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Research Article

On the reciprocal distance Laplacian spectral radius of graphs

Ummer Mushtaq[†], **Shariefuddin Pirzada**[*]

Department of Mathematics, University of Kashmir, Srinagar, India
[†]ummermushtaq1995@gmail.com
[*]pirzadasd@kashmiruniversity.ac.in

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Abstract: The reciprocal distance Laplacian matrix of a connected graph G is defined as $RD^L(G) = RTr(G) - RD(G)$, where $RTr(G)$ is the diagonal matrix whose i-th element $RTr(v_i) = \sum_{i \neq j \in V(G)} \frac{1}{d_{ij}}$ and $RD(G)$ is the Harary matrix. $RD^L(G)$ is a real symmetric matrix and we denote its eigenvalues as $\lambda_1(RD^L(G)) \geq \lambda_2(RD^L(G)) \geq \cdots \geq \lambda_n(RD^L(G))$. The largest eigenvalue $\lambda_1(RD^L(G))$ of $RD^L(G)$ is called the reciprocal distance Laplacian spectral radius. In this paper, we obtain upper bounds for the reciprocal distance Laplacian spectral radius. We characterize the extremal graphs attaining this bound.

Keywords: distance Laplacian matrix, reciprocal distance Laplacian matrix, Harary index; reciprocal distance Laplacian eigenvalues, reciprocal distance Laplacian spectral radius.

AMS Subject classification: 05C50, 05C12, 15A18

1. Introduction

Let $G = (V, E)$ be a simple connected graph with n vertices and let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices in G. The degree of a vertex v, denoted by $d(v)$, is the number of edges incident to the vertex v. A complete bipartite graph with cardinalities of the vertex sets in two parts as p and q is denoted by $K_{p,q}$. A graph G is called a split graph if its vertex set $V(G)$ can be partitioned into sets K and S such that the induced graph on K is a clique and S is an independent set (stable set). A split graph is said to be complete if there is an edge from each vertex of K to every vertex of S. The complete split graph is denoted by $CS_{(n,\alpha)}$. For other notations see [4].

* *Corresponding Author*

The adjacency matrix of a graph G is $n \times n$ matrix whose rows and columns are indexed by vertices and is defined as $A(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge from } v_i \textit{ to } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

For a graph G, the Laplacian matrix is defined as $L(G) = Deg(G) - A(G)$, where $Deg(G) = diag(d_1, d_2, \ldots, d_n)$ is the diagonal matrix of the vertex degrees. The Laplacian matrix is a real symmetric positive semi-definite matrix. The eigenvalues of $L(G)$ are called the Laplacian eigenvalues of G, which are denoted by $\mu_1(G), \mu_2(G), \ldots, \mu_n(G)$ and are arranged as $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$. The distance between a pair of vertices v_i and v_j is defined as the length of a shortest path between v_i and v_j and is denoted by $d_G(v_i, v_j)$ or $d_{i,j}$. The diameter of G, denoted by $d(G)$, is the largest distance between any two vertices of G. The distance matrix of G is denoted by $D(G)$ and is defined as $D(G) = (d(v_i, v_j))_{v_i, v_j \in V(G)}$.

The *transmission or the total distance* $Tr_G(v_i)$ (or briefly Tr_i if graph G is understood) of a vertex v_i is defined as the sum of the distances from v_i to all other vertices in G, that is,

$$Tr_G(v_i) = \sum_{v_j \in V(G)} d(v_i, v_j).$$

The total reciprocal distance of a vertex v is defined as

$$RH_G(v) = \sum_{u \in V(G)} \frac{1}{d_G(u, v)}, u \neq v.$$

Let $Tr(G) = diag(Tr_1, Tr_2, \ldots, Tr_n)$ be the diagonal matrix of vertex transmissions of G. In [2], Aouchiche and Hansen introduced the Laplacian for the distance matrix of a connected graph. The matrix $D^L = Tr(G) - D(G)$ is called the distance Laplacian matrix of G.

