Communications in Combinatorics and Optimization
Vol. 11, No. 1 (2026), pp. 227-245
https://doi.org/10.22049/cco.2024.26112.1974

CCO
Commun. Comb. Optim.

Research Article

On cozero divisor graphs of ring $\mathbb{Z}_n$

Zahid Raza[1], Bilal Ahmad Rather[2], Modjtaba Ghorbani[3,*]

[1]Department of Mathematics, College of Sciences, University of Sharjah, UAE
zraza@sharjah.ac.ae

[2]Mathematical Sciences Department, College of Science, United Arab Emirates University, Al Ain 15551, Abu Dhabi, UAE
bilalahmadrr@gmail.com

[3]Department of Mathematics, Faculty of Science, Shahid Rajaee Teacher Training University, Tehran, 16785-163, I.R. Iran
*mghorbani@sru.ac.ir

Received: 24 January 2024; Accepted: 10 July 2024
Published Online: 20 July 2024

Abstract: The cozero divisor graph $\Gamma'(R)$ of a commutative ring R is a simple graph with vertex set as non-zero zero divisor elements of R such that two distinct vertices x and y are adjacent iff $x \notin Ry$ and $y \notin Rx$, where xR is the ideal generated by x. In this article we find the spectra of $\Gamma'(\mathbb{Z}_n)$ for $n \in \{q_1q_2, q_1q_2q_3, q_1^{n_1}q_2\}$, where q_i's are primes. As a consequence we obtain the bounds for the largest (smallest) eigenvalues, bounds for spread, rank and inertia of $\Gamma'(\mathbb{Z}_{q_1^{n_1}q_2})$ along with the determinant, inverse and square of trace of its quotient matrix. We present the extremal bounds for the energy of $\Gamma'(\mathbb{Z}_n)$ for $n = q_1^{n_1}q_2$ and characterize the extremal graphs attaining them. We close article with conclusion for furtherance.

Keywords: spectra, energy, cozero divisor graphs, commutative rings

AMS Subject classification: 05C25, 05C50

1. Introduction

We consider finite, simple and undirected graphs. A graph is denoted by G with vertex set $V(G)$ and edge set $E(G)$. The numbers $n = |V(G)|$ and $m = |E(G)|$ is order and size of G, respectively. An edge between vertices u and v is denoted by uv. A vertex of null degree is the isolated vertex and a vertex of degree one is a pendent vertex.

---

* *Corresponding Author*

The *degree* $d_{v_i}(G)$ (or simply d_i, if G is clear) of a vertex v_i is the number of vertices incident on it. The *union* of two graphs G_1 with vertex set V_1 edge set E_1 and G_2 with vertex set V_2, and edge set E_2, is denoted by $G_1 \cup G_2$, is a graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The *join* of G_1 and G_2 with vertex sets V_1 and V_2, denoted by $G_1 \vee G_2$, is a graph with vertex set $V_1 \cup V_2$ and edge set $E(G_1) \cup E(G_2) \cup \{u, v \mid u \in V(G_1), v \in V(G_2)\}$.
The adjacency matrix $A(G) = (e_{ij})_{n\times n}$ of G is a $(0,1)$-matrix, with (i,j)-th term 1, if $v_iv_j \in E(G)$ and 0, otherwise.
The set of all eigenvalues of $A(G)$ with repetitions is the *spectrum* of G. The eigenvalues $\lambda_i(A(G))$ (or λ_i) can be written in natural partial order as:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n.$$

Th eigenvalue λ_1 is the *spectral radius* of G. Furthermore, for a connected graph, Perron Frobenius theory guarantees that λ_1 is unique and components of its eigenvector are positive. Clearly, $\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2 = \|A(G)\|_F^2 = 2m$. The summation of the absolute value of λ_i's of $A(G)$ is the energy [10] of G, that is, $\mathcal{E}(G) = \sum_{i=1}^{n} \Big|\lambda_i\Big|$.
The invariant $\mathcal{E}(G)$ originate from quantum chemistry and is used for approximating the π-electron energy of alkanes. Besides its chemical importance, mathematically it represents trace norm of $A(G)$. For more about the energy of G, see book [12]. The spectral properties of graphs are very well studied, for some recent work see [15].
Section 2 gives the eigenvalues of $\Gamma'(\mathbb{Z}_n)$ for $n \in \{q_1q_2, q_1q_2q_3, q_1^{n_1}q_2\}$. In particular, we discuss the spectral properties of $\Gamma'(\mathbb{Z}_{q_1^{n_1}q_2})$ in more detail, the invariants like inertia, spread, bounds for λ_1 and λ_n are presented. In Section 3, bounds for $\mathcal{E}(\Gamma'(\mathbb{Z}_{q_1^{n_1}q_2}))$ are given in terms of the determinant and square of the trace of its quotient matrix and identify the candidate graphs attaining these bounds. We end up the article with conclusion for the future work.

2. Eigenvalues of $\Gamma'(\mathbb{Z}_n)$

We first discuss the structure of cozero divisor graphs. The cozero divisor graph are motivated by zero divisor graphs, which are defined as graph $\Gamma(R)$ associated to a ring R, with vertex set equal as non-zero zero divisors of R such that two vertices x and y form an edge xy iff $x \cdot y = 0$. The cozero divisor graph of a commutative ring R (with unity $1 \neq 0$) is a simple graph with vertex set as non-zero non-unit elements of R such that two vertices u and v $(u \neq v)$ are adjacent iff $u \notin Rv$ and $v \notin Ru$, where uR is the ideal generated by u. The cozero divisor graph of R is denoted by $\Gamma'(R)$. The basic properties of cozero divisor graphs including their complement graphs, planarity, characterization of rings with structures like forest, star or unicyclic cozero divisor graphs, their relations with comaximal graphs of rings and zero divisor graph were investigated by Afkhami and Khashyarmanesh [1–4]. Cozero divisor graphs of polynomial rings were discussed in [5], spectral analysis of cozero divisor graphs were carried in [13]. For some other progress of cozero divisor, see [6, 7, 14].

