

On the energy of the line graph of unitary Cayley graphs

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Abstract: The energy of a graph G is the sum of the absolute values of the eigenvalues of its adjacency matrix. The energy of the line graph of graph G is denoted by $E(L(G))$. The unitary Cayley graph X_n is a graph with the vertex set $Z_n = \{0, 1, \dots, n-1\}$ and the edge set $\{(a, b) : \gcd(a-b, n) = 1\}$. In this paper, we focus on the line graph of the unitary Cayley graph X_n and compute the spectrum of line graphs of X_n and its complement graph $\overline{X_n}$. We also obtain the energy of the line graph of X_n and $\overline{X_n}$.

Keywords: graph energy, unitary Cayley graph, spectrum, complement, line graph

AMS Subject classification: 05C50, 05C25

1. Introduction

Let G be a simple graph of order n with m edges. The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of the graph G are the eigenvalues of the adjacency matrix of graph G . Assume that $\lambda_1 > \lambda_2 > \dots > \lambda_t$ are a non-increasing sequence of eigenvalues of G and their multiplicities are m_1, m_2, \dots, m_t , respectively. The spectrum of G is written as

$$\text{Spec}(G) = \begin{pmatrix} \lambda_1 & \dots & \lambda_t \\ m_1 & \dots & m_t \end{pmatrix}.$$

In [7, 13], the energy of graph G is defined as $E(G) = \sum_{i=1}^t m_i |\lambda_i|$. The line graph of G , denoted by $L(G)$ is the graph that each vertex of it corresponding to an edge of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges in G have a common vertex [9]. The energy of the line graph of graph G is denoted by $E(L(G))$. That is, $E(L(G))$ is the sum of the absolute values of eigenvalues of $A(L(G))$ [1]. Some results are obtained on the energy of the line graphs that can be found in [6, 8, 14].

Let $n > 1$ be a positive integer. The unitary Cayley graph $X_n = \text{Cay}(Z_n, U_n)$ is defined by the additive symmetric group $Z_n = \{0, 1, \dots, n-1\}$ of integers modulo n

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and the multiplicative group U_n of its units. The vertex set of X_n is the elements of Z_n and the edge set is as $\{(a, b) : a, b \in Z_n, \gcd(a - b, n) = 1\}$ [12].

In [16], the energy of graph X_n is computed. The spectrum of X_n for $n = p^\alpha$ where p is a prime and $n \geq 1$ is obtained [16]. Ilić [11] calculated the energy of the complement of the unitary Cayley graph X_n . In [3], the authors obtained the minimum edge dominating energy of some Cayley graphs including the unitary Cayley graphs. In [15] by computing the eigenvalues and Laplacian eigenvalues of the unitary addition Cayley graph and its complement, some bounds for energy and Laplacian energy for these graphs were obtained. Chen and Huang [2] computed the formulas for the eigenvalues of the Unitary Cayley graph and using these values presented the energy, the Kirchhoff index, and the number of spanning trees of this graph.

In this paper, we focus on the line graphs of X_n and its complement $\overline{X_n}$. We compute the spectrum of $L(X_n)$ and $L(\overline{X_n})$. We also obtain the energy of the line graph of the unitary Cayley graph X_n and its complement $\overline{X_n}$.

2. Main Results

In this section, we compute the eigenvalues of the line graph of the unitary Cayley graph X_n and its complement graph. Then, we calculate the energy of the graphs $L(X_n)$ and $L(\overline{X_n})$ via their eigenvalues. To do this, we recall the following result on the eigenvalues of the line graph of a regular graph.

Lemma 1. [5] *Let G be a regular graph of degree $r \geq 2$ with n vertices and m edges. Then the following relations hold.*

$$(i) \text{ For } 1 \leq i \leq n, \lambda_i(L(G)) = \lambda_i(G) + r - 2,$$

$$(ii) \text{ for } n + 1 \leq i \leq m, \lambda_i(L(G)) = -2.$$

In the following theorem, we obtain the line graph spectrum of the unitary Cayley graph X_n for $n = p^\alpha$ where p is prime.

