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Research Article

On the A_α-spectrum of the k-splitting signed graph and neighbourhood coronas

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Abstract: Let $\Sigma = (G, \sigma)$ be a signed graph with adjacency matrix $A(\Sigma)$ and $D(G)$ be the diagonal matrix of its vertex degrees. For any real $\alpha \in [0, 1]$, the A_α-matrix of a signed graph Σ is defined as $A_\alpha(\Sigma) = \alpha D(G) + (1 - \alpha)A(\Sigma)$. Given a signed graph Σ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, the k-splitting signed graph $S_k(\Sigma)$ of Σ is obtained by adding to each vertex $v \in V(\Sigma)$ new k vertices say $u^1, u^2, \ldots, u^k$ and joining every neighbour say u of the vertex v to u^i, $1 \leq i \leq k$ by an edge which inherits the sign from uv. In this paper, we determine the A_α-spectrum of $S_k(\Sigma)$ in case of Σ being a regular signed graph. For $k = 1$, we introduce two distinct coronas of signed graphs Σ_1 and Σ_2 based on $S_1(\Sigma_1)$, namely the splitting V-vertex neighbourhood corona and the splitting S-vertex neighbourhood corona. By examining the A_α-characteristic polynomial of the resulting signed graphs, we derive their A_α-spectra under certain regularity conditions on the constituent signed graphs. As applications, we use these results to construct infinite pairs of nonregular A_α-cospectral signed graphs.

Keywords: signed graph; k-splitting signed graph, regular signed graph, net-regular signed graph, A_α-matrix, cospectrality.

AMS Subject classification: 05C22, 05C50

1. Introduction

Let G be a simple graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E(G)$. The signed graph $\Sigma = (G, \sigma)$ is a graph G together with a function $\sigma \colon E(G) \longrightarrow \{+1, -1\}$ called the signature of G. If $\sigma(e) = 1$ (respectively, $\sigma(e) = -1$) for every edge e, then σ is called the all-positive (respectively, all-negative) signature and $\Sigma = (G, \sigma)$ is called an all-positive (respectively, all-negative) signed graph. The underlying graph G is interpreted as a signed graph where all its edges are positive. The degree of a vertex v in Σ is its degree in G. The number of positive edges incident

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with a vertex v is the positive degree of v, denoted by d_v^+ and the number of negative edges incident with v is the negative degree of v, denoted by d_v^-. The net degree d_v^{net} is the difference between positive and the negative edges incident with v. Accordingly, Σ is s-net-regular if d_v^{net}=s, for every vertex v of Σ. Finally, Σ is co-regular (or (r, s)-co-regular) if the underlying graph G is r-regular and Σ is s-net-regular [4]. For other basic notions and concepts, see [15].

For a signed graph Σ with vertex set V and $U \subset V$, Σ^U denotes the signed graph obtained from Σ by reversing the sign of every edge between U and $V(G) \setminus U$. We say that Σ and Σ^U are switching equivalent. In matrix terminology, the signed graphs Σ and Σ' are switching equivalent if there exists a diagonal matrix X with ± 1 on the main diagonal such that $A(\Sigma') = X^{-1}A(\Sigma)X$. Two signed graphs are switching isomorphic if one of them switches to a signed graph that is isomorphic to the other one.

Let $A(G)$ be the adjacency matrix of G and $D(G)$ the diagonal matrix of vertex degrees of G. In [11], Nikiforov introduced the A_α-matrix as the convex linear combination of $D(G)$ and $A(G)$, that is $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$ where $\alpha \in [0, 1]$. Various results on A_α-matrix can be seen in [7, 8, 12, 13]. The adjacency matrix $A(\Sigma) = (a_{ij})$ of a signed graph Σ is an $n \times n$ matrix in which $a_{ij} = \sigma(v_i v_j)$ if v_i and v_j are adjacent and 0 otherwise. The eigenvalues of Σ are identified to be the eigenvalues of $A(\Sigma)$ and they form the spectrum of Σ. The eigenvalues of the adjacency matrix $A(\Sigma)$ of a signed graph Σ are denoted by $\lambda_1(\Sigma), \lambda_2(\Sigma), \ldots, \lambda_n(\Sigma)$. In [2], Belardo et al. introduced the notion of A_α-matrix in signed graphs and defined it as $A_\alpha(\Sigma) = \alpha D(G) + (1-\alpha)A(\Sigma)$ where $\alpha \in [0, 1]$. Pasten *et al.* [14], studied some basic properties of $A_\alpha(\Sigma)$ and obtained some bounds for its eigenvalues. The A_α-characteristic polynomial $|xI - A_\alpha(\Sigma)|$ and the eigenvalues of the A_α-matrix of a signed graph Σ are denoted by $\phi_\Sigma(x)$ and $\lambda_1(A_\alpha(\Sigma)), \lambda_2(A_\alpha(\Sigma)), \ldots, \lambda_n(A_\alpha(\Sigma))$, respectively. The set of all eigenvalues of $A_\alpha(\Sigma)$ together with their multiplicities is called the A_α-spectrum of Σ. Two signed graphs are cospectral (resp. A_α-cospectral) if they are not switching isomorphic, but share the same spectrum (A_α-spectrum).

Until now, researchers have explored the A_α-spectrum of various graph operations. For instance, in [6], Li et al. studied the A_α-spectrum of graph products, Tahir et al. [20], studied the A_α- eigenvalues of coronae graphs. Some other results on A_α-spectrum of graph operations can be seen in [1, 16]. Recently, the spectra of some graph operations based on splitting graph have been studied in [5, 9]. Also some recent work on spectra of signed graphs can be seen in [18, 19].

Motivated by the above works, in this paper we first define the k-splitting signed graph $S_k(\Sigma)$ of Σ and determine its A_α-spectrum in case of Σ being regular. We introduce two distinct coronas of signed graphs Σ_1 and Σ_2 based on $S_1(\Sigma_1)$, namely $S_1(\Sigma_1)\dot{\vee}\Sigma_2$-the splitting V-vertex neighbourhood corona and $S_1(\Sigma_1)\overline{\vee}\Sigma_2$-the splitting S-vertex neighbourhood corona. By examining the A_α-characteristic polynomial of the resulting signed graphs, we derive their A_α-spectra under certain regularity conditions on the constituent signed graphs. As applications, we use these results to construct infinite pairs of nonregular A_α-cospectral signed graphs.

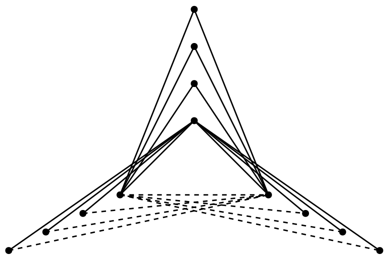

Figure 1. The k-splitting signed graph of a signed triangle with one negative edge and $k = 3$. Negative edges are dashed.

2. A_α-spectrum of k-splitting signed graph

We will use the symbols O, I and $\mathbf{j}$ to denote the all-zero matrix, the identity matrix and the all-one column vector, respectively. In all cases, the size may be given in the subscript.