The *Harary matrix* of a graph G, which is also called as the reciprocal distance matrix, denoted by $RD(G)$, is defined as [6]

$$RD_{ij}(G) = \begin{cases} \frac{1}{d(v_i, v_j)}, & \text{if} \quad i \neq j, \\ 0, & \text{if} \quad i = j. \end{cases}$$

The *Harary index* of G is

$$H(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} RD_{i,j}(G) = \sum_{i<j} \frac{1}{d_{i,j}}.$$

Clearly,

$$H(G) = \frac{1}{2} \sum_{v_i \in V(G)} RTr_G(v_i),$$

where $RTr_G(v_i)$ is the reciprocal distance degree of a vertex v_i and is defined as

$$RTr_G(v_i) = \sum_{v_j \in V(G)_{v_i \neq v_j}} \frac{1}{d_G(v_i, v_j)}.$$

In [3], Bapat and Panda defined the reciprocal distance Laplacian matrix as $RD^L(G) = RT(G) - RD(G)$. Since the sum of each row of $RD^L(G)$ is zero, it follows that 0 is an eigenvalue of $RD^L(G)$. Also the reciprocal distance Laplacian matrix $RD^L(G)$ is a real symmetric and positive semidefinite matrix, we denote the eigenvalues of $RD^L(G)$ as

$$\lambda_1(RD^L(G)) \geq \lambda_2(RD^L(G)) \geq \cdots \geq \lambda_n(RD^L(G)) = 0.$$

We will denote the spectral radius of $RD^L(G)$ by $\lambda(G) = \lambda_1(RD^L(G))$, called the reciprocal distance Laplacian spectral radius. As 0 is always a simple eigenvalue of the reciprocal distance Laplacian matrix, we define the reciprocal distance Laplacian spread of a connected graph G as

$$RDLS(G) = \lambda_1(RD^L(G)) - \lambda_{n-1}(RD^L(G)),$$

where $\lambda_1(RD^L(G))$ and $\lambda_{n-1}(RD^L(G))$ are, respectively, the largest and the second smallest reciprocal distance Laplacian eigenvalues of G. Some recent work on this topic can be seen in [1, 5].
The rest of the paper is organized as follows. In Section 2, we obtain upper bounds for the first two largest reciprocal distance Laplacian eigenvalues. In Section 3, we find upper bound for the reciprocal distance Laplacian spectral radius $\lambda_1(RD^L(G))$ for bipartite graphs. We also provide the sufficient conditions under which these bounds can be obtained.

2. Upper bound for the first two largest reciprocal distance Laplacian eigenvalues

Let M be a complex matrix of order n described in the following block form

$$M = \begin{pmatrix} M_{11} & \dots & M_{1t} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ M_{t1} & \dots & M_{tt} \end{pmatrix}$$

where the blocks M_{ij} are $n_i \times n_j$ matrices for any $1 \leq i, j \leq t$ and $n = n_1 + n_2 + \cdots + n_t$. For $1 \leq i, j \leq t$, let b_{ij} denote the average row sum of M_{ij}. Then $B(M) = (b_{ij})$ (or simply B) is called the quotient matrix of M. The following lemmas will be used in the sequel.

Lemma 1. [8] Let M be the matrix as defined above such that $M_{ij} = s_{ij} J_{n_i,n_j}$ for $i \neq j$ and $M_{ii} = s_{ii} J_{n_i,n_i} + p_i I_{n_i}$. Then the equitable quotient matrix of M is $B = (b_{ij})$ with $b_{ij} = s_{ij} n_j$, if $i \neq j$ and $b_{ii} = s_{ii} n_i + p_i$. Moreover,

$$\sigma(M) = \sigma(B) \cup \{p_1^{n_1-1}, p_2^{n_2-1} \ldots, p_t^{n_t-1}\},$$

where $n_i - 1$ denotes the multiplicity of eigenvalue p_i .

Lemma 2. [3] Let G be a connected graph on n vertices with $m \geq n$ edges and let $G' = G - e$ be the connected graph obtained from G by deletion of an edge e. Then $\lambda_i(RD^L(G)) \geq \lambda_i(RD^L(G'))$ for all $i = 1, 2 \ldots, n$.

Lemma 3. [7] Let X and Y be two $n \times n$ Hermitian matrices. Suppose that $Z = X + Y$, and we arrange the eigenvalues of a matrix by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then the following inequalities hold true:

$$\lambda_k(Z) \leq \lambda_j(X) + \lambda_{k-j+1}(Y), n \geq k \geq j \geq 1,$$
$$\lambda_k(Z) \geq \lambda_j(X) + \lambda_{k-j+n}(Y), n \geq j \geq k \geq 1.$$

Here, λ_i is the i-th largest eigenvalue of a given matrix. In any of these inequalities above, equality is attained if and only if there exists a unit eigenvector associated with each of the three eigenvalues involved.