In general it is not easy to find the structure of $\Gamma'(R)$ completely, though for some special cases we can have some information about the structure of $\Gamma'(R)$ (especially for $\Gamma'(\mathbb{Z}_n)$), where $\mathbb{Z}_n$ is the integral modulo ring. Depending on the proper divisors $d_i, i \notin \{1, n\}$ of n, we divide $V(\Gamma'(\mathbb{Z}_n))$ into mutually disjoint vertex cells as:

$$A_{d_i} = \{a \in \mathbb{Z}_n \ : \ (a, n) = d_i\},$$

where (a, n) is the greatest common divisor of a and n. Clearly A_{d_i} are mutually pairwise disjoint and $V(\Gamma'(\mathbb{Z}_n)) = \bigcup_{i=1}^{t} A_{d_i}$, where t is the number of proper divisor of n. Furthermore, for $a, b \in A_{d_i}$, we have $\langle a\rangle = \langle b\rangle$. The cardinality of A_{d_i} is $\Theta\left(\frac{n}{d_i}\right)$ (see [16]), for $i = 1, 2, \ldots, t$, where $\Theta(\cdot)$ is an Euler function. Also, if $a \in A_{d_i}$ and $b \in A_{d_j}$ then a and b are adjacent in $\Gamma'(\mathbb{Z}_n)$ if and only if $d_i \nmid d_j$ and $d_j \nmid d_i$, for $i, j \in \{1, 2, \ldots, \tau(n)-2\}$, where $\tau(\cdot)$ is divisor function. For $i \in \{1, 2, \ldots, \tau(n)-2\}$, the induced subgraph of A_{d_i} is $\overline{K}_{\Theta\left(\frac{n}{d_i}\right)}$. Note that the order of $\Gamma'(\mathbb{Z}_n)$ is $N = n-\Theta(n)-1$. For more above the structural properties of $\Gamma'(\mathbb{Z}_n)$, we refer to [13].

Lemma 1 ([9]). *Let G be a graph with independent vertices $\{u_1, u_2, \ldots, u_k\}$ sharing the same set of neighbors. Then 0 is the eigenvalues of $A(G)$ with multiplicity at least $k-1$.*

For $n = q_1q_2$ with $q_1 < q_1$ and q_1, q_2 are primes, the cozero divisor of is a complete bipartite graph with partite sets $\Theta(q_1)$ and $\Theta(q_2)$ and its spectrum is known as

$$\left\{0^{[q_1q_2-2]}, \pm\sqrt{\Theta(q_1)\Theta(q_2)}\right\}. \tag{2.1}$$

The very first result gives the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ when n is a product of three primes.

Theorem 1. *The spectrum of $\Gamma'(\mathbb{Z}_n)$ for $n = q_1q_2q_3$ with primes $q_1 < q_2 < q_3$ consists of the eigenvalue 0 with multiplicity $N-6$ and the eigenvalues of the following matrix*

$$\begin{pmatrix} 0 & \Theta(q_1q_3) & \Theta(q_1q_2) & 0 & 0 & \Theta(q_1) \\ \Theta(q_2q_3) & 0 & \Theta(q_1q_2) & 0 & \Theta(q_2) & 0 \\ \Theta(q_2q_3) & \Theta(q_1q_3) & 0 & \Theta(q_3) & 0 & 0 \\ 0 & 0 & \Theta(q_1q_2) & 0 & \Theta(q_2) & \Theta(q_1) \\ 0 & \Theta(q_1q_3) & 0 & \Theta(q_3) & 0 & \Theta(q_1) \\ \Theta(q_2q_3) & 0 & 0 & \Theta(q_3) & \Theta(q_2) & 0 \end{pmatrix}. \tag{2.2}$$

Proof. We partition the vertex set of $\Gamma'(\mathbb{Z}_n)$ on the basis of the proper divisor of n as

$$\begin{aligned} A_{q_1} &= \{kq_1 \mid k = 1, 2, \ldots, q_2q_3-1, q_2 \nmid k, q_3 \nmid k\}, \\ A_{q_2} &= \{kq_2 \mid k = 1, 2, \ldots, q_1q_3-1, q_1 \nmid k, q_3 \nmid k\}, \\ A_{q_3} &= \{kq_3 \mid k = 1, 2, \ldots, q_1q_2-1, q_1 \nmid k, q_2 \nmid k\}, \\ A_{q_1q_2} &= \{kq_1q_2 \mid k = 1, 2, \ldots, q_3-1\}, \\ A_{q_1q_3} &= \{kq_1q_3 \mid k = 1, 2, \ldots, q_2-1\}, \\ A_{q_2q_3} &= \{kq_2q_3 \mid k = 1, 2, \ldots, q_1-1\}. \end{aligned}$$