The Kronecker product $A \otimes B$ of two matrices $A = (a_{ij})_{m\times n}$ and $B_{p\times q}$ is the $mp \times nq$ matrix obtained from A by replacing each element a_{ij} by $a_{ij}B$. This is an associative operation with the property that $(A\otimes B)^T = A^T \otimes B^T$ and $(A\otimes B)(C\otimes D) = AC\otimes BD$ whenever the product AC and BD exist. The later implies $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ for non-singular matrices A and B. Moreover, if A and B are $n\times n$ and $p\times p$ matrices, then $\det(A \otimes B) = (\det A)^p(\det B)^n$.

The M-coronal $\chi_M(x)$ of an $n \times n$ square matrix M is defined to be the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is, $\chi_M(x) = \mathbf{j}_n^{\intercal}(xI_n - M)^{-1}\mathbf{j}_n$ [10]. If M has a constant row sum l, then $\chi_M(x) = \frac{n}{x-l}$.

Lemma 1 (Schur complement formula, [3, Lemma 2.2]). *Let A_1, A_2, A_3, A_4 be, respectively, $p \times p$, $p \times q$, $q \times p$, $q \times q$ matrices, with A_1 and A_4 invertible. Then*

$$\det\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \det A_4 \cdot \det(A_1 - A_2{A_4}^{-1}A_3)$$
$$= \det A_1 \cdot \det(A_4 - A_3{A_1}^{-1}A_2).$$

Let $\Sigma = (G, \sigma)$ be a signed graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$ with $|E(G)| = m$. The k-splitting signed graph $S_k(\Sigma)$ of Σ is obtained by adding to each vertex $v \in V(\Sigma)$ new k vertices say $u^1, u^2, \ldots, u^k$ and joining every neighbour say u of the vertex v to u^i, $1 \le i \le k$ by an edge which inherits the sign from uv. The signed graph $S_k(\Sigma)$ has $n(k+1)$ vertices and $m(2k+1)$ edges. Note that for $k = 1$, the signed graph $S_1(\Sigma)$ is called the splitting signed graph of Σ [17]. An example of k-splitting signed graph is illustrated in Figure 1. We label the vertices of $S_k(\Sigma)$ as follows. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $\{u_i^1, u_i^2, \ldots, u_i^k\}$ denote the vertex set added corresponding to vertex v_i for $1 \le i \le n$. Let $V^j(G) = \{u_1^j, v_2^j, \ldots, v_n^j\}$,

$1 \leq j \leq k$. Then
$$V(G) \cup V^1(G) \cup V^2(G) \cup \ldots \cup V^k(G) \tag{2.1}$$
is the partition of $V(S_k(\Sigma))$. The degree of the vertices of $S_k(\Sigma)$ are

$$d_{S_k(\Sigma)}(v_i) = (k+1)d_\Sigma(v_i), \quad \text{for} \quad i = 1, 2, \ldots, n \text{ and}$$
$$d_{S_k(\Sigma)}(u_i^j) = d_\Sigma(v_i), \quad \text{for} \quad i = 1, 2, \ldots, n \text{ and } 1 \leq j \leq k.$$

In the following theorem, we show that the operation on Σ, resulting in a k-splitting signed graph $S_k(\Sigma)$ preserves the switching equivalence.

Theorem 1. *If Σ_1 and Σ_2 are switching equivalent signed graphs, then $S_k(\Sigma_1)$ and $S_k(\Sigma_2)$ are also switching equivalent.*

Proof. Given that Σ_1 and Σ_2 are switching equivalent, therefore $A(\Sigma_1) = X^{-1}A(\Sigma_2)X$, for some switching matrix X. We have

$$\begin{aligned}
A(S_k(\Sigma_1)) &= \begin{pmatrix} A(\Sigma_1) & A(\Sigma_1) & \ldots & A(\Sigma_1) \\ A(\Sigma_1) & O_{n\times n} & \ldots & O_{n\times n} \\ \vdots & \vdots & \vdots & \\ A(\Sigma_1) & O_{n\times n} & \ldots & O_{n\times n} \end{pmatrix} \\
&= \begin{pmatrix} X^{-1}A(\Sigma_2)X & X^{-1}A(\Sigma_2)X & \ldots & X^{-1}A(\Sigma_2)X \\ X^{-1}A(\Sigma_2)X & O_{n\times n} & \ldots & O_{n\times n} \\ \vdots & \vdots & \vdots & \\ X^{-1}A(\Sigma_2)X & O_{n\times n} & \ldots & O_{n\times n} \end{pmatrix} \\
&= \begin{pmatrix} X^{-1} & O & \ldots & O \\ O & X^{-1} & \ldots & O \\ \vdots & \vdots & \vdots & \\ O & O & \ldots & X^{-1} \end{pmatrix} \begin{pmatrix} A(\Sigma_2) & A(\Sigma_2) & \ldots & A(\Sigma_2) \\ A(\Sigma_2) & O_{n\times n} & \ldots & O_{n\times n} \\ \vdots & \vdots & \vdots & \\ A(\Sigma_2) & O_{n\times n} & \ldots & O_{n\times n} \end{pmatrix} \begin{pmatrix} X & O & \ldots & O \\ O & X & \ldots & O \\ \vdots & \vdots & \vdots & \\ O & O & \ldots & X \end{pmatrix} \\
&= D^{-1}A(S_k(\Sigma_2))D,
\end{aligned}$$

and we are done, since the adjacency matrices are switching similar. □

Next, we compute the A_α-spectrum of $S_k(\Sigma)$ when Σ is r-regular. Observe that if $\lambda(\Sigma)$ is an eigenvalue of r-regular signed graph Σ, then $\alpha r + (1-\alpha)\lambda(\Sigma)$ is the A_α-eigenvalue of Σ.

Theorem 2. *Let Σ be the r-regular signed graph with n vertices and eigenvalues $\lambda_1(\Sigma)$, $\lambda_2(\Sigma), \ldots, \lambda_n(\Sigma)$. The A_α-spectrum of $S_k(\Sigma)$ consists of*

(i) αr with multiplicity $(k-1)n$ and

(ii) the roots of $x^2 - \big((k+2)\alpha r + (1-\alpha)\lambda_i(\Sigma)\big)x + \alpha r\big((k+1)\alpha r + (1-\alpha)\lambda_i(\Sigma)\big) - k(1-\alpha)^2\lambda_i(\Sigma)^2$, for $i = 1, 2, \ldots, n$.

