whose Sombor indices are integers.

2. The construction

Given a graph G and an integer $k \geq 2$, the R_k*-construction* produces a graph $R_k(G)$ from G as follows: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. We consider k copies of G as

$$V(G^i) = \{v_1^i, v_2^i, \ldots, v_n^i\}, \quad E\big(G^i\big) = \{v_r^i v_s^i | v_r v_s \in E(G)\}$$

for $i = 1, 2, \ldots, k$. Then

$$V(R_k(G)) = \bigcup_{i=1}^{k} V\big(G^i\big)$$

and

$$E(R_k(G)) = \bigcup_{i=1}^{k} E\big(G^i\big) \bigcup \{v_r^i v_s^j | i \neq j, v_r v_s \in E(G)\}.$$

We illustrate here an example of the R_k-construction.

Example 1. Suppose that G is a graph with $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ as shown below:

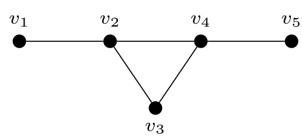

Figure 1. **Graph G.**

If $k = 3$, then $R_3(G)$ is constructed in Figure 2.

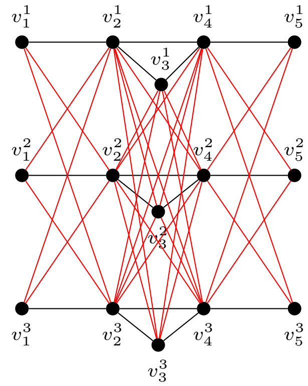

Figure 2. **Graph** $R_3(G)$.

Also, we define $R_k^l(G)$ iteratively as

$$R_k^l(G) = R_k\big(R_k^{l-1}(G)\big) \quad \text{for} \quad l \geq 2.$$

The main result of this section is as follows.

Theorem 1. *For each $l \geq 1$ and $k \geq 2$,*

$$SO\big(R_k^l(G)\big) = \left(\frac{k^3+k^2}{2}\right)^l SO(G).$$

Proof. We only prove for $l = 1$. Then the result follows by an induction on l. So, assume that $l = 1$. We will prove that

$$SO(R_k(G)) = \left(\frac{k^3+k^2}{2}\right) SO(G).$$

From the definition of Sombor index, we know that

$$SO(R_k(G)) = \sum_{uv \in E(R_k(G))} \sqrt{\deg^2_{R_k(G)}(u) + \deg^2_{R_k(G)}(v)}.$$

Notice that, for each vertex v_j^i (for $i = 1, 2, \ldots, k, j = 1, 2, \ldots, n$), we have

$$\deg_{R_k(G)}\big(v_j^i\big) = k \deg_G\big(v_j\big).$$

Now, we have

$$\begin{aligned}
SO(R_k(G)) &= \sum_{uv\in E(R_k(G))} \sqrt{\deg^2_{R_k(G)}(u)+\deg^2_{R_k(G)}(v)} \\
&= \sum_{l=1}^{k} \sum_{v_iv_j\in E(G)} \sqrt{\deg^2_{R_k(G)}(v_i^l)+\deg^2_{R_k(G)}(v_j^l)} \\
&+ \sum_{1\le l,l'\le k, l\neq l'} \sum_{v_iv_j\in E(G)} \sqrt{\deg^2_{R_k(G)}(v_i^l)+\deg^2_{R_k(G)}(v_j^{l'})}. \\
&= k\sum_{l=1}^{k} \sum_{v_iv_j\in E(G)} \sqrt{\deg^2_{(G)}(v_i)+\deg^2_{(G)}(v_j)} \\
&+ k \sum_{1\le l,l'\le k, l\neq l'} \sum_{v_iv_j\in E(G)} \sqrt{\deg^2_{(G)}(v_i)+\deg^2_{(G)}(v_j)}. \\
&= k^2 SO(G) + k\binom{k}{2} SO(G) = \left(\frac{k^3+k^2}{2}\right) SO(G).
\end{aligned}$$

□

Since $k^3 + k^2$ is even, for each integer k, we obtain the following.

Corollary 1. *If G is a graph with integer Sombor index, then for each $l \geq 1$, $k \geq 2$, $R_k^l(G)$ is a graph with integer Sombor index.*

3. Concluding Remarks

The Randić index of a graph G is defined by

$$R(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{\deg(u)\deg(v)}}.$$

The reciprocal Randić index is defined by

$$RR(G) = \sum_{uv\in E(G)} \sqrt{\deg(u)\deg(v)}.$$

Similar to Theorem 1, we can obtain the following:

Theorem 2. *For each $l \geq 1, k \geq 2$,*

$$RR\big(R_k^l(G)\big) = \left(\frac{k^3+k^2}{2}\right)^l RR(G) \quad \text{and} \quad R\big(R_k^l(G)\big) = \left(\frac{k+1}{2}\right)^l R(G).$$

The proofs are similar to that of Theorem 1 and are thus omitted.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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