Theorem 1. *Let X_n be the unitary Cayley graph where $n = p^\alpha$ and p be prime. If $L(X_n)$ is the line graph of the unitary Cayley graph X_n , then the spectrum of $L(X_n)$ is given as follows*

$$\text{Spec}(L(X_n)) = \begin{pmatrix} 2\phi(n) - 2 & 2(\phi(n) - 1) - n & \phi(n) - 2 & -2 \\ 1 & p - 1 & n - p & \frac{n\phi(n)}{2} - n \end{pmatrix}.$$

where $\phi(n)$ is the Euler function.

Proof. Assume that X_n is the unitary Cayley graph of order $n = p^\alpha$ and size $m = \frac{n\phi(n)}{2}$ whose the degree of all vertices is $\phi(n)$.

If $\alpha = 1$, then the unitary Cayley graph X_p is the complete graph K_p . Note that the

spectrum of graph K_p is $p-1$ with multiplicity $p-1$. Since $n = p$ and $\phi(p) = p-1$, we can consider the following from $\text{Spec}(K_n)$.

$$\text{Spec}(K_n) = \begin{pmatrix} p-1 & -1 \\ 1 & p-1 \end{pmatrix} = \begin{pmatrix} \phi(n) & \phi(n)-n \\ 1 & n-1 \end{pmatrix}.$$

Since K_n is $(n-1)$ -regular graph, then using Lemma 1, we obtain the eigenvalues of the line graph $L(K_n)$ for $1 \leq i \leq \frac{n(n-1)}{2} = m$.

- i) $\lambda_1(L(K_n)) = \lambda_1(K_n) + \phi(n) - 2 = 2\phi(n) - 2$.
- ii) For $2 \leq i \leq p-1$, we have $\lambda_i(L(K_n)) = \lambda_i(K_n) + \phi(n) - 2 = \phi(n) - n + \phi(n) - 2 = 2\phi(n) - n - 2$.
- iii) For $n \leq i \leq \frac{n(n-1)}{2} = m$, we have $\lambda_i(L(K_n)) = -2$.

Therefore, the $\text{Spec}(L(K_n))$ is as follows

$$\text{Spec}(L(K_n)) = \begin{pmatrix} 2\phi(n) - 2 & 2(\phi(n) - 1) - n & -2 \\ 1 & p-1 & \frac{n\phi(n)}{2} - n \end{pmatrix}.$$

Now, we assume $\alpha \geq 2$. In the proof of theorem 3.1 in [16], the spectrum of the unitary Cayley graph X_n for $n = p^\alpha$ is obtained as follows

$$\text{Spec}(X_{p^\alpha}) = \begin{pmatrix} p^\alpha - p^{\alpha-1} & -p^{\alpha-1} & 0 \\ 1 & p-1 & p^\alpha - p \end{pmatrix}.$$

Since $n = p^\alpha$ and $\phi(n) = p^\alpha - p^{\alpha-1} = n - p^{\alpha-1}$, we can consider the following $\text{Spec}(X_n)$ where $n = p^\alpha$.

$$\text{Spec}(X_{p^\alpha}) = \begin{pmatrix} \phi(n) & \phi(n) - n & 0 \\ 1 & p-1 & n-p \end{pmatrix}.$$

Using Lemma 1, X_n is a $\phi(n)$ -regular graph. Thus we can obtain the eigenvalues of the line graph $L(X_n)$.

- i) $\lambda_1(L(X_n)) = \lambda_1(X_n) + \phi(n) - 2 = 2\phi(n) - 2$.
- ii) For $2 \leq i \leq p$, we have $\lambda_i(L(X_n)) = \lambda_i(X_n) + \phi(n) - 2 = \phi(n) - n + \phi(n) - 2 = 2(\phi(n) - 1) - n$.
- iii) For $p+1 \leq i \leq n$, we have $\lambda_i(L(X_n)) = \lambda_i(X_n) + \phi(n) - 2 = 0 + \phi(n) - 2 = \phi(n) - n$.
- iv) For $n+1 \leq i \leq m = \frac{n\phi(n)}{2}$, the eigenvalues of $L(X_n)$ are $\lambda_i(L(X_n)) = -2$.