Proof. With the partition (2.1), the A_α-matrix of $S_k(\Sigma)$ is

$$A_\alpha(S_k(\Sigma)) = \begin{pmatrix} (k+1)\alpha r I_n + (1-\alpha)A(\Sigma) & (1-\alpha)A(\Sigma) & \dots & (1-\alpha)A(\Sigma) \\ (1-\alpha)A(\Sigma) & \alpha r I_n & \dots & O_{n\times n} \\ \vdots & \vdots & \vdots & \\ (1-\alpha)A(\Sigma) & O_{n\times n} & \dots & \alpha r I_n \end{pmatrix}.$$

The corresponding A_α-characteristic polynomial is given by

$$\begin{aligned} \phi_{S_k(\Sigma)}(x) &= \det\left(xI_{(k+1)n} - A_\alpha(SP_k(\Sigma))\right) \\ &= \det\begin{pmatrix} (x-(k+1)\alpha r)I_n - (1-\alpha)A(\Sigma) & -(1-\alpha)A(\Sigma) & \dots & -(1-\alpha)A(\Sigma) \\ -(1-\alpha)A(\Sigma) & (x-\alpha r)I_n & \dots & O_{n\times n} \\ \vdots & \vdots & \vdots & \\ -(1-\alpha)A(\Sigma) & O_{n\times n} & \dots & (x-\alpha r)I_n \end{pmatrix}. \end{aligned}$$

By performing row operations $R_1 + \frac{1-\alpha}{x-\alpha r}A(\Sigma)R_i \longrightarrow R_1$, for $i = 2, 3, \dots, k+1$, we have

$$\begin{aligned} \phi_{S_k(\Sigma)}(x) &= \det\begin{pmatrix} (x-(k+1)\alpha r)I_n - (1-\alpha)A(\Sigma) - \frac{k(1-\alpha)^2}{x-\alpha r}A(\Sigma)^2 & O_{n\times n} & \dots & O_{n\times n} \\ -(1-\alpha)A(\Sigma) & (x-\alpha r)I_n & \dots & O_{n\times n} \\ \vdots & \vdots & \vdots & \\ -(1-\alpha)A(\Sigma) & O_{n\times n} & \dots & (x-\alpha r)I_n \end{pmatrix} \\ &= (x-\alpha r)^{(k-1)n}\det\begin{pmatrix} (x-(k+1)\alpha r)I_n - (1-\alpha)A(\Sigma) - \frac{k(1-\alpha)^2}{x-\alpha r}A(\Sigma)^2 & O_{n\times n} \\ -(1-\alpha)A(\Sigma) & (x-\alpha r)I_n \end{pmatrix} \\ &= (x-\alpha r)^{kn}\det\left((x-(k+1)\alpha r)I_n - (1-\alpha)A(\Sigma) - \frac{k(1-\alpha)^2}{x-\alpha r}A(\Sigma)^2\right) \\ &= (x-\alpha r)^{(k-1)n}\prod_{i=1}^{n}\Big(x^2 - \big((k+2)\alpha r + (1-\alpha)\lambda_i(\Sigma)\big)x + \alpha r\big((k+1)\alpha r + (1-\alpha)\lambda_i(\Sigma)\big) \\ &\quad - k(1-\alpha)^2\lambda_i(\Sigma)^2\Big), \end{aligned}$$

completing the proof. □

From Theorem 2, we observe the following.

Remark 1. If Σ_1 and Σ_2 are cospectral r-regular signed graphs, then $S_k(\Sigma_1)$ and $S_k(\Sigma_2)$ are A_α-cospectral for all $k \in \mathbb{N}$ and $\alpha \in [0, 1]$.

It is worth mentioning that every pair of regular graphs, say G_1 and G_2, with the same number of vertices and the same vertex degree gives rise to a pair of cospectral regular signed graphs constructed in the following way: (1) insert a parallel negative edge between every pair of adjacent vertices of both graphs, (2) their signed line graphs are cospectral. This construction is obtained in [18, 19], and to our knowledge, there is no analogous counterpart for this method within the scope of ordinary graphs.

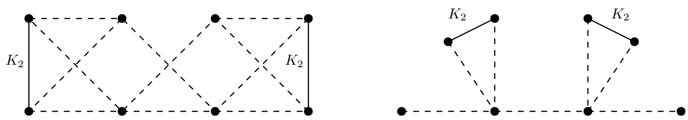

Figure 2. The splitting-V vertex neighbourhood corona and the splitting-S vertex neighbourhood corona.

3. Neighbourhood coronas based on splitting signed graph

Throughout this section, we deal with two signed graphs, $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ and assume that Σ_i has n_i vertices and m_i edges, for $i \in \{1, 2\}$. Also, let $S(\Sigma_1) = V(S_1(\Sigma_1)) \backslash V(\Sigma_1)$.

Definition 1. The *splitting-V vertex neighbourhood corona* $S_1(\Sigma_1)\dot{\vee}\Sigma_2$ of Σ_1 and Σ_2 is the signed graph obtained from $S_1(\Sigma_1)$ and n_1 copies of Σ_2 by joining each neighbour, say u, of the vertex $v_i \in V(\Sigma_1)$ to every vertex in the ith copy of Σ_2 by an edge which inherits the sign from $v_i u$. The signed graph $S_1(\Sigma_1)\dot{\vee}\Sigma_2$ has $n_1(n_2+2)$ vertices and $m_1(4n_2+3)+n_1m_2$ edges.

Definition 2. The *splitting-S vertex neighbourhood corona* $S_1(\Sigma_1)\overline{\vee}\Sigma_2$ of Σ_1 and Σ_2 is the signed graph obtained from $S_1(\Sigma_1)$ and n_1 copies of Σ_2 by joining each neighbour, say u, of the vertex $u_i \in S(\Sigma_1)$ to every vertex in the ith copy of Σ_2 by an edge which inherits the sign from $u_i u$. The signed graph $S_1(\Sigma_1)\overline{\vee}\Sigma_2$ has $n_1(n_2+2)$ vertices and $m_1(2n_2+3)+n_1m_2$ edges.

The above definitions are illustrated in Figure 2 with $\Sigma_1 = K_2^-$ and $\Sigma_2 = K_2$.

3.1. A_α-spectrum of the splitting-V vertex neighbourhood corona

Let $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ be two signed graphs on disjoint sets of n_1 and n_2 vertices, respectively. We label the vertices of $S_1(\Sigma_1)$ as $V(\Sigma_1) = \{v_1, v_2, \ldots, v_{n_1}\}$, $S(\Sigma_1) = \{u_1, u_2, \ldots, u_{n_1}\}$, and the vertices of Σ_2 as $V(\Sigma_2) = \{w_1, w_2, \ldots, w_{n_2}\}$. Let $V_j(\Sigma_2) = \{w_1^j, w_2^j, \ldots, w_{n_2}^j\}$ denote the vertex set of jth copy of Σ_2. Then the partition of vertices of $S_1(\Sigma_1)\dot{\vee}\Sigma_2$ is given by

$$V(\Sigma_1) \cup S(\Sigma_1) \cup V_1(\Sigma_2) \cup \ldots \cup V_{n_2}(\Sigma_2), \tag{3.1}$$

where $V_i(\Sigma_2) = \{w_i^1, w_i^2, \ldots, w_i^{n_1}\}$, $1 \leq i \leq n_2$. The degree of the vertices of $S_1(\Sigma_1)\dot{\vee}\Sigma_2$ are

$$d_{S_1(\Sigma_1)\dot{\vee}\Sigma_2}(v_i) = (n_2+2)d_{\Sigma_1}(v_i), \quad \text{for} \quad i = 1, 2, \ldots, n_1,$$
$$d_{S_1(\Sigma_1)\dot{\vee}\Sigma_2}(u_i) = (n_2+1)d_{\Sigma_1}(v_i), \quad \text{for} \quad i = 1, 2, \ldots, n_1 \text{ and}$$
$$d_{S_1(\Sigma_1)\dot{\vee}\Sigma_2}(w_j^i) = 2d_{\Sigma_1}(v_i) + d_{\Sigma_2}(w_j), \quad \text{for} \quad i = 1, 2, \ldots, n_1, 1 \leq j \leq n_2.$$

Now, we compute the A_α-characteristic polynomial of $S_1(\Sigma_1)\dot{\vee}\Sigma_2$ in case of Σ_1 being regular and Σ_2 any arbitrary signed graph.