As each induced subgraph of A_{d_i} is $\overline{K}_{\Theta\left(\frac{n}{d_i}\right)}$ and their cardinalities are $|A_{q_1}| = \Theta(q_2q_3) = (q_2-1)(q_3-1), |A_{q_2}| = (q_1-1)(q_3-1), |A_{q_3}| = (q_1-1)(q_2-1), |A_{q_1q_2}| = q_3-1, |A_{q_1q_3}| = q_2-1$ and $|A_{q_2q_3}| = q_1-1$. Also q_1 does not divide q_2, q_2 and q_2q_3, so each vertex of A_{q_1} is adjacent to all vertices of A_{q_2}, A_{q_3} and $A_{q_2q_3}$. Similarly q_2 does not divides q_1, q_3 and q_1q_3 and it implies that each vertex of A_{q_2} is adjacent to every vertex of A_{q_1}, A_{q_3} and $A_{q_1q_3}$. The divisor q_3 divides q_1q_3 and q_2q_3, so every vertex of A_{q_3} is adjacent to every vertex of A_{q_1}, A_{q_2} and $A_{q_1q_3}$. Likewise there are edges between each vertex of $A_{q_1q_3}$ with each vertex of $A_{q_1q_2}, A_{q_2q_3}$ and A_{q_2}, and edges between each vertex of $A_{q_2q_3}$ with each vertex of $A_{q_1q_2}$. This gives the structure of $\Gamma'(\mathbb{Z}_n)$ completely. As there vertices of A_{d_i} are independent set and each vertex of such an independent set have the common neighbourhood, so Lemma 1 gives that 0 is the eigenvalue of $\Gamma'(\mathbb{Z}_n)$ with multiplicity $N-6$.
Let $X = (x_1, \ldots, x_N)$ be the eigenvector of $A(\Gamma'(\mathbb{Z}_n))$ with $x_i = X(v_i)$, for $i = 1, 2, 3, \ldots, n$. Then with the structure of $\Gamma'(\mathbb{Z}_n)$ and its common neighbourhood sharing properties, we see that (see, [8]) each component of X that corresponds to all vertices of A_{d_i} is equal to x_i, for $i = 1, 2, \ldots, 6$. Therefore, from the eigenequation $A(G)X = \lambda X$, we have

$$\begin{aligned}
\lambda x_1 &= 0\cdot x_1 + \Theta(q_1q_3)x_2 + \Theta(q_1q_2)x_3 + 0\cdot x_4 + 0\cdot x_5 + \Theta(q_1)x_6,\\
\lambda x_2 &= \Theta(q_2q_3)x_1 + 0\cdot x_2 + \Theta(q_1q_2)x_3 + 0\cdot x_4 + \Theta(q_2)x_5 + 0\cdot x_6,\\
\lambda x_3 &= \Theta(q_2q_3)x_1 + \Theta(q_1q_3)x_2 + 0\cdot x_3 + \Theta(q_3)\cdot x_4 + 0\cdot x_5 + 0\cdot x_6,\\
\lambda x_4 &= 0\cdot x_1 + 0\cdot x_2 + \Theta(q_1q_2)x_3 + 0\cdot x_4 + \Theta(q_2)\cdot x_5 + \Theta(q_1)x_6,\\
\lambda x_5 &= 0\cdot x_1 + \Theta(q_1q_3)x_2 + 0\cdot x_3 + 0\cdot x_4 + \Theta(q_2)\cdot x_5 + \Theta(q_1)x_6,\\
\lambda x_6 &= \Theta(q_2q_3)x_1 + 0\cdot x_2 + 0\cdot x_3 + \Theta(q_3)x_4 + \Theta(q_2)x_5 + 0\cdot x_6,
\end{aligned}$$

The coefficient matrix of the right side of the above system of equations is

$$\begin{pmatrix}
0 & \Theta(q_1q_3) & \Theta(q_1q_2) & 0 & 0 & \Theta(q_1)\\
\Theta(q_2q_3) & 0 & \Theta(q_1q_2) & 0 & \Theta(q_2) & 0\\
\Theta(q_2q_3) & \Theta(q_1q_3) & 0 & \Theta(q_3) & 0 & 0\\
0 & 0 & \Theta(q_1q_2) & 0 & \Theta(q_2) & \Theta(q_1)\\
0 & \Theta(q_1q_3) & 0 & \Theta(q_3) & 0 & \Theta(q_1)\\
\Theta(q_2q_3) & 0 & 0 & \Theta(q_3) & \Theta(q_2) & 0
\end{pmatrix}.$$

□

Proposition 1. *The nullity of* $\Gamma'(\mathbb{Z}_n)$ *for* $n = q_1q_2q_3$ *with primes* $q_1 < q_2 < q_3$ *is* $N-4$.

Proof. By above theorem the multiplicity of $\Gamma'(\mathbb{Z}_n)$ is at least $n-6$. Also it is easy to verify that

$$\Big(\frac{\Theta(q_1)}{\Theta(q_2q_3)}, 0, -\frac{\Theta(q_1)}{\Theta(q_1q_2)}, -\frac{\Theta(q_1)}{\Theta(q_3)}, 0, 1\Big)$$

and

$$\Big(0, \frac{\Theta(q_2)}{\Theta(q_1q_3)}, -\frac{\Theta(q_2)}{\Theta(q_1q_2)}, -\frac{\Theta(q_2)}{\Theta(q_3)}, 1, 0\Big)$$

are the eigenvectors corresponding to th eigenvalue 0 and the result follows. □

The next consequence gives the number of (positive, negative and zero) eigenvalues of adjacency matrix, known as inertia of $A(\Gamma'(\mathbb{Z}_n))$.

Corollary 1. *The inertia of* $\Gamma'(\mathbb{Z}_n)$ *for* $n = q_1q_2q_3$ *with primes* $q_1 < q_2 < q_3$ *is* $(2, 2, N-4)$.

We will explain Theorem 1 with the help of the following example.