Therefore, according to the above eigenvalues, the result is completed. \square

Theorem 2. *The energy of the line graph of the unitary Cayley graph X_n for $n = p^\alpha$ where p is prime, is given as follows*

$$E(L(X_n)) = 2n(\phi(n) - 2).$$

Proof. Let X_n be the $\phi(n)$ -regular graph with $n = p^\alpha$ vertices and $m = \frac{n\phi(n)}{2}$ edges. According to the definition, the energy of the line graph of a graph is equal to the sum of the absolute values of the eigenvalues of the adjacency matrix of its line graph. Using Theorem 1, we obtain the energy of $L(X_n)$, where $n = p^\alpha$.

For $\alpha = 1$ where $n = p$, we obtained the spectrum of $L(X_n)$ as follows

$$\text{Spec}(L(X_n)) = \begin{pmatrix} 2\phi(n) - 2 & 2(\phi(n) - 1) - n & -2 \\ 1 & p - 1 & \frac{p\phi(n)}{2} - p \end{pmatrix}.$$

Therefore, we get

$$\begin{aligned} E(L(X_p)) &= \sum_{i=1}^m |\lambda_i(X_p)| \\ &= 2\phi(n) - 2 + (p - 1)(2(\phi(n) - 1) - 2) \\ &\quad + \left(\frac{p\phi(n)}{2} - p\right)(| - 2|) \\ &= 2p\phi(n) - np + p\phi(n) - 3p \\ &= p(\phi(n) - n) + p(2\phi(n) - 3). \end{aligned} \quad (2.1)$$

With considering $n = p$ and $\phi(n) = p - 1$, we have from (2.1),

$$\begin{aligned} E(L(X_p)) &= p(\phi(n) - n) + p(2\phi(n) - 3) \\ &= p(p - 1 - p) + p(2\phi(n) - 3) \\ &= -p + p(2\phi(n) - 3) \\ &= p(2\phi(n) - 4) \\ &= 2n(\phi(n) - 2). \end{aligned}$$

For $\alpha \geq 2$, using Theorem 1, the spectrum of line graph $L(X_n)$ is as follows

$$\text{Spec}(L(X_n)) = \begin{pmatrix} 2\phi(n) - 2 & 2(\phi(n) - 1) - n & \phi(n) - 2 & -2 \\ 1 & p - 1 & n - p & \frac{n\phi(n)}{2} - n \end{pmatrix}.$$

Therefore, we get

$$\begin{aligned} E(L(X_{p^\alpha})) &= \sum_{i=1}^m |\lambda_i(X_{p^\alpha})| \\ &= 2\phi(n) - 2 + (p - 1)(2\phi(n) - 2 - n) + (n - p)(\phi(n) - 2) \\ &\quad + \left(\frac{n\phi(n)}{2} - n\right)(| - 2|) \\ &= p\phi(n) - pn - 3n + 2n\phi(n) \end{aligned}$$

$$= p(\phi(n) - n) + n(2\phi(n) - 3). \quad (2.2)$$

With putting $n = p^\alpha$ and $\phi(n) = p^\alpha - p^{\alpha-1}$ in (2.2), we have

$$\begin{aligned} E(L(X_{p^\alpha})) &= p(\phi(n) - n) + n(2\phi(n) - 3) \\ &= p(p^\alpha - p^{\alpha-1} - p^\alpha) + n(2\phi(n) - 3) \\ &= p(-p^{\alpha-1}) + n(2\phi(n) - 3) \\ &= -n + n(2\phi(n) - 3) \\ &= n(2\phi(n) - 4) \\ &= 2n(\phi(n) - 2). \end{aligned}$$

□

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. The direct product of G_1 and G_2 is the graph $G = (V, E)$ denoted by $G_1 \otimes G_2$ where $V = V_1 \times V_2$, the direct product of V_1 and V_2 with (v_1, v_2) and (u_1, u_2) are adjacent in G if and only if v_1, u_1 are adjacent in G_1 and v_2, u_2 are adjacent in G_2 . The following result is obtained about the direct product of the unitary Cayley graphs.

Lemma 2. [16] Assume that $(m, n) = 1$. If X_m and X_n are the unitary Cayley graphs, then $X_m \otimes X_n \simeq X_{mn}$.