Theorem 3. *Let Σ_1 be the r_1-regular signed graph with n_1 vertices and eigenvalues $\lambda_1(\Sigma_1), \lambda_2(\Sigma_1), \dots, \lambda_{n_1}(\Sigma_1)$, and Σ_2 be the signed graph with n_2 vertices having A_α-eigenvalues $\lambda_1(A_\alpha(\Sigma_2)), \lambda_2(A_\alpha(\Sigma_2)), \dots, \lambda_{n_2}(A_\alpha(\Sigma_2))$. Let $\chi_{A_\alpha(\Sigma_2)}(x)$ be the $A_\alpha(\Sigma_2)$-coronal of Σ_2. Then, for each $\alpha \in [0,1]$, the A_α-characteristic polynomial of $S_1(\Sigma_1)\dot{\vee}\Sigma_2$ is*

$$\begin{aligned}\phi_{S_1(\Sigma_1)\dot{\vee}\Sigma_2}(x) = &\prod_{i=1}^{n_2}\left(x - 2\alpha r_1 - \lambda_i(A_\alpha(\Sigma_2))\right)^{n_1}\prod_{i=1}^{n_1}\Big(x^2 - \big(\alpha r_1(n_2+1) + \alpha(n_2+2) + 2(1-\alpha)^2\\ &\cdot \chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)\lambda_i(\Sigma_1)^2 + (1-\alpha)\lambda_i(\Sigma_1)\big)x + \big(\alpha(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1)\\ &+ (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)\lambda_i(\Sigma_1)^2\big)\big(\alpha r_1(n_2+1) + (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)\\ &\cdot \lambda_i(\Sigma_1)^2\big) - \big((1-\alpha)\lambda_i(\Sigma_1) + (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)\lambda_i(\Sigma_1)^2\big)^2\Big).\end{aligned}$$

Proof. With respect to the partition (3.1), the adjacency matrix of $S_1(\Sigma_1)\dot{\vee}\Sigma_2$ can be written as

$$A(S_1(\Sigma_1)\dot{\vee}\Sigma_2) = \begin{pmatrix} A(\Sigma_1) & A(\Sigma_1) & A(\Sigma_1)\otimes \mathbf{j}_{\mathbf{n_2}}^\intercal \\ A(\Sigma_1) & O_{n\times n} & A(\Sigma_1)\otimes \mathbf{j}_{\mathbf{n_2}}^\intercal \\ A(\Sigma_1)\otimes \mathbf{j}_{\mathbf{n_2}} & A(\Sigma_1)\otimes \mathbf{j}_{\mathbf{n_2}} & I_{n_1}\otimes A(\Sigma_2)\end{pmatrix}.$$

Let D be the diagonal matrix of vertex degrees of Σ_2. The diagonal matrix of vertex degrees of $S_1(\Sigma_1)\dot{\vee}\Sigma_2$ is given by

$$D(S_1(\Sigma_1)\dot{\vee}\Sigma_2) = \begin{pmatrix} r_1(n_2+2)I_{n_1} & O & O \\ O & r_1(n_2+1)I_{n_1} & O \\ O & O & I_{n_1}\otimes(2r_1 I_{n_2} + D)\end{pmatrix}.$$

Therefore, the A_α-matrix of $S_1(\Sigma_1)\dot{\vee}\Sigma_2$ is given by $A_\alpha(SP_1(\Sigma_1)\dot{\vee}\Sigma_2) =$

$$\begin{pmatrix} \alpha r_1(n_2+2)I_{n_1} + (1-\alpha)A(\Sigma_1) & (1-\alpha)A(\Sigma_1) & (1-\alpha)A(\Sigma_1)\otimes \mathbf{j}_{\mathbf{n_2}}^\intercal \\ (1-\alpha)A(\Sigma_1) & \alpha r_1(n_2+1)I_{n_1} & (1-\alpha)A(\Sigma_1)\otimes \mathbf{j}_{\mathbf{n_2}}^\intercal \\ (1-\alpha)A(\Sigma_1)\otimes \mathbf{j}_{\mathbf{n_2}} & (1-\alpha)A(\Sigma_1)\otimes \mathbf{j}_{\mathbf{n_2}} & I_{n_1}\otimes(2\alpha r_1 I_{n_2} + A_\alpha(\Sigma_2))\end{pmatrix}.$$

The A_α-characteristic polynomial of $S_1(\Sigma_1)\dot{\vee}\Sigma_2$ is

$$\phi_{S_1(\Sigma_1)\dot{\vee}\Sigma_2}(x) = \det\left(xI_{2n_1+n_1n_2} - A_\alpha(S_1(\Sigma_1)\dot{\vee}\Sigma_2)\right) =$$

$$\det\begin{pmatrix} (x-\alpha r_1(n_2+2))I_{n_1} - (1-\alpha)A(\Sigma_1) & -(1-\alpha)A(\Sigma_1) & -(1-\alpha)A(\Sigma_1)\otimes \mathbf{j}_{\mathbf{n_2}}^\intercal \\ -(1-\alpha)A(\Sigma_1) & (x-\alpha r_1(n_2+1))I_{n_1} & -(1-\alpha)A(\Sigma_1)\otimes \mathbf{j}_{\mathbf{n_2}}^\intercal \\ -(1-\alpha)A(\Sigma_1)\otimes \mathbf{j}_{\mathbf{n_2}} & -(1-\alpha)A(\Sigma_1)\otimes \mathbf{j}_{\mathbf{n_2}} & I_{n_1}\otimes((x-2\alpha r_1)I_{n_2} - A_\alpha(\Sigma_2))\end{pmatrix}.$$