Example 1. For $n = 2 \cdot 3 \cdot 7 = 41$, the spectrum of $\Gamma'(\mathbb{Z}_n)$ consist of the eigenvalue 0 with multiplicity 23 and the eigenvalues of Q given below

$$Q = \begin{pmatrix} 0 & 6 & 2 & 0 & 0 & 1 \\ 12 & 0 & 2 & 2 & 0 & 0 \\ 12 & 6 & 0 & 0 & 6 & 0 \\ 0 & 6 & 0 & 0 & 6 & 1 \\ 0 & 0 & 2 & 2 & 0 & 1 \\ 12 & 0 & 0 & 2 & 6 & 0 \end{pmatrix}.$$

For $n = 42$, the order of $\Gamma'(\mathbb{Z}_n)$ is $N = n - \Theta(n) - 1 = 29$ and A_{d_i}'s are

$$\begin{aligned} A_2 &= \{2, 4, 8, 10, 16, 20, 22, 26, 32, 34, 38, 40\}, \\ A_3 &= \{3, 9, 15, 27, 33, 39\}, A_7 = \{7, 14\}, \\ A_{2\cdot 3} &= \{6, 12, 18, 24, 30, 36\}, \\ A_{2\cdot 7} &= \{14, 28\}, A_{3\cdot 7} = \{21\}. \end{aligned}$$

By Theorem 1, 0 is the eigenvalues of $\Gamma'(\mathbb{Z}_{105})$ with multiplicity 15 and the eigenvalues of Q are

$$\{12.9874, -10.0154, -6.71108, 3.7391, 0, 0\}.$$

Thus the spectrum of $\Gamma'(\mathbb{Z}_{105})$ is completely determined and inertia is $(2, 2, 17)$.
Next subsequent result gives the spectrum of $\Gamma'(\mathbb{Z}_n)$ for $n = q_1^{n_1}q_2$ and $n = q_1q_2^{n_2}$ can be similarly discussed.
For this, we have the following result.

Theorem 2. *Let $n = q_1^{n_1}q_2$ (or $n = q_1q_2^{n_2}$, n_2 is a positive integer) where q_1, q_2 and primes and n_1 is a positive integer. Then $\Gamma'(\mathbb{Z}_n)$ is a bipartite graph.*

Proof. For $n = q_1^{n_1}q_2$ with $q_1 < q_2$, the proper divisors of n are

$$q_1^i, i = 1, 2, \dots, n_1, q_2$$

and

$$q_1^j q_2, j = 1, 2, \dots, n_1 - 1.$$

We consider the following sets based on these divisors

$$A_{q_1^i} = \{a \in \mathbb{Z}_n \ : \ (a, n) = q_1^i\} \quad \text{and} \quad A_{q_1^{i-1}q_2} = \{b \in \mathbb{Z}_n \ : \ (b, n) = q_1^{i-1}q_2\},$$

where $i = 1, 2, \dots, n_1$. Labelling these sets by $A_i = A_{q_1^{n_1-i+1}}$ and $B_i = A_{q_1^{i-1}q_2}$, for $i = 1, 2, \dots, n_1$. The cardinality of A_i is $\Theta(p^{i-1}q)$ and that of B_i is $\Theta(p^{n_1-i+1})$, for $i = 1, 2, \dots, n_1$. Also A_i's induce a totally disconnected graph of order $\sum_{i=1}^{n_1} \Theta(p^{n_1-i}q) = p^{n_1}(q-1)$, since $\sum_{i=1}^{\eta} \Theta(p^i) = p^\eta - 1$, for prime p. B_i's induce a totally disconnected graph of order $\sum_{i=1}^{n_1} p^i = p^{n_1} - 1$. This implies that no vertex of any A_i is adjacent to any vertex of A_j, for each $i < j$, since $p^j = cp^i$, where c is some scaler. Similarly, no vertex of B_i is adjacent to any vertex of B_j, for each i and j. Therefore the vertex of set of $\Gamma'(\mathbb{Z}_n)$ can be partitioned in to subsets $\bigcup_i A_i$ and $\bigcup_i B_j$ and there are edges only between them. Thus, we obtain that $\Gamma'(\mathbb{Z}_n)$ is a bipartite graph. Similar analysis if true for $n = q_1q_2^{n_2}$, where n_2 is a positive integer. □

The above result along with (2.1) implies that $\Gamma'(\mathbb{Z}_n)$ has 3 distinct eigenvalues iff $n = q_1q_2$, a parallel of well know fact that a bipartite graph has three distinct λ_i's iff it is complete bipartite.
The following result gives the spectrum of $\Gamma'(\mathbb{Z}_n)$ for $n = q_1^{n_1}q_2$.

Theorem 3. *If $n = q_1^{n_1}q_2$, then the spectrum of $\Gamma'(\mathbb{Z}_n)$ consist of the eigenvalue 0 with multiplicity $(q_2 - 1)q_1^{n_1-1} + q_1^{n_1} - 1 - 2n_1$ and the eigenvalues of (2.3).*

Proof. From Theorem 2, there are adjacency relation only between A_i's and B_j's for some i and j. The divisor $q_1^{n_1}$ is not multiple of any $q_1^{n_1-i}q_2$, for $i = 1, 2, \dots, n_1$. So, the vertices of A_1 are adjacent to all the vertices of B_i, $i = 1, 2, \dots, n_1$. For $i = 1, 2, \dots, n_1 - 2$, the divisor $q_1^{n_1-1}$ is adjacent to $q_1^{n_1-i}q_2$ except $q_1^{n_1-1}q_2$, it implies that the vertices of A_2 are adjacent to all B_i except $i = n_1$. Similarly, the set A_{n_1} containing some multiples of q_1 is adjacent only to set B_1, the set A_{n_1-1} is adjacent to sets B_1 and B_2 and so on. Thus, in general the adjacency among A_i's and B_i's can be represented by the relation: each vertex of A_i is adjacent to every vertex of

