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Research Article

Remarks on the bounds of graph energy

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Abstract: Let G be a graph of order n with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The energy of G is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$. In the present paper, new bounds on $E(G)$ are provided. In addition, some bounds of $E(G)$ are compared.

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1. Introduction

Let $G = (V, E)$ be a simple graph with n vertices and m edges, where $V = \{v_1, v_2, \ldots, v_n\}$. If v_i and v_j are two adjacent vertices of G, it is denoted by $i \sim j$. Denote by $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$ the vertex degree sequence of G. The Randić index of G is one of the most important graph topological indices defined as $R(G) = \sum_{i \sim j} \frac{1}{\sqrt{d_i d_j}}$ [31] (see also [21]).
Let $A(G)$ be the $(0,1)$ −adjacency matrix of a graph G. Eigenvalues of $A(G)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, are the eigenvalues of G. Denote by $|\lambda_1^*| \geq |\lambda_2^*| \geq \cdots \geq |\lambda_n^*|$ the non-increasing arrangement of the absolute values of eigenvalues of G. For the spectral radius λ_1 of G, it is a well known fact that $\lambda_1 = |\lambda_1^*|$. Evidently,

$$|\lambda_1^*|^2 + |\lambda_2^*|^2 + \cdots + |\lambda_n^*|^2 = 2m$$

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and

$$\prod_{i=1}^{n} |\lambda_i^*| = |\det A| .$$

One of the most studied graph spectrum-based invariants in graph theory is the graph energy defined in [19]. It is calculated as

$$E(G) = \sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{n} |\lambda_i^*| .$$

Details on the theory and applications of $E(G)$ including its basic properties and various bounds can be found in monograph [23] and recent papers [5, 9, 15, 16, 20, 26, 27]. We now list some bounds on $E(G)$, reported earlier in the literature.
Two of the present authors [27] proved that

$$E(G) \geq \frac{2m + n|\lambda_1^*||\lambda_n^*|}{|\lambda_1^*| + |\lambda_n^*|} \tag{1.1}$$

and obtained the following inequality as a corollary of (1.1)

$$E(G) \geq \frac{2\sqrt{2mn|\lambda_1^*||\lambda_n^*|}}{|\lambda_1^*| + |\lambda_n^*|} , \tag{1.2}$$

which was established in [12]. However, the equality case was not given properly in [27]. This was corrected in [9]. Nine years after paper [12] was published, the inequality (1.2) was again proved by Oboudi [30]. More interestingly, the author [30] proved (1.1) as an intermediate result, while proving (1.2). In [20], the inequality (1.2) was named as Oboudi–type inequality. It is worth mentioning here that the inequalities (1.1) and (1.2) were obtained as special case of one more general result reported in [25].
Very recently, Filipovski [15] obtained that

$$E(G) \geq \frac{2m}{\Delta} \tag{1.3}$$

and for triangle-free graphs

$$E(G) \leq \sqrt{2n}R(G) , \tag{1.4}$$

where $R(G)$ is Randić index of G.
In this paper, we obtain new bounds for $E(G)$. In addition, we compare some bounds of $E(G)$.

2. Lemmas

In this section, we list some preliminary lemmas that will be used in the subsequent section.

Lemma 1. [7] *Let $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ be a sequence of positive real numbers. Then*

$$a_1 + \cdots + a_n \geq n\,(a_1 a_2 \cdots a_n)^{1/n} \left(\frac{(a_1 + a_n)^2}{4a_1 a_n} \right)^{1/n} . \tag{2.1}$$

Equality holds if $a_2 = a_3 = \cdots = a_{n-1} = \frac{a_1+a_n}{2}$.

Lemma 2. [18] *For $a_1, a_2, \ldots, a_n \geq 0$ and $p_1, p_2, \ldots, p_n \geq 0$ such that $\sum_{i=1}^{n} p_i = 1$,*

$$\sum_{i=1}^{n} p_i a_i - \prod_{i=1}^{n} a_i^{p_i} \geq n\lambda \left(\frac{1}{n} \sum_{i=1}^{n} a_i - \prod_{i=1}^{n} a_i^{1/n} \right), \tag{2.2}$$

where $\lambda = \min\{p_1, p_2, \ldots, p_n\}$. Moreover, the equality in (2.2) holds if and only if $a_1 = a_2 = \cdots = a_n$.