By performing row operations
$R_i + \left((1-\alpha)A(\Sigma_1) \otimes \mathbf{j}_{\mathbf{n_2}}^{\intercal}\right)\left(I_{n_1} \otimes ((x-2\alpha r_1)I_{n_2} - A_\alpha(\Sigma_2))\right)^{-1} R_3 \to R_i, i \in \{1,2\}$ and using Lemma 1, we obtain

$$\begin{aligned}\phi_{S_1(\Sigma_1)\dot{\vee}\Sigma_2}(x) &= \det\Big(I_{n_1} \otimes ((x-2\alpha r_1)I_{n_2} - A_\alpha(\Sigma_2))\Big)\det(M)\\ &= \prod_{i=1}^{n_2}\big(x - 2\alpha r_1 - \lambda_i(A_\alpha(\Sigma_2))\big)^{n_1}\det(M), \qquad (3.2)\end{aligned}$$

where $\det(M) = \det\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4\end{pmatrix}$ with
$M_1 = (x - \alpha r_1(n_2+2))I_{n_1} - (1-\alpha)A(\Sigma_1) - (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)A(\Sigma_1)^2$,
$M_2 = M_3 = -(1-\alpha)A(\Sigma_1) - (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)A(\Sigma_1)^2$ and
$M_4 = (x-\alpha r_1(n_2+1))I_{n_1} - (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)A(\Sigma_1)^2$.
Again, using Lemma 1, we have $\det(M) = \det(M_4)\det(M_1 - M_2M_4^{-1}M_3)$, that is

$$\begin{aligned}\det(M) &= \det\Big((x-\alpha r_1(n_2+1))I_{n_1} - (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)A(\Sigma_1)^2\Big)\\ &\quad\cdot\det\Big((x-\alpha r_1(n_2+2))I_{n_1} - (1-\alpha)A(\Sigma_1) - (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)A(\Sigma_1)^2\\ &\qquad - \big((1-\alpha)A(\Sigma_1) + (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)A(\Sigma_1)^2\big)\\ &\qquad\cdot\big((x-\alpha r_1(n_2+1))I_{n_1} - (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)A(\Sigma_1)^2\big)^{-1}\\ &\qquad\cdot\big((1-\alpha)A(\Sigma_1) + (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)A(\Sigma_1)^2\big)\Big)\\ &= \prod_{i=1}^{n_1}\Big(\big(x-\alpha r_1(n_2+2) - (1-\alpha)\lambda_i(\Sigma_1) - (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)\lambda_i(\Sigma_1)^2\big)\\ &\qquad\cdot\big(x-\alpha r_1(n_2+1) - (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)\lambda_i(\Sigma_1)^2\big)\\ &\qquad - \big((1-\alpha)\lambda_i(\Sigma_1) + (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)\lambda_i(\Sigma_1)^2\big)^2\Big)\\ &= \prod_{i=1}^{n_1}\Big(x^2 - \big(\alpha r_1(n_2+1) + \alpha(n_2+2) + 2(1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)\lambda_i(\Sigma_1)^2 + (1-\alpha)\\ &\qquad\cdot\lambda_i(\Sigma_1)\big)x + \big(\alpha(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1) + (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)\lambda_i(\Sigma_1)^2\big)\\ &\qquad\cdot\big(\alpha r_1(n_2+1) + (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)\lambda_i(\Sigma_1)^2\big)\\ &\qquad - \big((1-\alpha)\lambda_i(\Sigma_1) + (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)\lambda_i(\Sigma_1)^2\big)^2\Big).\end{aligned}$$

Using the value of $\det(M)$ in equality (3.2), the result follows. □

If Σ_2 is a co-regular signed graph, then we have the following observation.

Corollary 1. *Assume that under the assumptions of Theorem 3, Σ_2 is co-regular signed graph with co-regularity pair (r_2, s_2) and $\lambda_k(A_\alpha(\Sigma_2)) = \alpha r_2 + (1-\alpha)s_2$ for some fixed k $(1 \le k \le n_2)$. The A_α-spectrum of $S_1(\Sigma_1)\dot{\vee}\Sigma_2$ consists of*

(i) $2\alpha r_1 + \lambda_i(A_\alpha(\Sigma_2))$ *with multiplicity n_1, for $i \in \{1, 2, \dots, k-1, k+1, \dots, n_2\}$ and*

(ii) the roots of $x^3 - \Big(2\alpha r_1 + \alpha r_2 + (1-\alpha)s_2 + \alpha r_1(n_2+1) + \alpha(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1)\Big)x^2 + \Big(\big(\alpha r_1(n_2+1) + \alpha(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1)\big)\big(2\alpha r_1 + \alpha r_2 + (1-\alpha)s_2\big) - 2n_2(1-\alpha)^2\lambda_i(\Sigma_1)^2 + \alpha r_1(n_2+1)\big(\alpha(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1)\big) - (1-\alpha)^2\lambda_i(\Sigma_1)^2\Big)x - \alpha r_1(n_2+1)(2\alpha r_1 + \alpha r_2 + (1-\alpha)s_2)\big(\alpha(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1)\big) + n_2(1-\alpha)^2\lambda_i(\Sigma_1)^2\big(\alpha(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1)\big) + \alpha n_2 r_1(1-\alpha)^2(n_2+1)\lambda_i(\Sigma_1)^2 + (1-\alpha)^2(2\alpha r_1 + \alpha r_2 + (1-\alpha)s_2)\lambda_i(\Sigma_1)^2 - 2n_2(1-\alpha)^3\lambda_i(\Sigma_1)^3$, *for $i \in \{1, 2, \dots, n_1\}$.*

Proof. Since Σ_2 is (r_2, s_2)-co-regular and hence each row sum of the matrix $A_\alpha(\Sigma_2)$ equals $\alpha r_2 + (1-\alpha)s_2$, the coronal of the $A_\alpha(\Sigma_2)$-matrix is $\chi_{A_\alpha(\Sigma_2)}(x - 2r_1\alpha) = \frac{n_2}{x - 2r_1\alpha - r_2\alpha - (1-\alpha)s_2}$. For brevity, we put $\beta = 2r_1\alpha + r_2\alpha + (1-\alpha)s_2$ and using the value of $\chi_{A_\alpha(\Sigma_2)}(x - 2r_1\alpha)$ in Theorem 3, we have