Before we proceed, we have the following lemmas in matrix theory that will address the problem of obtaining the extremal graphs for which the bound can be achieved.

Lemma 4. Let $A = (a_{ij})$ be matrix of order n such that $a_{ii} = -\sum_{i \neq j=1}^{n} a_{ij}$ and $a_{ks} = -1, k \neq s, k = 1, 2, \ldots, n$, and $a_{sl} = -1, l \neq s, l = 1, 2, \ldots, n$, then n is an eigenvalue of matrix A with corresponding eigenvector $(-1, -1, \ldots, \underbrace{n-1}_{\text{s-th}}, \ldots, -1)^t$.

Proof. The matrix A can be written as

$$A = \begin{pmatrix} -(a_{12} + \cdots - 1 + \cdots + a_{1n}) & \ldots & -1 & \ldots & a_{1n} \\ a_{21} & \ldots & -1 & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -1 & \ldots & n-1 & \ldots & -1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & \ldots & -1 & \ldots & -(a_{n1} + \cdots - 1 + \cdots + a_{n.n-1}) \end{pmatrix}$$

Now consider the vector $\mathbf{x} = (-1, -1, \ldots, \underbrace{n-1}_{\text{s-th}}, \ldots, -1)^t$. We have

$$A\mathbf{x} = \begin{pmatrix} -n \\ -n \\ \cdots \\ n(n-1) \\ \cdots \\ -n \end{pmatrix} = n \begin{pmatrix} -1 \\ -1 \\ \cdots \\ (n-1) \\ \cdots \\ -1 \end{pmatrix} = n\mathbf{x}.$$

This shows that n is an eigenvalue of A with corresponding eigenvector $\mathbf{x} = (-1, -1, \ldots, \underbrace{n-1}_{\text{s-th}}, \ldots, -1)^t$. This completes the proof. □

Lemma 5. Let $A = (a_{ij})$ be matrix of order n such that $a_{ii} = -\sum_{i \neq j=1}^{n} a_{ij}$ and there exist $s, m, s \neq m$ such that $1 \leq s, m \leq n$ satisfying $a_{ks} = -1, k \neq s, k = 1, 2, \ldots, n$, and $a_{sl} = -1, l \neq s, l = 1, 2, \ldots, n$, and $a_{pm} = -1, m \neq p, p = 1, 2, \ldots, n$, and $a_{mq} = -1, l \neq s, q = 1, 2, \ldots, n$. Then n is an eigenvalue of the matrix A with algebraic multiplicity at least two and corresponding eigenvectors $(-1, -1, \ldots, \underbrace{n-1}_{\text{s-th}}, \ldots, -1)^t$ and $(-1, -1, \ldots, \underbrace{n-1}_{\text{m-th}}, \ldots, -1)^t$.

Proof. Proceeding as in Lemma 4, we observe that $(-1, -1, \ldots, \underbrace{n-1}_{\text{s-th}}, \ldots, -1)^t$ and $(-1, -1 \ldots, \underbrace{n-1}_{\text{m-th}}, \ldots, -1)^t$ are two linearly independent eigenvectors of A corresponding to the eigenvalue n. Therefore the geometric multiplicity of n is at least two. Since the algebraic multiplicity is always greater than or equal to geometric multiplicity, so n will be an eigenvalue of algebraic multiplicity at least two. □

The following theorem gives an upper bound for the reciprocal distance spectral radius and sufficient condition to obtain the equality.

Theorem 1. *Let G be a connected graph of order $n \geq 3$ with at least two non-pendant vertices. Then $\lambda_1(RD^L(G)) \leq n$. Moreover, if G has at least one vertex of degree $n-1$, the equality holds.*

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. Assume that G has a vertex say v_i of degree $n-1$. This implies that all the non-diagonal entries of the i^{th} row and the i^{th} column of $RD^L(G)$ will be -1 and the i^{th} diagonal element d_{ii} will be $n-1$. So the matrix $RD^L(G)$ will be of the same form as that of matrix A defined in Lemma 4. Therefore, by Lemma 4, n is an eigenvalue of $RD^L(G)$. So the equality occurs in this case.
Let H be the graph obtained from G by deleting an edge incident at v_i and some

non-pendant vertex v_{i+1}. Clearly the graph H is connected and has no vertex of degree $n-1$. Using Lemma 2 , we have $\lambda_1(RD^L(H)) \leq \lambda_1(RD^L(G))$. □

The requirement of a vertex of degree $n-1$ is only a sufficient condition to have a reciprocal distance Laplacian eigenvalue n. For example, the complete bipartite graph $K_{2,2}$ has one reciprocal distance Laplacian eigenvalue 4, but has no vertex of degree 3. In Lemma 7, we have proved that every complete bipartite graph $K_{p,q}$ has one eigenvalue n.
The requirement of two non-pendant vertices in Theorem 1 can be relaxed. All we need is a vertex of degree $n-1$. In the next theorem, we show that if the graph has a vertex of degree $n-1$, then n is a Laplacian reciprocal distance eigenvalue. But for this we need the following lemma.