$\bigcup_{j=1}^{n_1-(i-1)} B_j$, for $i = 1, 2, \ldots, n_1$. Thus, the relations of adjacency between A_i's and B_j's in $\Gamma'(\mathbb{Z}_n)$ are completely known and its order is

$$N = n - \Theta(n) - 1 = q_1^{n_1} + q_1^{n_1-1} q_2 - q_1^{n_1-1} - 1.$$

By Lemma 1, 0 is the eigenvalue of $\Gamma'(\mathbb{Z}_n)$ with repetition

$$\sum_{i=0}^{n_1-1} \Theta(q_1^i q_2) - n_1 + \sum_{i=1}^{n_1} \Theta(q_1^i) - n_1 = (q_2 - 1) q_1^{n_1-1} + q_1^{n_1} - 1 - 2n_1,$$

since Θ is multiplicative and $\sum_{i=1}^{t} \Theta(p^i) = p^t - 1$, where p is prime. Labelling the vertices from A_i's to B_j's. Let $X = (x_1, \ldots, x_N)$ be the eigenvector of $A(\Gamma'(\mathbb{Z}_n))$. Then with the structure of $\Gamma'(\mathbb{Z}_n)$ and keep in view the common neighbourhood sharing vertices of A_i's, each component of X (see, [8]) that relates to each vertex of A_i is equal to x_i, for $i = 1, 2, \ldots, n_1$ and every vertex of A_j is equal to x_j, that is,

$$\begin{aligned} X = (&\underbrace{x_1, \ldots, x_1}_{\Theta(q_2)}, \underbrace{x_2, \ldots, x_2}_{\Theta(q_1 q_2)}, \ldots, \underbrace{x_{n_1}, \ldots, x_{n_1}}_{\Theta(q_1^{n_1-1} q_2)}, \underbrace{x_{n_1+1}, \ldots, x_{n_1+1}}_{\Theta(q_1^{n_1})}, \\ &\underbrace{x_{n_1+2}, \ldots, x_{n_1+2}}_{\Theta(q_1^{n_1-1})}, \ldots, \underbrace{x_{2n_1}, \ldots, x_{2n_1}}_{\Theta(q_1)}). \end{aligned}$$

Thus by $A(G)X = \lambda X$, we have

$$\begin{aligned}
\lambda x_1 &= \Theta(q_1^{n_1}) x_1 + \Theta(q_1^{n_1-1}) x_2 + \Theta(q_1^{n_1-2}) x_3 + \cdots + \Theta(q_1^3) x_{n_1-2} \\
&\quad + \Theta(q_1^2) x_{n_1-1} + \Theta(q_1) x_{n_1} \\
\lambda x_2 &= \Theta(q_1^{n_1}) x_1 + \Theta(q_1^{n_1-1}) x_2 + \Theta(q_1^{n_1-2}) x_3 + \cdots + \Theta(q_1^3) x_{n_1-2} \\
&\quad + \Theta(q_1^2) x_{n_1-1} \\
\lambda x_3 &= \Theta(q_1^{n_1}) x_1 + \Theta(q_1^{n_1-1}) x_2 + \Theta(q_1^{n_1-2}) x_3 + \cdots + \Theta(q_1^3) x_{n_1-2} \\
&\quad \vdots \\
\lambda x_{n_1-2} &= \Theta(q_1^{n_1}) x_1 + \Theta(q_1^{n_1-1}) x_2 + \Theta(q_1^{n_1-2}) x_3 \\
\lambda x_{n_1-1} &= \Theta(q_1^{n_1}) x_1 + \Theta(q_1^{n_1-1}) x_2, \quad \lambda x_{n_1} = \Theta(q_1^{n_1}) x_1 \\
\lambda x_{n_1+1} &= \Theta(q_2) x_{n_1+1} + \Theta(q_1 q_2) x_{n_1+2} + \Theta(q_1^2 q_2) x_{n_1+3} + \ldots \\
&\quad + \Theta(q_1^{n_1-3} q_2) x_{n_1-2} + \Theta(q_1^{n_1-2} q_2) x_{n_1-1} + \Theta(q_{n_1-1} q_2) x_{n_1} \\
\lambda x_{n_1+2} &= \Theta(q_2) x_{n_1+1} + \Theta(q_1 q_2) x_{n_1+2} + \Theta(q_1^2 q_2) x_{n_1+3} + \ldots \\
&\quad + \Theta(q_1^{n_1-3} q_2) x_{n_1-2} + \Theta(q_1^{n_1-2} q_2) x_{n_1-1} \\
\lambda x_{n_1+3} &= \Theta(q_2) x_{n_1+1} + \Theta(q_1 q_2) x_{n_1+2} + \Theta(q_1^2 q_2) x_{n_1+3} + \ldots \\
&\quad + \Theta(q_1^{n_1-3} q_2) x_{n_1-2} \\
&\quad \vdots \\
\lambda x_{2n_1-2} &= \Theta(q_2) x_{n_1+1} + \Theta(q_1 q_2) x_{n_1+2} + \Theta(q_1^2 q_2) x_{n_1+3} \\
\lambda x_{2n_1-1} &= \Theta(q_2) x_{n_1+1} + \Theta(q_1 q_2) x_{n_1+2}, \quad \lambda x_{2n_1} = \Theta(q_2) x_{n_1+1}.
\end{aligned}$$

The coefficient matrix of the above system of equations is given by

$$M = \begin{pmatrix} \mathbf{0}_{n_1\times n_1} & A_{n_1\times n_1} \\ B_{n_1\times n_1} & \mathbf{0}_{n_1\times n_1} \end{pmatrix}, \tag{2.3}$$

where

$$A = \begin{pmatrix} \Theta(q_1^{n_1}) & \Theta(q_1^{n_1-1}) & \Theta(q_1^{n_1-2}) & \dots & \Theta(q_1^{3}) & \Theta(q_1^{2}) & \Theta(q_1) \\ \Theta(q_1^{n_1}) & \Theta(q_1^{n_1-1}) & \Theta(q_1^{n_1-2}) & \dots & \Theta(q_1^{3}) & \Theta(q_1^{2}) & 0 \\ \Theta(q_1^{n_1}) & \Theta(q_1^{n_1-1}) & \Theta(q_1^{n_1-2}) & \dots & \Theta(q_1^{3}) & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \Theta(q_1^{n_1}) & \Theta(q_1^{n_1-1}) & \Theta(q_1^{n_1-2}) & \dots & 0 & 0 & 0 \\ \Theta(q_1^{n_1}) & \Theta(q_1^{n_1-1}) & 0 & \dots & 0 & 0 & 0 \\ \Theta(q_1^{n_1}) & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} \Theta(q_2) & \Theta(q_1q_2) & \Theta(q_1^2q_2) & \dots & \Theta(q_1^{n_1-3}q_2) & \Theta(q_1^{n_1-2}q_2) & \Theta(q_1^{n_1-1}q_2) \\ \Theta(q_2) & \Theta(q_1q_2) & \Theta(q_1^2q_2) & \dots & \Theta(q_1^{n_1-3}q_2) & \Theta(q_1^{n_1-2}q_2) & 0 \\ \Theta(q_2) & \Theta(q_1q_2) & \Theta(q_1^2q_2) & \dots & \Theta(q_1^{n_1-3}q_2) & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \Theta(q_2) & \Theta(q_1q_2) & \Theta(q_1^2q_2) & \dots & 0 & 0 & 0 \\ \Theta(q_2) & \Theta(q_1q_2) & 0 & \dots & 0 & 0 & 0 \\ \Theta(q_2) & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