$$
\begin{aligned}
& x^2 - \Big(\alpha r_1(n_2+1) + \alpha(n_2+2) + 2(1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)\lambda_i(\Sigma_1)^2 + (1-\alpha)\lambda_i(\Sigma_1)\Big)x \\
&\quad + \Big(\alpha(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1) + (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)\lambda_i(\Sigma_1)^2\Big)\Big(\alpha r_1(n_2+1) \\
&\quad + (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)\lambda_i(\Sigma_1)^2\Big) - \Big((1-\alpha)\lambda_i(\Sigma_1) + (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)\lambda_i(\Sigma_1)^2\Big)^2 \\
&= x^2 - \Big(\alpha r_1(n_2+1) + \alpha(n_2+2) + 2(1-\alpha)^2\frac{n_2}{x-\beta}\lambda_i(\Sigma_1)^2 + (1-\alpha)\lambda_i(\Sigma_1)\Big)x \\
&\quad + \Big(\alpha(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1) + (1-\alpha)^2\frac{n_2}{x-\beta}\lambda_i(\Sigma_1)^2\Big)\Big(\alpha r_1(n_2+1) + (1-\alpha)^2\frac{n_2}{x-\beta}\lambda_i(\Sigma_1)^2\Big) \\
&\quad - \Big((1-\alpha)\lambda_i(\Sigma_1) + (1-\alpha)^2\frac{n_2}{x-\beta}\lambda_i(\Sigma_1)^2\Big)^2 \\
&= \frac{1}{(x-\beta)^2}\Big(x^2(1-\beta)^2 - \big(\alpha r_1(n_2+1) + \alpha(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1)\big)(x-\beta)^2 x \\
&\quad - 2n_2(1-\alpha)^2\lambda_i(\Sigma_1)^2(x-\beta)x + \alpha r_1(n_2+1)\big(\alpha(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1)\big)(x-\beta)^2 + n_2(1-\alpha)^2 \\
&\quad \cdot \lambda_i(\Sigma_1)^2\big(\alpha(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1)\big)(x-\beta) + \alpha r_1 n_2(n_2+1)(1-\alpha)^2\lambda_i(\Sigma_1)^2(x-\beta) + n_2^2 \\
&\quad \cdot (1-\alpha)^4\lambda_i(\Sigma_1)^4 - (1-\alpha)^2\lambda_i(\Sigma_1)^2(x-\beta)^2 - n_2^2(1-\alpha)^4\lambda_i(\Sigma_1)^4 - 2n_2(1-\alpha)^3\lambda_i(\Sigma_1)^3(x-\beta)\Big) \\
&= \frac{1}{x-\beta}\Big(x^2(1-\beta) - \big(\alpha r_1(n_2+1) + \alpha(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1)\big)(x^2 - \beta x) - 2n_2(1-\alpha)^2\lambda_i(\Sigma_1)^2 \\
&\quad \cdot x + \alpha r_1(n_2+1)\big(\alpha(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1)\big)(x-\beta) + n_2(1-\alpha)^2\lambda_i(\Sigma_1)^2\big(\alpha(n_2+2) + (1-\alpha) \\
&\quad \cdot \lambda_i(\Sigma_1)\big) + \alpha r_1 n_2(n_2+1)(1-\alpha)^2\lambda_i(\Sigma_1)^2 - (1-\alpha)^2\lambda_i(\Sigma_1)^2(x-\beta) - 2n_2(1-\alpha)^3\lambda_i(\Sigma_1)^3\Big) \\
&= \frac{1}{x-\beta}\Big(x^3 - \Big(\beta + \alpha r_1(n_2+1) + \alpha(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1)\Big)x^2 + \Big(\beta\big(\alpha r_1(n_2+1) + \alpha(n_2+2) \\
&\quad + (1-\alpha)\lambda_i(\Sigma_1)\big) - 2n_2(1-\alpha)^2\lambda_i(\Sigma_1)^2 + \alpha r_1(n_2+1)\big(\alpha(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1)\big) - (1-\alpha)^2 \\
&\quad \cdot \lambda_i(\Sigma_1)^2\Big)x - \beta\alpha r_1(n_2+1)\big(\alpha(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1)\big) + n_2(1-\alpha)^2\lambda_i(\Sigma_1)^2\big(\alpha(n_2+2) \\
&\quad + (1-\alpha)\lambda_i(\Sigma_1)\big) + \alpha n_2 r_1(1-\alpha)^2(n_2+1)\lambda_i(\Sigma_1)^2 + \beta(1-\alpha)^2\lambda_i(\Sigma_1)^2 - 2n_2(1-\alpha)^3\lambda_i(\Sigma_1)^3\Big).
\end{aligned} \tag{3.3}
$$

Also

$$\prod_{i=1}^{n_2}\big(x-2\alpha r_1-\lambda_i(A_\alpha(\Sigma_2))\big)^{n_1}=(x-\beta)^{n_1}\prod_{\substack{i=1\\ i\neq k}}^{n_2}\big(x-2\alpha r_1-\lambda_i(A_\alpha(\Sigma_2))\big)^{n_1}. \quad (3.4)$$

In view of equalities (3.3) and (3.4), the result follows. □

Now in the following corollary, we obtain the A_α-eigenvalues of $S_1(\Sigma_1)\dot{\vee}\Sigma_2$, where $\Sigma_2 = K^-_{p,q}$.

Corollary 2. *Suppose that under the assumptions of Theorem 3, $\Sigma_2 = K^-_{p,q}$, a complete bipartite signed graph with all negative signature. The A_α-spectrum of $S_1(\Sigma_1)\dot{\vee}\Sigma_2$ consists of*

(i) $\alpha(2r_1+p)$ with multiplicity $n_1(q-1)$,

(ii) $\alpha(2r_1+q)$ with multiplicity $n_1(p-1)$ and

(iii) the four roots of the equation $P_i(x)=0$ for each $i\in\{1,2,\dots,n_1\}$, where $P_i(x)$ is given by (3.3) with $\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r_1)=\frac{(p+q)(x-2\alpha r_1)-\alpha(p^2+q^2)-2(1-\alpha)pq}{(x-2\alpha r_1)^2-\alpha(p+q)(x-2\alpha r_1)+(2\alpha-1)pq}$.

Proof. The A_α-matrix of $K^-_{p,q}$ is given by

$$A_\alpha(K^-_{p,q})=\begin{pmatrix}\alpha q I_p & -(1-\alpha)J_{p\times q}\\ -(1-\alpha)J_{q\times p} & \alpha p I_q\end{pmatrix},$$

where $J_{p\times q}$ is a matrix of all ones. Let

$$X=\begin{pmatrix}\big(y-\alpha p-(1-\alpha)q\big)I_p & O_{p\times q}\\ O_{q\times p} & \big(y-\alpha q-(1-\alpha)p\big)I_q\end{pmatrix}.$$

We have

$$\begin{aligned}(yI_{p+q}-A_\alpha(K^-_{p,q}))X\mathbf{J_{p+q}}&=\begin{pmatrix}(y-\alpha q)I_p & (1-\alpha)J_{p\times q}\\ (1-\alpha)J_{q\times p} & (y-\alpha p)I_q\end{pmatrix}\begin{pmatrix}\big(y-\alpha p-(1-\alpha)q\big)\mathbf{j_p}\\ \big(y-\alpha q-(1-\alpha)p\big)\mathbf{j_q}\end{pmatrix}\\ &=\begin{pmatrix}\big(y^2-\alpha(p+q)y+(2\alpha-1)pq\big)\mathbf{j_p}\\ \big(y^2-\alpha(p+q)y+(2\alpha-1)pq\big)\mathbf{j_q}\end{pmatrix}\\ &=\big(y^2-\alpha(p+q)y+(2\alpha-1)pq\big)\mathbf{J_{p+q}},\end{aligned}$$

implying that $\big(yI_{p+q}-A_\alpha(K^-_{p,q})\big)^{-1}\mathbf{J_{p+q}}=\frac{X\mathbf{J_{p+q}}}{y^2-\alpha(p+q)y+(2\alpha-1)pq}$. Hence, the coronal of the $A_\alpha(K^-_{p,q})$-matrix is given by

$$\begin{aligned}\chi_{A_\alpha(\Sigma_2)}(y)&=\mathbf{J}^\intercal_{\mathbf{p+q}}\big(yI_{p+q}-A_\alpha(K^-_{p,q})\big)^{-1}\mathbf{J_{p+q}}\\ &=\frac{\mathbf{J}^\intercal_{\mathbf{p+q}}X\mathbf{J_{p+q}}}{y^2-\alpha(p+q)y+(2\alpha-1)pq}\\ &=\frac{(p+q)y-\alpha(p^2+q^2)-2(1-\alpha)pq}{y^2-\alpha(p+q)y+(2\alpha-1)pq}. \quad (3.5)\end{aligned}$$