Lemma 6. *The reciprocal distance Laplacian spectrum of a complete split graph $CS(n,\alpha)$, $\alpha \leq n-1$, is $\{(n-\frac{\alpha}{2})^{\alpha-1}, n^{n-\alpha}, 0^1\}$.*

Proof. Let $V = \{v_1, v_2, \ldots, v_\alpha, \ldots, v_n\}$ be the vertex set of $CS(n,\alpha)$ and $V_1 = \{v_1, v_2, \ldots, v_\alpha\}$ be the set of the vertices in its largest independent set. Therefore the reciprocal distance Laplacian matrix of $CS(n,\alpha)$ is

$$RD^L(CS(n,\alpha)) = \begin{pmatrix} ((n-\frac{\alpha}{2})I - \frac{1}{2}J)_{\alpha\times\alpha} & -J_{\alpha\times n-\alpha} \\ -J_{n-\alpha\times\alpha} & (nI-J)_{n-\alpha\times n-\alpha} \end{pmatrix}.$$

Therefore, by Lemma 1, $\sigma(RD^L(CS(n,\alpha))) = \{(n-\frac{\alpha}{2})^{\alpha+1}, n^{n-(\alpha-1)}\} \cup \sigma(B)$, where

$$B = \begin{pmatrix} n-\alpha & -n+\alpha \\ -\alpha & \alpha \end{pmatrix}.$$

By direct calculations, we see that the eigenvalues of B are 0 and n. Therefore the complete reciprocal distance Laplacian spectrum of $CS(n,\alpha)$ is $\{(n-\frac{\alpha}{2})^{\alpha-1}, n^{n-\alpha}, 0^1\}$. □

Theorem 2. *Let G be a connected graph of order $n \geq 3$. Then $\lambda_1(RD^L(G)) \leq n$ if G has at least one vertex of degree $n-1$ and n is a reciprocal distance Laplacian eigenvalue of G.*

Proof. We consider the following two cases.

Case (i). If G has at least two non-pendant vertices, then the result follows by Theorem 1.

Case (ii). If G has only one non-pedant vertex, then the only possible connected graph in this case is $CS(n, n-1) \cong K_{n-1,1}$. By substituting $\alpha = n-1$ in Lemma 6, we get the required result. □

The following theorem gives an upper bound for the second largest reciprocal distance Laplacian eigenvalue and the sufficient condition for obtaining the bound.

Theorem 3. *If G is a connected graph of order $n \geq 3$ with at least two non-pendant vertices, then $\lambda_2(RD(G)) \leq n$. If G has at least two vertices of degree $n-1$, then the equality holds.*

Proof. Let $V = \{v_1, v_2, \dots, v_n\}$ be the vertex set of G and assume that G has two vertices, say v_i and v_k, of degree $n-1$. This implies that all the non-diagonal entries of i^{th} and k^{th} row and i^{th} and k^{th} column of the Laplacian reciprocal distance matrix $RD(G)$ will be -1. Also, the i^{th} and k^{th} diagonal element d_{ii} and d_{kk} will be $n-1$. Therefore, the matrix $RD^L(G)$ will be of the same form as that of matrix A, which we constructed in Lemma 5. Therefore, by Lemma 5, n is an eigenvalue of $RD(G)$ of multiplicity at least two. Thus, $\lambda_2(RD^L(G))$, and so the equality occurs in this case. Now, assume that v_i and v_k are the only vertices of degree $n-1$ in G. Let H be the graph obtained from G by deleting an edge incident at v_i or v_k. Clearly, the graph H is connected. Using Lemma 2, we have $\lambda_2(RD^L(H)) \leq \lambda_2(RD^L(G))$. □

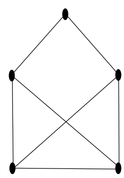

Figure 1. **Graph G**

The requirement of at least two vertices of degree $n-1$ is only a sufficient condition. The reciprocal Laplacian distance spectrum of the graph G in Figure 1 is $\{5^2, 2^2, 0\}$. In fact, the second reciprocal Laplacian distance eigenvalue can be n even if the graph has no vertex of degree $n-1$.
In the next theorem, we obtain a lower bound for the second smallest reciprocal distance Laplacian eigenvalue λ_{n-1} of graph having at least one vertex of degree $n-1$.