Lemma 3. [22] *Let $p = (p_i)$ and $a = (a_i)$, $i = 1, 2, \ldots, n$, be sequences of positive real numbers such that*

$$\sum_{i=1}^{n} p_i = 1 \qquad \text{and} \qquad 0 < r \leq a_i \leq R\,.$$

Then

$$\sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} \frac{p_i}{a_i} \leq \left(\sqrt{\frac{R}{r}} + \sqrt{\frac{r}{R}} \right)^2 . \tag{2.3}$$

Lemma 4. [29, 32] *Let $p = (p_i)$ and $a = (a_i)$, $i = 1, 2, \ldots, n$, be real number sequences such that*

$$\sum_{i=1}^{n} p_i = 1 \qquad \text{and} \qquad 0 < r \leq a_i \leq R\,.$$

Then

$$\sum_{i=1}^{n} p_i a_i + rR \sum_{i=1}^{n} \frac{p_i}{a_i} \leq r + R\,. \tag{2.4}$$

Remark 1. From the inequality between arithmetic and geometric means (AM–GM), we obtain

$$2\sqrt{rR \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} \frac{p_i}{a_i}} \leq \sum_{i=1}^{n} p_i a_i + rR \sum_{i=1}^{n} \frac{p_i}{a_i} \leq r + R\,. \tag{2.5}$$

Having this in mind, the inequality (2.3) can be obtained from (2.4), that is (2.3) is a corollary of (2.4).

Lemma 5. [28] *Let $x = (x_i)$, $i = 1, 2, \dots, n$, be a real number sequence with the properties*

$$\sum_{i=1}^{n} x_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{n} |x_i| = 1\,.$$

Then for any real number sequence $a = (a_i)$, $i = 1, 2, \dots, n$, holds

$$\left|\sum_{i=1}^{n} a_i x_i\right| \le \frac{1}{2}\left(\max_{1\le i\le n} a_i - \min_{1\le i\le n} a_i\right)\,. \tag{2.6}$$

Lemma 6. [29] *Let $p = (p_i)$, $i = 1, 2, \dots, n$, be a sequence of non–negative real numbers and $a = (a_i)$, $i = 1, \dots, n$, a sequence of positive real numbers. Then for any real r, $r \le 0$ or $r \ge 1$, holds*

$$\left(\sum_{i=1}^{n} p_i\right)^{r-1} \sum_{i=1}^{n} p_i a_i^r \ge \left(\sum_{i=1}^{n} p_i a_i\right)^{r}\,. \tag{2.7}$$

When $0 \le r \le 1$, the opposite inequality is valid. Equality holds if and only if either $r = 0$, or $r = 1$, or $a_1 = \cdots = a_n$, or $p_1 = \cdots = p_t = 0$ and $a_{t+1} = \cdots = a_n$, or $a_1 = \cdots = a_t$ and $p_{t+1} = \cdots = p_n$, for some t, $1 \le t \le n-1$.

Lemma 7. [28] *Let $p = (p_i)$, $a = (a_i)$ and $b = (b_i)$, $i = 1, 2, \dots, n$, be positive real number sequences such that $a = (a_i)$ and $b = (b_i)$ are of similar monotonicity. Then*

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i \ge \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i\,. \tag{2.8}$$

Equality holds if and only if $a_1 = \cdots = a_n$ or $b_1 = \cdots = b_n$.

Lemma 8. [11] *Let G be a graph with n vertices, m edges and vertex degree sequence $d_1 \ge d_2 \ge \cdots \ge d_n$. Then*

$$E(G) \le \sum_{i=1}^{n} \sqrt{d_i}\,. \tag{2.9}$$

Lemma 9. [13] *Let G be a triangle-free graph with n vertices and m edges. Then,*

$$\lambda_1 \le \sqrt{m} \le R(G)\,,$$

where $R(G)$ is Randić index of G.

3. Main Results

Theorem 1. *Let G be a non-singular graph with n vertices and m edges and let $|\lambda_1^*| \geq |\lambda_2^*| \geq \cdots \geq |\lambda_n^*| > 0$ be a non-increasing arrangement of the absolute values of eigenvalues of G. Then*

$$E(G) \geq \lambda_1 + \frac{2m - \lambda_1^2}{|\lambda_2^*|} \tag{3.1}$$

Equality in (3.1) holds if and only if $|\lambda_2^*| = \cdots = |\lambda_n^*|$.