Theorem 3. Let $L(X_m)$ and $L(X_n)$ be the line graph of the unitary Cayley graphs X_m and X_n , respectively. If $(m, n) = 1$, then

$$L(X_m) \otimes L(X_n) \simeq L(X_{mn}).$$

Proof. Since for two graphs G_1 and G_2 , if $G_1 \simeq G_2$ then $L(G_1) \simeq L(G_2)$. Thus by applying Lemma 2.2, it is sufficient to prove $L(X_m) \otimes L(X_n) \simeq L(X_m \otimes X_n)$. Therefore, we show that there is a one-to-one correspondence between the vertices and edges of two graphs $L(X_m) \otimes L(X_n)$ and $L(X_m \otimes X_n)$.

Assume that $e_1 = a_1b_1 \in V(L(Z_m)) = E(Z_m)$ and $e_2 = a_2b_2 \in V(L(Z_n)) = E(Z_n)$. Thus, $(a_1 - b_1, m) = 1 = (a_2 - b_2, n)$. Since $a_1 \sim b_1$ in Z_m and $a_2 \sim b_2$ in Z_n thus $(a_1, a_2) \sim (b_1, b_2)$ in $Z_m \otimes Z_n$.

Therefore, $e = ((a_1, a_2), (b_1, b_2)) \in E(Z_m \otimes Z_n) = V(L(Z_m \otimes Z_n))$. Consequently, any vertex $(e_1, e_2) = (a_1b_1, a_2b_2)$ in graph $L(Z_m) \otimes L(Z_n)$ corresponding to the vertex $e = ((a_1, a_2), (b_1, b_2))$ in $L(Z_m \otimes Z_n)$. Therefore, there is the bijective correspondence between vertex sets of $L(Z_m) \otimes L(Z_n)$ and $L(Z_m \otimes Z_n)$.

We show that there is a bijective correspondence between edges in the line graph $(Z_m \otimes Z_n)$ and the direct product of $L(Z_m)$ and $L(Z_n)$. Let $k \in E(L(Z_m \otimes Z_n))$. Thus, there are two vertices e and e' in $L(Z_m \otimes Z_n)$ such that $k = ee'$ where $e =$

$((a_1, b_1), (a_2, b_2))$ and $e' = ((a'_1, b'_1), (a'_2, b'_2))$ for $a_1, a_2, a'_1, a'_2 \in Z_m$ and $b_1, b_2, b'_1, b'_2 \in Z_n$. According to the line graph of a graph and without loss of generality assume that $(a_1, b_1) = (a'_1, b'_1)$. Therefore, $a_1 = a'_1$ and $b_1 = b'_1$.

On the other hand, the corresponding with vertices e and e' in $L(Z_m \otimes Z_n)$, we have $k_1 = (a_1 a_2, b_1 b_2) \in L(Z_m) \otimes L(Z_n)$ and $k_2 = (a'_1 a'_2, b'_1 b'_2) \in L(Z_m) \otimes L(Z_n)$. Therefore, it is sufficient to show that k_1 and k_2 are adjacent in $L(Z_m) \otimes L(Z_n)$. That is, we show $a_1 a_2 \sim a'_1 a'_2$ in $L(Z_m)$ and $b_1 b_2 \sim b'_1 b'_2$ in $L(Z_n)$. Using the definition of the line graph $L(Z_m)$ and since $a_1 = a'_1$ and $b_1 = b'_1$, then it is clear to have k_1 and k_2 are adjacent in $L(Z_m) \otimes L(Z_n)$. This completes the proof. \square

Corollary 1. *If $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$, then the direct product of unitary Cayley graph $L(X_{P_1^{\alpha_1}}) \otimes \dots \otimes L(X_{P_k^{\alpha_k}}) \simeq L(X_n)$.*

The tensor product $A \otimes B$ of the $r \times s$ matrix $A = (a_{ij})$ and the $t \times u$ matrix $B = (b_{ij})$ is defined as $rt \times su$ matrix got by replacing each entry a_{ij} of A with the double array $a_{ij}B$. For two graphs G_1 and G_2 , $A(G_1 \otimes G_2) = A(G_1) \otimes A(G_2)$.