□

By repeated cofactor expansion across $2n_1$-th row of M, then $2n_1 - 1$ and so on, we obtain the determinant formulae for Matrix (2.3). We make it precise in the following result.

Proposition 2. *The determinant of the matrix M given in (2.3) is*

$$\det(M) = (-1)^{n_1} \prod_{i=1}^{n_1} \Theta(q_1^{n_i})\Theta(q_1^{n_i-1}q_2).$$

Proposition 3. *The square of the trace of the matrix M given in (2.3) is*

$$tr(M^2) = 2\sum_{j=1}^{n_1} \sum_{i=1}^{n_1-(j-1)} \Theta(q_1^i)\Theta(q_1^{j-1}q_2).$$

Proof. By definition of $M^2 = (m_{ij}^2)$, we have

$$\begin{aligned} m_{11}^2 = \Theta(q_2)\Big(&\Theta(q_1^{n_1}) + \Theta(q_1^{n_1-1}) + \Theta(q_1^{n_1-2}) + \dots + \Theta(q_1^3) \\ &+ \Theta(q_1^2) + \Theta(q_1)\Big) \end{aligned}$$

$$
\begin{aligned}
m^2_{22} &= \Theta(q_1q_2)\Big(\Theta(q_1^{n_1}) + \Theta(q_1^{n_1-1}) + \Theta(q_1^{n_1-2}) + \dots \\
&\quad + \Theta(q_1^3) + \Theta(q_1^2)\Big) \\
&\ \vdots \\
m^2_{n_1n_1} &= \Theta(q_1^{n_1})\Theta(q_1^{n_1-1}q_2) \\
m^2_{(n_1+1)(n_1+1)} &= \Theta(q_1^{n_1})\Big(\Theta(q_2) + \Theta(q_1q_2) + \Theta(q_1^2q_2) + \dots + \Theta(q_1^{n_1-3}q_2) \\
&\quad + \Theta(q_1^{n_1-2}q_2) + \Theta(q_1^{n_1-1}q_2)\Big) \\
m^2_{(n_1+2)(n_1+2)} &= \Theta(q_1^{n_1-1})\Big(\Theta(q_2) + \Theta(q_1q_2) + \Theta(q_1^2q_2) + \dots + \Theta(q_1^{n_1-3}q_2) \\
&\quad + \Theta(q_1^{n_1-2}q_2)\Big) \\
&\ \vdots \\
m^2_{2n_12n_1} &= \Theta(q_2)\Theta(q_1).
\end{aligned}
$$

Now, sum all above expression, we obtain the result. □

It is very interesting and challenging to find the inertia of a general Hermitian matrix. In this direction, we have the following consequence of Theorems 2 and 3 for $\Gamma'(\mathbb{Z}_n)$

Corollary 2. *The inertia of $A(\Gamma'(\mathbb{Z}_n))$ for $n = q_1^{n_1}q_2$ is $(n_1, n_2, N - 2n_1)$.*

Corollary 3. *The rank of M given in (2.3) is $2n_1$.*

Next result gives the inverse of the matrix M given in (2.3).

Proposition 4. *Let M be defined as in (2.3). Then the inverse of M is*

$$
\mathcal{M}' = \begin{pmatrix} \boldsymbol{0}_{n_1\times n_1} & A'_{n_1\times n_1} \\ B'_{n_1\times n_1} & \boldsymbol{0}_{n_1\times n_1} \end{pmatrix},
$$

where

$$
A' = \begin{pmatrix}
0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{\Theta(q_2)} \\
0 & 0 & 0 & \dots & 0 & \frac{1}{\Theta(q_1q_2)} & -\frac{1}{\Theta(q_1q_2)} \\
0 & 0 & 0 & \dots & \frac{1}{\Theta(q_1^2q_2)} & -\frac{1}{\Theta(q_1^2q_2)} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \frac{1}{\Theta(q_1^{n_1-3}q_2)} & \dots & 0 & 0 & 0 \\
0 & \frac{1}{\Theta(q_1^{n_1-2}q_2)} & -\frac{1}{\Theta(q_1^{n_1-2}q_2)} & \dots & 0 & 0 & 0 \\
\frac{1}{\Theta(q_1^{n_1-1}q_2)} & -\frac{1}{\Theta(q_1^{n_1-1}q_2)} & 0 & \dots & 0 & 0 & 0
\end{pmatrix}
$$

$$B' = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{\Theta(q_1^{n_1})} \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{\Theta(q_1^{n_1-1})} & -\frac{1}{\Theta(q_1^{n_1-1})} \\ 0 & 0 & 0 & \dots & \frac{1}{\Theta(q_1^{n_1-2})} & -\frac{1}{\Theta(q_1^{n_1-2})} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \frac{1}{\Theta(q_1^3)} & \dots & 0 & 0 & 0 \\ \frac{1}{\Theta(q_1^2)} & -\frac{1}{\Theta(q_1^2)} & 0 & \dots & 0 & 0 & 0 \\ \frac{1}{\Theta(q_1)} & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