Proof. Observe that

$$|\lambda_2^*| \sum_{i=2}^{n} |\lambda_i^*| \geq \sum_{i=2}^{n} |\lambda_i^*|^2 = 2m - \lambda_1^{*2}$$

that is,

$$E(G) - |\lambda_1^*| \geq \frac{2m - \lambda_1^{*2}}{|\lambda_2^*|},$$

wherefrom the inequality (3.1) is obtained. Moreover, the equality in (3.1) holds if and only if $|\lambda_2^*| = \cdots = |\lambda_n^*|$. □

Remark 2. We should note that

$$E(G) \geq \lambda_1 + \frac{2m - \lambda_1^2}{|\lambda_2^*|} \geq \frac{2m}{\lambda_1}$$

when $\lambda_1 = |\lambda_1^*| \neq |\lambda_2^*|$. By the above result and the fact that $\lambda_1 \leq \Delta$ [8],

$$E(G) \geq \lambda_1 + \frac{2m - \lambda_1^2}{|\lambda_2^*|} \geq \frac{2m}{\lambda_1} \geq \frac{2m}{\Delta}. \tag{3.2}$$

This implies that the lower bound (3.1) is stronger than the lower bound (1.3).

Remark 3. Notice that the following inequality is valid

$$\frac{2m + n\,|\lambda_1^*|\,|\lambda_n^*|}{|\lambda_1^*| + |\lambda_n^*|} \geq \frac{2m}{\lambda_1}, \tag{3.3}$$

since $\lambda_1 \geq \frac{2m}{n} \geq \sqrt{\frac{2m}{n}}$ [8] for all connected non-singular graphs. Considering (1.1), (3.2) and (3.3), we deduce that the lower bound (1.1) is stronger than the lower bound (1.3) for connected non-singular graphs.

Corollary 1. *Let G be a graph with n vertices and m edges. Then*

$$E(G) \geq \frac{4m}{\lambda_1 - \lambda_n}. \tag{3.4}$$

Equality holds if and only if $\lambda_1 = \cdots = \lambda_p = -\lambda_{p+1} = \cdots = -\lambda_n$, $n = 2p$.

The inequality (3.4) is a special case of one inequality proved in [10].

Remark 4. From (3.2) and (3.4), the following is valid

$$E(G) \geq \frac{4m}{\lambda_1 - \lambda_n} \geq \frac{2m}{\lambda_1} \geq \frac{2m}{\Delta},$$

which implies that the lower bound (3.4) is stronger than the lower bound (1.3).

Remark 5. Caporossi et al. [6] presented the following lower bound based on the number of edges as:

$$E(G) \geq 2\sqrt{m}. \tag{3.5}$$

Considering (1.1) and (3.3) with Lemma 9, we have that

$$E(G) \geq \frac{2m + n\,|\lambda_1^*|\,|\lambda_n^*|}{|\lambda_1^*| + |\lambda_n^*|} \geq \frac{2m}{\lambda_1} \geq 2\sqrt{m}.$$

This implies that the lower bound (1.1) is stronger than the lower bound (3.5) for connected non-singular triangle-free graphs.

Remark 6. McClelland [24] obtained the following upper bound for graph energy involving the number of vertices and the number of edges:

$$E(G) \leq \sqrt{2mn}. \tag{3.6}$$

From (3.6) and Lemma 9, one can easily arrive at the upper bound (1.4) obtained in [15]. Moreover, it can be concluded that (3.6) is stronger than (1.4) for triangle-free graphs.

Theorem 2. *Let G be a non-singular graph with n vertices, m edges and maximum degree Δ. Let $|\lambda_1^*| \geq |\lambda_2^*| \geq \cdots \geq |\lambda_n^*| > 0$ be a non-increasing arrangement of the absolute values of eigenvalues of G. Then*

$$E(G) \geq \Delta + \frac{2m - \Delta^2 + (n-1)\,|\lambda_2^*|\,|\lambda_n^*|}{|\lambda_2^*| + |\lambda_n^*|}. \tag{3.7}$$

Equality in (3.7) holds if and only if G is regular graph with the property $|\lambda_i^*| = |\lambda_n^*|$ or $|\lambda_i^*| = |\lambda_2^*|$ for any $i = 2, \ldots, n$.