Further, the A_α-characteristic polynomial of $K^-_{p,q}$ is given by

$$\begin{aligned}
\phi_{K^-_{p,q}}(y) &= \det\begin{pmatrix}(y-\alpha q)I_p & (1-\alpha)J_{p\times q}\\ (1-\alpha)J_{q\times p} & (y-\alpha p)I_q\end{pmatrix}\\
&= \det\big((y-\alpha q)I_p\big)\det\big((y-\alpha p)I_q-(1-\alpha)J_{q\times p}\frac{1}{y-\alpha q}(1-\alpha)J_{p\times q}\big)\\
&= (y-\alpha q)^p\det\big((y-\alpha p)I_q-\frac{p(1-\alpha)^2}{y-\alpha q}J_{q\times q}\big)\\
&= (y-\alpha p)^{q-1}(y-\alpha q)^{p-1}\big(y^2-\alpha(p+q)y+(2\alpha-1)pq\big). \qquad (3.6)
\end{aligned}$$

Using (3.5) and (3.6) in Theorem 3, the result follows. □

Finally, to conclude this subsection, we provide a construction of new pairs of A_α-cospectral signed graphs.

Remark 2. Let Σ_1 and Σ_1' be two cospectral r-regular signed graphs, and Σ_2 be any arbitrary signed graph. Then the signed graphs $S_1(\Sigma_1)\dot{\vee}\Sigma_2$ and $S_1(\Sigma_1')\dot{\vee}\Sigma_2$ are A_α-cospectral for all $\alpha\in[0,1]$.
Let Σ_1 be r-regular signed graph, Σ_2 and Σ_2' be two A_α-cospectral signed graphs with $\chi_{A_\alpha(\Sigma_2)}(x-2\alpha r)=\chi_{A_\alpha(\Sigma_2')}(x-2\alpha r)$ for all $\alpha\in[0,1]$. Then the signed graphs $S_1(\Sigma_1)\dot{\vee}\Sigma_2$ and $S_1(\Sigma_1)\dot{\vee}\Sigma_2'$ are A_α-cospectral.

3.2. A_α-spectrum of the splitting-S vertex neighbourhood corona

We use the vertex labelling fixed at the beginning of the subsection 3.1. The degree of the vertices of $S_1(\Sigma_1)\overline{\vee}\Sigma_2$ are

$$\begin{aligned}
d_{S_1(\Sigma_1)\overline{\vee}\Sigma_2}(v_i) &= (n_2+2)d_{\Sigma_1}(v_i), \quad \text{for} \quad i=1,2,\ldots,n_1,\\
d_{S_1(\Sigma_1)\overline{\vee}\Sigma_2}(u_i) &= d_{\Sigma_1}(v_i), \quad \text{for} \quad i=1,2,\ldots,n_1 \text{ and}\\
d_{S_1(\Sigma_1)\overline{\vee}\Sigma_2}(w_j^i) &= d_{\Sigma_1}(v_i)+d_{\Sigma_2}(w_j), \quad \text{for} \quad i=1,2,\ldots,n_1, 1\le j\le n_2.
\end{aligned}$$

We compute the A_α-characteristic polynomial of $S_1(\Sigma_1)\overline{\vee}\Sigma_2$, but with less details in the proof.

Theorem 4. *Let Σ_1 be the r_1-regular signed graph with n_1 vertices and eigenvalues $\lambda_1(\Sigma_1),\lambda_2(\Sigma_1),\ldots,\lambda_{n_1}(\Sigma_1)$, and Σ_2 be the signed graph with n_2 vertices having A_α-eigenvalues $\lambda_1(A_\alpha(\Sigma_2)),\lambda_2(A_\alpha(\Sigma_2)),\ldots,\lambda_{n_2}(A_\alpha(\Sigma_2))$. Let $\chi_{A_\alpha(\Sigma_2)}(x)$ be the $A_\alpha(\Sigma_2)$-coronal of Σ_2. Then, for each $\alpha\in[0,1]$, the A_α-characteristic polynomial of $S_1(\Sigma_1)\overline{\vee}\Sigma_2$ is*

$$\begin{aligned}
\phi_{S_1(\Sigma_1)\overline{\vee}\Sigma_2}(x) = \prod_{i=1}^{n_1}\Big((x-\alpha r_1)\big(x-\alpha r_1(n_2+2)-(1-\alpha)\lambda_i(\Sigma_1)-(1-\alpha)^2\lambda_i(\Sigma_1)^2\\
\cdot\chi_{A_\alpha(\Sigma_2)}(x-\alpha r_1)\big)-(1-\alpha)^2\lambda_i(\Sigma_1)^2\Big)\prod_{i=1}^{n_2}\big(x-\alpha r_1-\lambda_i(A_\alpha(\Sigma_2))\big)^{n_1}.
\end{aligned}$$

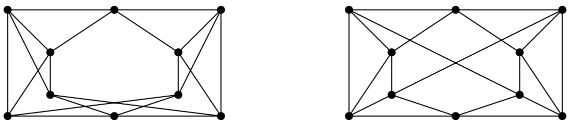

Figure 3. **G_2 and G_2' are cospectral 4-regular graphs.**

Proof. With respect to the partition (3.1), the A_α-matrix of $S_1(\Sigma_1)\nabla\Sigma_2$ can be written as

$$A_\alpha(S_1(\Sigma_1)\nabla\Sigma_2) = \begin{pmatrix} \alpha r_1(n_2+2)I_{n_1} + (1-\alpha)A(\Sigma_1) & (1-\alpha)A(\Sigma_1) & (1-\alpha)A(\Sigma_1)\otimes \mathbf{j}_{\mathbf{n_2}}^{\intercal} \\ (1-\alpha)A(\Sigma_1) & \alpha r_1 I_{n_1} & O_{n_1\times n_1}\otimes \mathbf{j}_{\mathbf{n_2}}^{\intercal} \\ (1-\alpha)A(\Sigma_1)\otimes \mathbf{j}_{\mathbf{n_2}} & O_{n_1\times n_1}\otimes \mathbf{j}_{\mathbf{n_2}} & I_{n_1}\otimes(\alpha r_1 I_{n_2} + A_\alpha(\Sigma_2)) \end{pmatrix}.$$