Lemma 3. [16] *If G_1 and G_2 are any two graphs, then*

$$E(G_1 \otimes G_2) = E(G_1)E(G_2).$$

Theorem 4. *If $n > 1$ and $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ where p_i are distinct primes and α_i are positive integers for $1 \leq i \leq k$, then*

$$E(X_n) = 2^k n (\phi(n) - 2^k).$$

Proof. Let X_n be the unitary Cayley graph. Using Lemma 3, Corollary 1 and Theorem 2 we get

$$\begin{aligned} E(L(X_n)) &= E(L(X_{P_1^{\alpha_1}}) \otimes \dots \otimes L(X_{P_k^{\alpha_k}})) \\ &= E(L(X_{P_1^{\alpha_1}})) \dots E(L(X_{P_k^{\alpha_k}})) \\ &= \left(2P_1^{\alpha_1} (\phi(P_1^{\alpha_1}) - 2)\right) \dots \left(2P_k^{\alpha_k} (\phi(P_k^{\alpha_k}) - 2)\right) \\ &= 2^k (p_1^{\alpha_1} \dots p_k^{\alpha_k}) (\phi(p_1^{\alpha_1}) \dots \phi(p_k^{\alpha_k}) - 2^k) \\ &= 2^k n (\phi(p_1^{\alpha_1} \dots p_k^{\alpha_k}) - 2^k). \end{aligned}$$

Therefore, the result holds. \square

Graph G is said to be hyperenergetic if its energy exceeds the energy of the complete graph K_n equivalently $E(G) > 2n - 2$ [10]. Using the following lemma, we obtain Theorem 5 about hyperenergetic from the unitary Cayley graph X_n .

Lemma 4. [10] Let G be a graph of order $n \geq 5$ and of size m . If $m \geq 2n$, then $L(G)$ is hyperenergetic.

Theorem 5. Let X_n be the unitary Cayley graph where $n \geq 5$. If $\phi(n) \geq 4$, then $L(X_n)$ is hyperenergetic.

Proof. X_n is a graph of the order $n \geq 5$ and size $m = \frac{n\phi(n)}{2} \geq 2n$, therefore using Lemma 4, the result is completed. \square

Lemma 5. [4] If G is a r -regular with n vertices then

$$P_{\overline{G}}(x) = (-1)^n \frac{x - n + r + 1}{x + r + 1} P_G(-x - 1),$$

where $P_{\overline{G}}$ is the characteristic polynomial of the complement of the graph G .

Theorem 6. Let X_n be the unitary Cayley graph with $n = p^\alpha$ where $\alpha \geq 2$ and $p \geq 2$. Then the spectrum of the line graph of X_n is as follows

$$\text{Spec}(L(\overline{X_n})) = \begin{pmatrix} 2(n - \phi(n) - 2) & n - \phi(n) - 4 & -2 \\ p & n - p & \frac{n\phi(n)}{2} - n \end{pmatrix}.$$

Proof. Let X_n be the $\phi(n)$ -regular graph with n vertices. Using Lemma 5 and Theorem 3.1 in [16], the characteristic polynomial of $\overline{X_n}$ is as follows

$$\begin{aligned} P_{\overline{X_n}}(\lambda) &= (-1)^n \left(\frac{\lambda - n + \phi(n) + 1}{\lambda + \phi(n) + 1} \right) P_{X_n}(-\lambda - 1) \\ &= (-1)^n \left(\frac{\lambda - n + \phi(n) + 1}{\lambda + \phi(n) + 1} \right) (-\lambda - \phi(n) - 1) \\ &\quad (-\lambda - \phi(n) + n - 1)^{p-1} (-\lambda - 1)^{n-p} \\ &= (-1)^{2n} (\lambda - n + \phi(n) + 1) (\lambda + \phi(n) - n + 1)^{p-1} (\lambda + 1)^{n-p} \\ &= (\lambda + \phi(n) - n + 1)^p (\lambda + 1)^{n-p}. \end{aligned}$$