Proof. Since $|\lambda_n^*| \leq |\lambda_i^*| \leq |\lambda_2^*|$ for any $i = 2, \ldots, n$, we have that

$$(|\lambda_i^*| - |\lambda_n^*|)\,(|\lambda_i^*| - |\lambda_2^*|) \leq 0.$$

From the above, we arrive at

$$\sum_{i=2}^{n} \left(|\lambda_i^*|^2 - |\lambda_i^*|\,(|\lambda_2^*| + |\lambda_n^*|) + |\lambda_2^*|\,|\lambda_n^*| \right) \leq 0,$$

that is

$$2m - \lambda_1^2 - (|\lambda_2^*| + |\lambda_n^*|)(E(G) - \lambda_1) + (n-1)|\lambda_2^*||\lambda_n^*| \leq 0,$$

i.e.

$$E(G) \geq \lambda_1 + \frac{2m - \lambda_1^2 + (n-1)|\lambda_2^*||\lambda_n^*|}{|\lambda_2^*| + |\lambda_n^*|}. \tag{3.8}$$

Now consider the function $f(x)$ defined by

$$f(x) = x + \frac{2m - x^2}{|\lambda_2^*| + |\lambda_n^*|}.$$

It can be easily shown that f is decreasing with respect to the x. Since $\lambda_1 \leq \Delta$ [8], we get that

$$f(\lambda_1) \geq f(\Delta) = \Delta + \frac{2m - \Delta^2}{|\lambda_2^*| + |\lambda_n^*|}. \tag{3.9}$$

Thus, by (3.8) and (3.9), we obtain (3.7). The equality in (3.7) holds if and only if all inequalities used in the derivation of (3.7) must be equalities. This implies that G is regular graph with the property $|\lambda_i^*| = |\lambda_n^*|$ or $|\lambda_i^*| = |\lambda_2^*|$ for any $i = 2, \ldots, n$. □

Corollary 2. *Let G be a non-singular graph with n vertices, m edges and maximum degree Δ. Let $|\lambda_1^*| \geq |\lambda_2^*| \geq \cdots \geq |\lambda_n^*| > 0$ be a non-increasing arrangement of the absolute values of eigenvalues of G. Then*

$$E(G) \geq \Delta + \frac{2\sqrt{2m(n-1)|\lambda_2^*||\lambda_n^*|} - \Delta^2}{|\lambda_2^*| + |\lambda_n^*|}. \tag{3.10}$$

Remark 7. Recall that the equality in (3.7) holds if and only if G is regular graph with the property $|\lambda_i^*| = |\lambda_n^*|$ or $|\lambda_i^*| = |\lambda_2^*|$ for any $i = 2, \ldots, n$. For instance, line graph of Petersen graph G_1 is a 4-regular graph with 15 vertices, 30 edges and spectrum

$$\left\{4,\ [\pm 2]^5,\ [-1]^4\right\}.$$

For this graph, $E(G_1) = 28$. On the other hand, the lower bounds (3.7) and (1.1) give the values 28 and 24, respectively.

Akbari and Hosseinzadeh [3] propose the following conjecture.

Conjecture 3.1. [3] For every non-singular graph G, $E(G) \geq \Delta + \delta$ and the equality holds if and only if G is a complete graph.

The proofs of special cases of this conjecture were given in recent papers [1, 2, 4, 17]. The lower bound (3.7) yields a new case when Conjecture 3.1 holds.

Corollary 3. *Let G be a non-singular graph with n vertices, m edges and maximum degree Δ. Let $|\lambda_1^*| \geq |\lambda_2^*| \geq \cdots \geq |\lambda_n^*| > 0$ be a non-increasing arrangement of the absolute values of eigenvalues of G. If G has the following property*

$$2m - \Delta^2 + (n-1)\,|\lambda_2^*|\,|\lambda_n^*| \geq \delta\left(|\lambda_2^*| + |\lambda_n^*|\right),$$

then

$$E\,(G) \geq \Delta + \delta.$$

The proof of the next theorem is analogous to that of Theorem 2, thus omitted.

Theorem 3. *Let G be a non-singular bipartite graph with n vertices, m edges and maximum degree Δ. Let $|\lambda_1^*| \geq |\lambda_2^*| \geq \cdots \geq |\lambda_n^*| > 0$ be a non-increasing arrangement of the absolute values of eigenvalues of G. Then*

$$E\,(G) \geq 2\Delta + \frac{2m - 2\Delta^2 + (n-2)\,|\lambda_3^*|\,|\lambda_n^*|}{|\lambda_3^*| + |\lambda_n^*|}\,. \tag{3.11}$$

Equality in (3.11) holds if and only if G is a bipartite regular graph with the property $|\lambda_i^*| = |\lambda_n^*|$ or $|\lambda_i^*| = |\lambda_3^*|$ for any $i = 3, \ldots, n$.