From this we obtain

$$\begin{aligned} \phi_{S_1(\Sigma_1)\nabla\Sigma_2}(x) &= \det\big(xI_{2n_1+n_1n_2} - A_\alpha(SP_1(\Sigma_1)\nabla\Sigma_2)\big) \\ &= \det\big(I_{n_1}\otimes((x-\alpha r_1)I_{n_2} - A_\alpha(\Sigma_2))\big)\det(M), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} \det(M) &= \det\begin{pmatrix} (x-\alpha r_1(n_2+2))I_{n_1} - (1-\alpha)A(\Sigma_1) - (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-\alpha r_1)A(\Sigma_1)^2 & -(1-\alpha)A(\Sigma_1) \\ -(1-\alpha)A(\Sigma_1) & (x-\alpha r_1)I_{n_1} \end{pmatrix} \\ &= (x-\alpha r_1)^{n_1}\det\Big((x-\alpha r_1(n_2+2))I_{n_1} - (1-\alpha)A(\Sigma_1) - (1-\alpha)^2\chi_{A_\alpha(\Sigma_2)}(x-\alpha r_1)A(\Sigma_1)^2 \\ &\quad - \frac{(1-\alpha)^2}{x-\alpha r_1}A(\Sigma_1)^2\Big) \\ &= \prod_{i=1}^{n_1}\Big((x-\alpha r_1)\big(x-\alpha r_1(n_2+2) - (1-\alpha)\lambda_i(\Sigma_1) - (1-\alpha)^2\lambda_i(\Sigma_1)^2\chi_{A_\alpha(\Sigma_2)}(x-\alpha r_1)\big) \\ &\qquad - (1-\alpha)^2\lambda_i(\Sigma_1)^2\Big). \end{aligned}$$

Using the value of $\det(M)$ in equality (3.7), the result follows. □

Corollary 3. *Assume that under the assumptions of Theorem 4, Σ_2 is a co-regular signed graph with co-regularity pair (r_2, s_2) and $\lambda_k(A_\alpha(\Sigma_2)) = \alpha r_2 + (1-\alpha)s_2$ for some fixed k $(1 \le k \le n_2)$. The A_α-spectrum of $S_1(\Sigma_1)\nabla\Sigma_2$ consists of*

(i) $\alpha r_1 + \lambda_i(A_\alpha(\Sigma_2))$ with multiplicity n_1, for $i \in \{1, 2, \dots, k-1, k+1, \dots, n_2\}$ and

(ii) the roots of $x^3 - \Big(\alpha r_1(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1) + \alpha r_1 + \alpha(r_1+r_2) + (1-\alpha)s_2\Big)x^2 + \Big((\alpha r_1 + \alpha(r_1+r_2) + (1-\alpha)s_2)\big(\alpha r_1(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1)\big) + \alpha r_1(\alpha(r_1+r_2) + (1-\alpha)s_2) - n_2(1-\alpha)^2\lambda_i(\Sigma_1)^2 - (1-\alpha)^2\lambda_i(\Sigma_1)^2\Big)x - \alpha r_1(\alpha(r_1+r_2) + (1-\alpha)s_2)\big(\alpha r_1(n_2+2) + (1-\alpha)\lambda_i(\Sigma_1)\big) + \alpha r_1 n_2(1-\alpha)^2\lambda_i(\Sigma_1)^2 + (\alpha(r_1+r_2) + (1-\alpha)s_2)(1-\alpha)^2\lambda_i(\Sigma_1)^2$, for $i \in \{1, 2, \dots, n_1\}$.

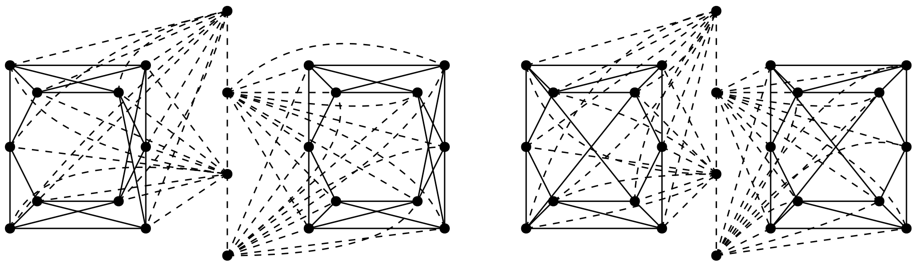

Figure 4. **Pair of A_α-cospectral signed graphs $S_1(K_2^-)\dot{\vee}G_2$ and $S_1(K_2^-)\dot{\vee}G_2'$.**

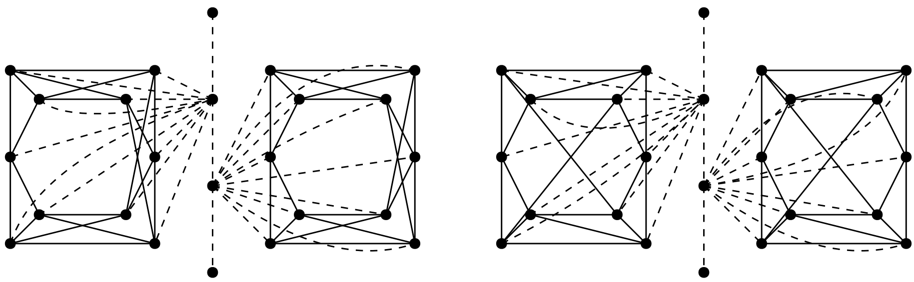

Figure 5. **Pair of A_α-cospectral signed graphs $S_1(K_2^-)\overline{\vee}G_2$ and $S_1(K_2^-)\overline{\vee}G_2'$.**

Proof. Given that Σ_2 is (r_2, s_2)-co-regular and hence $\chi_{A_\alpha(\Sigma_2)}(x - \alpha r_1) = \frac{n_2}{x-\alpha(r_1+r_2)-(1-\alpha)s_2}$. By plugging in the value of $\chi_{A_\alpha(\Sigma_2)}(x - \alpha r_1)$ in Theorem 4 and engaging in straightforward calculations yield the desired result. □

Remark 3. Let Σ_1 and Σ_1' be two cospectral r-regular signed graphs, and Σ_2 be any arbitrary signed graph. Then the signed graphs $S_1(\Sigma_1)\overline{\vee}\Sigma_2$ and $S_1(\Sigma_1')\overline{\vee}\Sigma_2$ are A_α-cospectral for all $\alpha \in [0, 1]$.
Let Σ_1 be r-regular signed graph, Σ_2 and Σ_2' be two A_α-cospectral signed graphs with $\chi_{A_\alpha(\Sigma_2)}(x - \alpha r) = \chi_{A_\alpha(\Sigma_2')}(x - \alpha r)$ for all $\alpha \in [0, 1]$. Then the signed graphs $S_1(\Sigma_1)\overline{\vee}\Sigma_2$ and $S_1(\Sigma_1)\overline{\vee}\Sigma_2'$ are A_α-cospectral.

Example: Let $\Sigma_1 = K_2^-$, $\Sigma_2 = G_2$ and $\Sigma_2' = G_2'$, where G_2 and G_2' are the graphs in Figure 3. It is known from ([21], preposition 3) that G_2 and G_2' are a pair of cospectral 4-regular graphs. In view of Remark 2 and 3, the signed graphs

(i) $S_1(K_2^-)\dot{\vee}G_2$ and $S_1(K_2^-)\dot{\vee}G_2'$ are A_α-cospectral shown in Figure 4 and

(ii) $S_1(K_2^-)\overline{\vee}G_2$ and $S_1(K_2^-)\overline{\vee}G_2'$ are A_α-cospectral shown in Figure 5.

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