Therefore, the eigenvalues of $\overline{X_n}$ are $n - \phi(n) - 1$ with multiplicity p and -1 with multiplicity $n - p$. Since $\overline{X_n}$ is an $(n - 1) - \phi(n)$ -regular graph, then the eigenvalues of $L(\overline{X_n})$ are as follows

i) For $1 \leq i \leq p$, we get

$$\begin{aligned} \lambda_i(L(\overline{X_n})) &= \lambda_i(\overline{X_n}) + (n - 1) - \phi(n) - 2 \\ &= n - \phi(n) - 1 + (n - 1) - \phi(n) - 2 \\ &= 2n - 2\phi(n) - 4 = 2(n - \phi(n) - 2). \end{aligned}$$

ii) For $p + 1 \leq i \leq n$, we get

$$\begin{aligned}\lambda_i(L(\overline{X_n})) &= \lambda_i(\overline{X_n}) + (n - 1) - \phi(n) - 2 \\ &= -1 + (n - 1) - \phi(n) - 2 \\ &= n - \phi(n) - 4.\end{aligned}$$

iii) For $n + 1 \leq i \leq \frac{n\phi(n)}{2}$, we have $\lambda_i(L(\overline{X_n})) = -2$.

Therefore, the result holds. \square

Theorem 7. Let X_n be the unitary Cayley graph with $n = p^\alpha$ where $\alpha \geq 2$ and $p \geq 2$. If $\overline{X_n}$ is the complement of X_n , then the energy of $L(\overline{X_n})$ is given as follows

i) If $n = 4$, then $E(L(\overline{X_4})) = 6$.

ii) If $n = 9$, then $E(L(\overline{X_9})) = 58$.

iii) If $\alpha \geq 2$ and $p \geq 5$, then $E(L(\overline{X_n})) = n(n - 5)$.

Proof. Using Theorem 6, we get

$$\begin{aligned}E(L(\overline{X_{p^\alpha}})) &= \sum_{i=1}^m |\lambda_i(L(\overline{X_{p^\alpha}}))| \\ &= 2p|n - \phi(n) - 2| + (n - p)|n - \phi(n) - 4| + \left(\frac{n\phi(n)}{2} - n\right) - 2|.\end{aligned}$$

We consider the following cases.

Case 1. If $\alpha = 2$ and $p = 2, 3$, then

$$E(L(\overline{X_{p^2}})) = 2p(n - \phi(n) - 2) + (n - p)(4 - n + \phi(n)) + n\phi(n) + 2n.$$

With putting $\alpha = 2$ and $p = 2, 3$, we obtain $E(L(\overline{X_{2^2}})) = 6$ and $E(L(\overline{X_{3^2}})) = 58$.

Case 2. Let $\alpha \geq 2$ and $p \geq 5$. Then

$$\begin{aligned}E(L(\overline{X_{p^\alpha}})) &= 2p(n - \phi(n) - 2) + (n - p)(n - \phi(n) - 4) + n\phi(n) - 2n \\ &= pn - p\phi(n) + n^2 - 6n \\ &= p(n - \phi(n)) + (n^2 - 6n) \\ &= p(p^\alpha - p^\alpha + p^{\alpha-1}) + (n^2 - 6n) \\ &= p(p^{\alpha-1}) + n^2 - 6n \\ &= p^\alpha + n^2 - 6n \\ &= n + n^2 - 6n.\end{aligned}$$

Therefore, the result holds. \square

Theorem 8. Let $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ where $\alpha \geq 2$ and $p \geq 5$. Then

$$E(L(\overline{X_n})) = n^2 - 5^k n.$$

Proof. Similar to the proof from Theorem 3 and using Corollary 1 and Lemma 3, we get

$$E(L(\overline{X_n})) = E(L(\overline{X_{p_1^{\alpha_1}}}))E(L(\overline{X_{p_2^{\alpha_2}}})) \dots E(L(\overline{X_{p_k^{\alpha_k}}})).$$

By applying Theorem 7 in the above relation, we have

$$\begin{aligned} E(L(\overline{X_n})) &= \left((p_1^{\alpha_1})^2 - 5p_1^{\alpha_1} \right) \left((p_2^{\alpha_2})^2 - 5p_2^{\alpha_2} \right) \dots \left((p_k^{\alpha_k})^2 - 5p_k^{\alpha_k} \right) \\ &= p_1^{\alpha_1} \dots p_k^{\alpha_k} \left(\prod_{i=1}^k (p_i^{\alpha_i} - 5) \right) \\ &= n \left(\prod_{i=1}^k p_i^{\alpha_i} - \prod_{i=1}^k 5 \right) \\ &= n(n - 5^k). \end{aligned}$$

□

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