Corollary 4. *Let G be a non-singular bipartite graph with n vertices, m edges and maximum degree Δ. Let $|\lambda_1^*| \geq |\lambda_2^*| \geq \cdots \geq |\lambda_n^*| > 0$ be a non-increasing arrangement of the absolute values of eigenvalues of G. Then*

$$E\,(G) \geq 2\Delta + \frac{2\sqrt{2m\,(n-2)\,|\lambda_3^*|\,|\lambda_n^*|} - 2\Delta^2}{|\lambda_3^*| + |\lambda_n^*|}. \tag{3.12}$$

Remark 8. The equality in (3.11) holds if and only if G is a bipartite regular graph with the property $|\lambda_i^*| = |\lambda_n^*|$ or $|\lambda_i^*| = |\lambda_3^*|$ for any $i = 3, \ldots, n$. Recall that Franklin graph G_2 is a 3-regular bipartite graph with 12 vertices, 18 edges and spectrum

$$\left\{\pm 3, \left[\pm\sqrt{3}\right]^2, \ \left[\pm 1\right]^3\right\}.$$

For graph G_2, $E\,(G_2) = 12 + 4\sqrt{3}$. Moreover, the lower bound (3.11) gives $12 + 4\sqrt{3}$ whereas the lower bound (1.1) gives 18.

For $a_i = |\lambda_i^*|$, $i = 2, 3, \ldots, n$, from (2.1) we obtain the following result.

Proposition 1. *Let G be a graph with n vertices. Let $|\lambda_1^*| \geq \cdots \geq |\lambda_n^*| > 0$ be a non–increasing arrangement of the absolute values of eigenvalues of G. Then*

$$E(G) \geq \lambda_1 + (n-1)\left(\frac{|\det A|}{\lambda_1}\right)^{1/(n-1)}\left(\frac{(|\lambda_2^*| + |\lambda_n^*|)^2}{4\,|\lambda_2^*|\,|\lambda_n^*|}\right)^{1/(n-1)}.$$

Equality holds when $|\lambda_3^*| = \cdots = |\lambda_{n-1}^*| = \frac{|\lambda_2^*| + |\lambda_n^*|}{2}$.

Theorem 4. *Let G be a graph with n vertices and m edges, where $2m \geq n$. Then for any real ξ, $\lambda_1 \geq \xi \geq \frac{2m}{n}$*

$$E(G) \geq \xi + (n-1)\left((k+1)\frac{|\det A|^{\frac{(k+1)n-k}{(k+1)n(n-1)}}}{\xi^{\frac{1}{(k+1)(n-1)}}} - k|\det A|^{1/n}\right). \tag{3.13}$$

Equality in (3.13) holds if and only if $G \cong \frac{n}{2}K_2$ (n is even).

Proof. Let us take, $a_i = |\lambda_i^*|$ for $i = 1, 2, \ldots, n$, $p_1 = \frac{k}{(k+1)n}$ and $p_i = \frac{(k+1)n-k}{(k+1)n(n-1)}$ for $i = 2, \ldots, n$, in (2.2), where $k \geq 0$ is a real number. Then, we get the following inequality

$$\begin{aligned}&\frac{k}{(k+1)n}\lambda_1 + \frac{(k+1)n-k}{(k+1)n(n-1)}\sum_{i=2}^{n}|\lambda_i^*| - \lambda_1^{\frac{k}{(k+1)n}}\prod_{i=2}^{n}|\lambda_i^*|^{\frac{(k+1)n-k}{(k+1)n(n-1)}}\\ \geq\ &\frac{k}{(k+1)n}\sum_{i=1}^{n}|\lambda_i^*| - \frac{k}{k+1}\prod_{i=1}^{n}|\lambda_i^*|^{1/n},\end{aligned}$$

that is,

$$E(G) \geq \lambda_1 + (k+1)(n-1)\frac{|\det A|^{\frac{(k+1)n-k}{(k+1)n(n-1)}}}{\lambda_1^{\frac{1}{(k+1)(n-1)}}} - k(n-1)|\det A|^{1/n}. \tag{3.14}$$