Proof. It is easy to see that $M\mathcal{M}' = I_{2n_1}$, where I is an identity matrix and result follows. □

Lemma 2 ([8]). *For a symmetric $Z \in \mathbb{R}^{n\times n}$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $\{1,2,\dots,n\} = W_1 \cup W_2 \cup \cdots \cup W_k$ be a partition such that $|W_i| = n_i > 0$, and consider block matrix $Z = (B_{i,j})$, where $B_{i,j}$ is an $n_i \times n_j$ block for $i,j = 1,2\dots k$. Let $b_{i,j}$ be the sum of rows in $B_{i,j}$ and form a new matrix $Q = \left(\frac{b_{i,j}}{n_i}\right)$ for $i,j = 1,2\dots,k$. The eigenvalues of Q and Z satisfy $\lambda_i(Z) \geq \xi_i(Q) \geq \lambda_n - k - i$ for $i = 1,2,\dot{;}k$ where ξ_i is the i-th largest eigenvalue of Q. Furthermore, if $B_{i,j}$ has constant row sums $b_{i,j}$, then the spectrum of Q is subset of the spectrum of Z.*

We note that the matrix M given in (2.3) is same as the quotient matrix of $A(\Gamma'(\mathbb{Z}_n))$ and the largest and the smallest eigenvalue of $A(\Gamma'(\mathbb{Z}_n))$ are the eigenvalues of M. Next, we establish the bounds for them in the following result.

Theorem 4. *Let M be the matrix given in (2.3) of Theorem 3. Then*

$$\lambda_1(A(\Gamma'(\mathbb{Z}_n))) \geq \frac{1}{n_1}\sqrt{\sum_{i=1}^{n_1}\ell_i \sum_{i=1}^{n_1} l_i} \quad \textit{and} \quad \lambda_n(A(\Gamma'(\mathbb{Z}_n))) \leq -\frac{1}{n_1}\sqrt{\sum_{i=1}^{n_1}\ell_i \sum_{i=1}^{n_1} l_i},$$

where ℓ_i is the i-th row sum of the matrix A and l_i is the i-th row sum of the matrix B given in (2.3). The equality holds iff $\ell_1 == \ell_2 = \dots \ell_{n_1}$ and $l_1 = l_2 = \cdots = l_{n_1}$, that is, same as saying n is the product of two distinct primes.

Proof. Consider the partition $\{\{1,2,\dots,n_1\},\{1,2,\dots,n_1\}\}$ of the index set $\{1,2,\dots,2n_1\}$ of the matrix M given in (2.3). The the quotient matrix of M with this partition is

$$M' = \begin{pmatrix} 0 & \frac{1}{n_1}\sum_{i=1}^{n_1}\ell_i \\ \frac{1}{n_1}\sum_{i=1}^{n_1} l_i & 0 \end{pmatrix},$$

where ℓ_i is the i-th row of the matrix A and l_i is the i-th row sum of the matrix B given in (2.3). Clearly, the eigenvalues of M' are $\lambda = \pm\frac{1}{n_1}\sqrt{\sum_{i=1}^{n_1}\ell_i\sum_{i=1}^{n_1}l_i}$. By Lemma 2 and its interlacing property, we have

$$\lambda_1(M) \geq \lambda_1(M') \geq \lambda_2(M) \geq \lambda_2(M') \geq \lambda_3(M) \geq \lambda_4(M) \geq \cdots \geq \lambda_{2n_1}(M),$$

which in turn gives us

$$\lambda_1(A(\Gamma'(\mathbb{Z}_n))) \geq \frac{1}{n_1}\sqrt{\sum_{i=1}^{n_1}\ell_i\sum_{i=1}^{n_1}l_i} \quad \text{and} \quad \lambda_n(A(\Gamma'(\mathbb{Z}_n))) \leq -\frac{1}{n_1}\sqrt{\sum_{i=1}^{n_1}\ell_i\sum_{i=1}^{n_1}l_i}.$$

The equality holds iff the partition $\{\{1,2,\ldots,n_1\},\{1,2,\ldots,n_1\}\}$ is equitable and in this case each eigenvalue of M' is the eigenvalue of M (Lemma 2), that is, the matrices A and B given in Theorem 3 have constant row sum. Thus with this condition, we must have $\ell_1 = \ell_2 = \cdots = \ell_{n_1}$ and $l_1 = l_2 = \cdots = l_{n_1}$. From $\ell_1 = \ell_2$, we get $\Theta(q_1^{n_1}) + \Theta(q_1^{n_1-1}) + \Theta(q_1^{n_1-2}) + \cdots + \Theta(q_1^3) + \Theta(q_1^2) + \Theta(q_1) = \Theta(q_1^{n_1}) + \Theta(q_1^{n_1-1}) + \Theta(q_1^{n_1-2}) + \cdots + \Theta(q_1^3) + \Theta(q_1^2)$, we obtain $\Theta(q_1) = 0$. Also, from $\ell_2 = \ell_3, \ell_3 = \ell_4, \ldots, \ell_{n_1-1} = \ell_{n_1}$, we have $\Theta(q_1^2) = 0, \Theta(q_1^3) = 0, \ldots, \Theta(q_1^{n_1-1}) = 0$ and in this case $\Theta(q_1^{n_1})$ remains arbitrary. Similarly, from $l_1 = l_2, l_2 = l_3, \ldots, l_{n_1-1} = l_{n_1}$, we obtain $\Theta(q_1^{n_1-1}q_2) = 0, \Theta(q_1^{n_1-2}q_2) = 0, \Theta(q_1q_2) = 0$ and $\Theta(q_2)$ remains non-zero. Thus, with this information, we see that there is only one non-empty set A_1 and only one non-empty set B_1 and there are edges from each vertex of A_1 to every vertex of B_1. In this case the matrix M' is an equitable quotient matrix and each eigenvalues of M' is the eigenvalues of $A(\Gamma'(\mathbb{Z}_n))$. By Theorem 2, it follows that $\Gamma'(\mathbb{Z}_n)$ is the complete bipartite graph. □

The spread of the adjacency matrix of a graph G with eigenvalues $\lambda_n(G) \leq \lambda_{n-1}(G) \leq \cdots \leq \lambda_1(G)$ is given as $S(G) = \lambda_1(G) - \lambda_n(G)$.