Theorem 4. *Let G be a connected graph of order $n \geq 3$ with at least one vertex of degree $n-1$. Then $\lambda_{n-1}(RD(G)) \geq \frac{n+1}{2}$. If $G \cong K_{n-1,1}$ then the equality holds.*

Proof. If G' is the graph obtained from G by adding an edge between a pair of non-adjacent vertices in G, then from Lemma 2, we have $\lambda_i(RD^L(G')) \geq \lambda_i(RD^L(G))$. With this observation, we start with the graph having a vertex of degree $n-1$ and having minimum number of edges, that is $K_{n-1,1}$. From Lemma 6, we get $\lambda_{n-1} RD^L(K_{n-1,1}) = \frac{n+1}{2}$. Therefore, by Lemma 2, the result follows. □

We now find an upper bound for the reciprocal distance Laplacian spread of a connected graph G.

Theorem 5. *Let G be a connected graph of order $n \geq 3$ with at least one vertex of degree $n-1$. Then $RDLS(G) \leq \frac{n-1}{2}$. If $G \cong K_{n-1,1}$ then the equality holds.*

Proof. Since G has a vertex of degree $n-1$, therefore from Theorems 1 and 4, we have $\lambda_1 RD^L(G) = n$ and $\lambda_{n-1} RD^L(G) \geq \frac{n+1}{2}$. Now, we have

$$\lambda_1 RD^L(G) - \lambda_{n-1} RD^L(G) \leq n - \frac{n+1}{2} = \frac{n-1}{2}.$$

□

3. Upper bound for the reciprocal distance Laplacian spectral radius of bipartite graphs

In this section, we obtain the complete characterization of bipartite graphs having the largest reciprocal distance Laplacian spectral radius n. We obtain the necessary and sufficient condition for a bipartite graph to have reciprocal distance Laplacian spectral radius n. The following lemma gives the reciprocal distance Laplacian spectrum of a complete bipartite graph.

Lemma 7. *The reciprocal distance Laplacian spectrum of a complete bipartite graph $K_{p,k}$ with $p+k=n$ is $\{(k+\frac{p}{2})^{p-1}, (p+\frac{k}{2})^{k-1}, n, 0\}$.*

Proof. The reciprocal distance Laplacian matrix of $K_{p,k}$, $p+k=n$, is

$$RD^L(K_{p,k}) = \begin{pmatrix} ((k+\frac{p}{2})I - \frac{1}{2}J)_{p\times p} & -J_{p\times k} \\ -J_{k\times p} & ((k+\frac{p}{2})I - \frac{1}{2}J)_{k\times k} \end{pmatrix}.$$

Then, by Lemma 1, $\sigma(RD^L(K_{p,k})) = \{(k+\frac{p}{2})^{p-1}, (p+\frac{k}{2})^{k-1}\} \cup \sigma(B)$, where

$$B = \begin{pmatrix} k & -k \\ -p & p \end{pmatrix}.$$

Therefore, by Lemma 1, the reciprocal distance Laplacian spectrum of $RD^L(K_{p,k})$ is $\{(k+\frac{p}{2})^{p-1}, (p+\frac{k}{2})^{k-1}\}, n, 0\}$. □

The following theorem gives an upper bound for the reciprocal distance Laplacian spectral radius of a bipartite graph and the extremal graph for the equality.

Theorem 6. *Let G be a bipartite graph of order n with p and q vertices in its independent sets. Then $\lambda_1(RD^L(G)) \leq n$ with equality if and only if $G \cong K_{p,q}$.*

Proof. Case (i). If $p = 1$ or $q = 1$, then the only possible bipartite graph in this case is $K_{n-1,1}$. Then, by Lemma 6, n is a reciprocal distance Laplacian eigenvalue of $K_{n-1,1}$. So the equality always holds in this case.
Case (ii). If $G \cong K_{p,q}$, $p \geq 2$ and $q \geq 2$, then by Lemma 7, n is an eigenvalue of multiplicity 1, so the equality holds in this case. Let H be the graph obtained from $K_{p,q}$ by deleting an edge. Clearly, H is always connected, because $p \geq 2$ and $q \geq 2$. Using Lemma 2, we have $\lambda_1(RD^L(H)) \leq \lambda_1(RD^L(K_{p,q}))$. So it only remains to show that $\lambda_1(RD^L(H)) < n$. We have the following relation between the Laplacian reciprocal distance matrices of $K_{p,q}$ and H.