Consider the function $f(x)$ defined as

$$f(x) = x + \frac{(k+1)(n-1)}{x^{\frac{1}{(k+1)(n-1)}}}|\det A|^{\frac{(k+1)n-k}{(k+1)n(n-1)}}.$$

It can be easily seen that

$$f'(x) = 1 - |\det A|^{\frac{(k+1)n-k}{(k+1)n(n-1)}}x^{-\frac{(k+1)n-k}{(k+1)(n-1)}},$$

and f is increasing for $x \geq |\det A|^{1/n}$. Then, for any real ξ, $\lambda_1 \geq \xi \geq \frac{2m}{n}$

$$\lambda_1 \geq \xi \geq \frac{2m}{n} \geq \sqrt{\frac{2m}{n}} \geq \frac{E(G)}{n} \geq |\det A|^{1/n}$$

(see, Theorem 2.2 in [5]). Thus

$$f(\lambda_1) \geq f(\xi) = \xi + (k+1)(n-1)\frac{|\det A|^{\frac{(k+1)n-k}{(k+1)n(n-1)}}}{\xi^{\frac{1}{(k+1)(n-1)}}}.$$

Combining this with (3.14), we get the desired lower bound (3.13). Assume that the equality in (3.13) holds. Then,

$$\lambda_1 = |\lambda_1^*| = \xi \text{ and } |\lambda_1^*| = |\lambda_2^*| = \cdots = |\lambda_n^*| .$$

The above conditions imply that the equality in (3.13) holds if and only if $G \cong \frac{n}{2} K_2$ (n is even) . □

Corollary 5. *Let G be a graph with n vertices and m edges, where $2m \geq n$. Then*

$$E(G) \geq \frac{2m}{n} + (n-1)\left((k+1)\, \frac{|\det A|^{\frac{(k+1)n-k}{(k+1)n(n-1)}}}{\left(\frac{2m}{n}\right)^{\frac{1}{(k+1)(n-1)}}} - k\,|\det A|^{1/n} \right) . \tag{3.15}$$

Equality in (3.15) holds if and only if $G \cong \frac{n}{2} K_2$ (n is even) .

Remark 9. The following inequalities were obtained in [5]

$$E(G) \geq \frac{2m}{n} + (n-1) \left(\frac{n\,|\det A|}{2m} \right)^{1/(n-1)} \tag{3.16}$$

and

$$E(G) \geq \xi + (n-1) \left(\frac{|\det A|}{\xi} \right)^{1/(n-1)} , \tag{3.17}$$

where ξ is a real number such that $\lambda_1 \geq \xi \geq \frac{2m}{n}$. Note that (3.16) and (3.17) are, respectively, obtained from (3.15) and (3.13) for $k = 0$.

Theorem 5. *Let G be a graph with n vertices. Then*

$$E(G) \leq n \left(|\lambda_1^*| + |\lambda_n^*| - |\lambda_1^*|\,|\lambda_n^*|\,|\det A|^{-1/n} \right) . \tag{3.18}$$

Equality holds if and only if $G \cong \frac{n}{2} K_2$, where n is even.

Proof. For $p_i = \frac{1}{n}$, $a_i = |\lambda_i^*|$, $R = |\lambda_1^*|$, $r = |\lambda_n^*|$, $i = 1, \ldots, n$, the inequality (2.4) becomes

$$\frac{1}{n} \sum_{i=1}^{n} |\lambda_i^*| + \frac{|\lambda_1^*|\,|\lambda_n^*|}{n} \sum_{i=1}^{n} \frac{1}{|\lambda_i^*|} \leq |\lambda_1^*| + |\lambda_n^*| ,$$

that is

$$E(G) + |\lambda_1^*|\,|\lambda_n^*| \sum_{i=1}^{n} \frac{1}{|\lambda_i^*|} \leq n\big(|\lambda_1^*| + |\lambda_n^*|\big) . \tag{3.19}$$

On the other hand, from the AM–GM inequality, we have that

$$\sum_{i=1}^{n} \frac{1}{|\lambda_i^*|} \geq \frac{n}{|\det A|^{1/n}} \,. \tag{3.20}$$

Now from (3.19) and (3.20) we arrive at (3.18).
Equality in (3.20) holds if and only if $|\lambda_1^*| = \cdots = |\lambda_n^*|$, which implies that equality in (3.18) holds if and only if $G \cong \frac{n}{2}K_2$, where n is even. □

Having in mind (2.5) we have the following corollary of Theorem 5.

Corollary 6. *Let G be a graph with n vertices. Then*

$$E(G) \leq \frac{n\left(|\lambda_1^*| + |\lambda_n^*|\right)^2 \left(|\det A|\right)^{1/n}}{4|\lambda_1^*|\,|\lambda_n^*|} \,. \tag{3.21}$$

The inequality (3.21) was proven in [16].
The proof of the next theorem in analogous to that of Theorem 5, hence omitted.

Theorem 6. *Let G be a graph with $n \geq 3$ vertices. Then*

$$E(G) \leq |\lambda_1^*| + (n-1)\left(|\lambda_2^*| + |\lambda_n^*| - |\lambda_2^*|\,|\lambda_n^*| \left(\frac{|\lambda_1^*|}{|\det A|}\right)^{1/(n-1)}\right) .$$

Equality holds when $|\lambda_2^*| = \cdots = |\lambda_n^*|$.

Corollary 7. *Let G be a graph with $n \geq 3$ vertices. Then*

$$E(G) \leq |\lambda_1^*| + \frac{n-1}{4}\left(\sqrt{\frac{|\lambda_2^*|}{|\lambda_n^*|}} + \sqrt{\frac{|\lambda_n^*|}{|\lambda_2^*|}}\right)^2 \left(\frac{|\det A|}{|\lambda_1^*|}\right)^{1/(n-1)} .$$

Equality holds when $|\lambda_2^*| = \cdots = |\lambda_n^*|$.

For $x_i = \frac{\lambda_i}{E(G)}$, $i = 1, 2, \ldots, n$, from (2.6) the following result is obtained.

Proposition 2. *Let G be a graph with n vertices and m edges. Then for any real number sequence $a = (a_i)$, $i = 1, 2, \ldots, n$, holds*

$$\left|\sum_{i=1}^{n} a_i \lambda_i\right| \leq \frac{\left(\max\limits_{1 \leq i \leq n} a_i - \min\limits_{1 \leq i \leq n} a_i\right) E(G)}{2} \,. \tag{3.22}$$

Corollary 8. *Let G be a graph with n vertices and vertex degree sequence $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$. Then*

$$\sum_{i=1}^{n} d_i\lambda_i \leq \frac{E(G)(\Delta-\delta)}{2}\,. \tag{3.23}$$

Equality holds if G is a regular graph.

The inequality (3.23) was proven in [14].

Theorem 7. *Let G be a graph with $n \geq 2$ vertices, m edges and without isolated vertices. Then*

$$E(G) \leq \frac{2m\left(\sqrt{\Delta}+\sqrt{\delta}-\sqrt{\frac{2m}{n}}\right)}{\sqrt{\Delta\delta}}\,. \tag{3.24}$$

Equality holds if and only if $\cong \frac{n}{2}K_2$, for even n.

Proof. For $p_i = \frac{d_i}{2m}$, $a_i = \sqrt{d_i}$, $i = 1, 2, \ldots, n$, $r = \sqrt{\delta}$, $R = \sqrt{\Delta}$, the inequality (2.4) transforms into

$$\sum_{i=1}^{n} d_i^{3/2} + \sqrt{\Delta\delta}\sum_{i=1}^{n}\sqrt{d_i} \leq 2m(\sqrt{\Delta}+\sqrt{\delta})\,. \tag{3.25}$$

On the other hand, for $r = \frac{3}{2}$, $p_i = 1$, $a_i = d_i$, $i = 1, 2, \ldots, n$, the inequality (2.7) becomes

$$\left(\sum_{i=1}^{n} 1\right)^{1/2}\sum_{i=1}^{n} d_i^{3/2} \geq \left(\sum_{i=1}^{n} d_i\right)^{3/2},$$

that is

$$\sum_{i=1}^{n} d_i^{3/2} \geq 2m\sqrt{\frac{2m}{n}}\,. \tag{3.26}$$

From (3.25) and (3.26) we obtain that

$$2m\sqrt{\frac{2m}{n}} + \sqrt{\Delta\delta}\sum_{i=1}^{n}\sqrt{d_i} \leq 2m(\sqrt{\Delta}+\sqrt{\delta})\,,$$

that is

$$\sum_{i=1}^{n}\sqrt{d_i} \leq \frac{2m\left(\sqrt{\Delta}+\sqrt{\delta}-\sqrt{\frac{2m}{n}}\right)}{\sqrt{\Delta\delta}}\,.$$

Now from the above and (2.9) we arrive at (3.24).
Equality in (3.26) holds if and only if $d_1 = d_2 = \cdots = d_n$, which implies that equality in (3.24) holds if and only if $G \cong \frac{n}{2}K_2$, for even n. □

Denote by $D = diag(d_1, d_2, \ldots, d_n)$ the diagonal degree matrix of graph G. In the next corollary, we give an upper bound for $E(G)$ in terms of m, Δ, δ and the determinant of the matrix D, $(det D)$.