$$RD^L(H) = RD^L(K_{p,q}) + \begin{pmatrix} 0_{p-1\times p-1} & 0_{p-1\times 2} & 0_{p-1\times q-1} \\ 0_{2\times p-1} & M_{2\times 2} & 0_{2\times q-1} \\ 0_{q1\times p-1} & 0_{q-1\times 2} & 0_{q-1\times q-1} \end{pmatrix} = RD^L(K_{p,q}) + Y,$$

where $M = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{pmatrix}$ and $Y = \begin{pmatrix} 0_{p-1\times p-1} & 0_{p-1\times 2} & 0_{p-1\times q-1} \\ 0_{2\times p-1} & M_{2\times 2} & 0_{2\times q-1} \\ 0_{q-1\times p-1} & 0_{q-1\times 2} & 0_{q-1\times q-1} \end{pmatrix} = RD^L(G) + Y$

Using Lemma 3, it follows that $\lambda_1(RD^L(H)) \leq \lambda_1(RD^L(G) + \lambda_1(Y)$. Clearly, it can be seen that $-\frac{4}{3}$ and 0^{n-1} are eigenvalues of Y. By substituting $\lambda_1(Y) = 0$, we have

$$\lambda_1(RD^L(H)) \leq \lambda_1(RD^L(K_{p,q}) = n. \tag{3.1}$$

By Lemma 3, equality in above is possible if and only if there exists a common unit eigenvector $\mathbf{x}$ for the eigenvalues $\lambda_1(RD^L(H))$, n and 0 of the matrices $RD^L(H)$, $RD^L(K_{p,q}))$ and Y respectively. We can see that any eigenvector of Y corresponding to the eigenvalue 0 should have p^{th} and $(p+1)^{th}$ coordinate same. But the eigenvector of $RD^L(K_{p,q})$ corresponding to n is

$$\mathbf{x} = \Big(-\frac{k}{p}, -\frac{k}{p}, \ldots, -\frac{k}{p}, 1, \ldots, 1 \Big)^t,$$

which has $-\frac{k}{p}$ at the p^{th} place and 1 at the $(p+1)^{th}$ place. Since n is a simple eigenvalue of $RD^L(K_{p,q})$, so every eigenvector corresponding to n is a scalar multiple of x. Therefore no common unit eigenvector is possible corresponding to these eigenvalues. This completes the proof. □

In this theorem, we find the relationship between reciprocal distance Laplacian eigenvalues of $K'_{p+1,q-1}$ and $K_{p,q}$, where $K'_{p+1,q-1}$ is the complete bipartite graph obtained from $K_{p,q}$ by shifting one vertex from one independent set to another.

Theorem 7. *Let $K_{p,q}$ with $p \geq q$, be a complete bipartite graph of order n and $K'_{p+1,q-1}$ be a complete bipartite graph obtained from $K_{p,q}$ by shifting a vertex from one independent set to other. Then $\lambda_i(RD(K'_{p+1,q-1})) > \lambda_i(RD^L(K_{p,q}))$, for $i = 2, 3 \ldots, q-1$.*

Proof. Let $V_1 = \{v_1, v_2, \ldots, v_p\}$ and $V_2 = \{v_{p+1}, v_{p+2}, \ldots, v_n\}$ be the independent set of vertices in $K_{p,q}$ and $p \geq q$. Let $V_1' = \{v_1', v_2', \ldots, v_p', v_{p+1}'\}$ and $V_2' = \{v_{p+2}', \ldots, v_n'\}$ be independent set of vertices in $K_{p+1,q-1}'$ obtained from the vertex set of $K_{p,q}$ by shifting a vertex from V_2 to V_1. Then from Lemma 7, the reciprocal distance Laplacian spectrum of $K_{p+1,q-1}'$ is $(\{k + \frac{p+1}{2})^p, (\{p + \frac{p-1}{2}\})^{q-2}, n, 0\}$. Now comparing the spectrum of both the complete graphs as obtained in Lemma 7, we get the required result. □

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