Corollary 9. *Let G be a graph with $n \geq 2$ vertices, m edges and without isolated vertices. Then*

$$E(G) \leq \frac{1}{\sqrt{\Delta\delta}} \left(2m(\sqrt{\Delta} + \sqrt{\delta}) - n(\det D)^{3/(2n)} \right) . \tag{3.27}$$

Equality holds if and only if $G \cong \frac{n}{2} K_2$, for even n.

Proof. Since

$$\sum_{i=1}^{n} d_i^{3/2} \geq n \left(\prod_{i=1}^{n} d_i^{3/2} \right)^{1/n} = n(\det D)^{3/(2n)} .$$

From the above and inequality (3.25) we obtain

$$\sum_{i=1}^{n} \sqrt{d_i} \leq \frac{1}{\sqrt{\Delta\delta}} \left(2m(\sqrt{\Delta} + \sqrt{\delta}) - n(\det D)^{3/(2n)} \right) .$$

From the above and inequality (2.9) we obtain (3.27). □

Theorem 8. *Let G be a graph with $n \geq 2$ vertices and m edges. Then*

$$E(G) \leq \frac{n}{4m}(2m + M_1(G)) . \tag{3.28}$$

Equality holds if and only if $G \cong \frac{n}{2} K_2$, for even n, or $G \cong \overline{K_n}$.

Proof. For $p_i = 1$, $a_i = |\lambda_i^*|$, $b_i = d_i$, $i = 1, 2, \ldots, n$, the inequality (2.8) becomes

$$n \sum_{i=1}^{n} |\lambda_i^*| d_i \geq \sum_{i=1}^{n} |\lambda_i^*| \sum_{i=1}^{n} d_i = 2mE(G) . \tag{3.29}$$

Bearing in mind the AM–GM inequality, we have that

$$n \sum_{i=1}^{n} |\lambda_i^*| d_i \leq \frac{n}{2} \sum_{i=1}^{n} \left(|\lambda_i^*|^2 + d_i^2 \right) = \frac{n}{2}(2m + M_1(G)) . \tag{3.30}$$

From (3.29) and (3.30) we obtain

$$2mE(G) \leq \frac{n}{2}(2m + M_1(G)) ,$$

from which (3.28) is obtained.
Equality in (3.29) holds if and only if $d_1 = \cdots = d_n$, or $|\lambda_1^*| = \cdots = |\lambda_n^*|$. Equality in (3.30) holds if and only if $|\lambda_i^*| = d_i$, for every $i = 1, 2, \ldots, n$. This implies that equality (3.28) holds if and only if $|\lambda_1^*| = \cdots = |\lambda_n^*|$, that is if and only if $G \cong \frac{n}{2}K_2$, for even n, or $G \cong \overline{K_n}$. □

Since $M_1(G) \leq 2m\Delta$ we have the next corollary of Theorem 8.

Corollary 10. *Let G be a graph with $n \geq 2$ vertices. Then*

$$E(G) \leq \frac{n}{2}(1+\Delta)\,. \tag{3.31}$$

Equality holds if and only if $G \cong \frac{n}{2}K_2$, for even n.

Remark 10. In [33, Theorem 2.1] the following upper bound on $E(G)$ was proven

$$E(G) \leq \frac{\sqrt{\Delta}}{\delta^2} M_1(G)\,. \tag{3.32}$$

The upper bounds (3.28) and (3.31) are incomparable with (3.32). Thus, for example, when $G \cong K_5$, the exact value is $E(G) = 8$, while the bound (3.32) is equal to 10, and both bounds (3.28) and (3.31) are equal to 12.5. However, when $G \cong P_5$, the exact value is $E(G) = 5.4641$, while the bound (3.32) is equal to 19.799, and bounds given by (3.28) and (3.31) are equal to 6.875 and 7.5, respectively.

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Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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