Corollary 4. *The spread of $\Gamma'(\mathbb{Z}_n)$ is given as*

$$S(\Gamma'(\mathbb{Z}_n)) \geq \frac{2}{n_1}\sqrt{\sum_{i=1}^{n_1}\ell_i\sum_{i=1}^{n_1}l_i},$$

with equality holding iff n is the product of two distinct primes.

We will illustrate all above results with the help of the following example.

Example 2. For $n = 48 = 2^4 \cdot 3$, the spectrum of $\Gamma'(\mathbb{Z}_n)$ consists of the eigenvalues 0 with multiplicity 23 and the eigenvalues of the matrix

For $n = 48$, the order of $\Gamma'(\mathbb{Z}_n)$ is $n - \Theta(n) - 1 = 31$. The independent vertex partitions of $\Gamma'(\mathbb{Z}_n)$ are

$$\begin{aligned} A_2 &= \{2, 10, 14, 22, 26, 34, 38, 46\}, A_{2^2} = \{4, 20, 28, 44\}, A_{2^3} = \{8, 40\}, \\ A_{2^4} &= \{16, 32\}, A_3 = \{3, 9, 15, 21, 27, 33, 39, 45\}, A_{2\cdot 3} = \{6, 18, 30, 42\}, \\ A_{2^2\cdot 3} &= \{12, 36\}, A_{2^3\cdot 3} = \{24\}. \end{aligned}$$

By Theorem 3, 0 is eigenvalue of $\Gamma'(\mathbb{Z}_{48})$ with multiplicity $(q_2-1)q_1^{n_1-1} + q_1^{n_1} - 1 - 2n_1 = 2\cdot 2^3 + 2^4 - 1 - 8 = 23$ and the other eigenvalues of $\Gamma'(\mathbb{Z}_{48})$ are the eigenvalues of matrix given below

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & \Theta(2^4) & \Theta(2^3) & \Theta(2^2) & \Theta(2) \\ 0 & 0 & 0 & 0 & \Theta(2^4) & \Theta(2^3) & \Theta(2^2) & 0 \\ 0 & 0 & 0 & 0 & \Theta(2^4) & \Theta(2^3) & 0 & 0 \\ 0 & 0 & 0 & 0 & \Theta(2^4) & 0 & 0 & 0 \\ \Theta(3) & \Theta(2\cdot 3) & \Theta(2^2\cdot 3) & \Theta(2^3\cdot 3) & 0 & 0 & 0 & 0 \\ \Theta(3) & \Theta(2\cdot 3) & \Theta(2^2\cdot 3) & 0 & 0 & 0 & 0 & 0 \\ \Theta(3) & \Theta(2\cdot 3) & 0 & 0 & 0 & 0 & 0 & 0 \\ \Theta(3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.4}$$

The determinant of above matrix is $\Theta(2)\Theta(2^2)\Theta(2^3)\Theta(2^4)\Theta(3)\Theta(2\cdot 3)\Theta(2^2\cdot 3)\Theta(2^3\cdot 3)$ and trace of $\mathcal{M}^2$ is $2\Theta(q_2)\sum_{i=1}^{4}\Theta(q_1^i) + 2\Theta(q_1q_2)\sum_{i=1}^{3}\Theta(q_1^i) + 2\Theta(q_1^2q_2)\sum_{i=1}^{2}\Theta(q_1^i) + 2\Theta(q_1^3q_2)\Theta(q_1)$. The inertia triplet is $(4, 4, 23)$. By Theorem 4, $\lambda_1(\Gamma'(\mathbb{Z}_{48})) \geq \frac{1}{4}\sqrt{49\cdot 30}$, $\lambda_n(\Gamma'(\mathbb{Z}_{48})) \leq -\frac{1}{4}\sqrt{49\cdot 30}$ and $S(A(\mathbb{Z}_{48}))) \geq \frac{1}{2}\sqrt{49\cdot 30}$. The inverse of $\mathcal{M}$ is

$$M' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\Theta(3)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\Theta(2\cdot 3)} & -\frac{1}{\Theta(2\cdot 3)} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\Theta(2^2\cdot 3)} & -\frac{1}{\Theta(2^2\cdot 3)} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\Theta(2^3\cdot 3)} & -\frac{1}{\Theta(2^3\cdot 3)} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\Theta(2^4)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\Theta(2^3)} & -\frac{1}{\Theta(2^3)} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\Theta(2^2)} & -\frac{1}{\Theta(2^2)} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\Theta(2)} & -\frac{1}{\Theta(2)} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3. Energy of cozero divisor graphs of $\mathbb{Z}_n$

By Theorem 3, $N-2$ eigenvalues of $\Gamma'(\mathbb{Z}_n), n = q_1^{n_1}q_2$ are zero and the non-zero eigenvalues only come from (2.3). Thus finding the energy of $\Gamma'(\mathbb{Z}_n)$ is same as finding the energy of M given in (2.3). In order to present the bounds for $\mathcal{E}(\Gamma'(\mathbb{Z}_n))$, we have the following known sequence of inequalities.

Let $S = \{t_1, t_2, t_3, \dots, t_r\}, t_i > 0$ be the set of r numbers in $\mathbb{R}$ and let S_s be the average of products of s-element subset of S, that is,

$$\begin{aligned} S_1 &= \frac{t_1 + t_2 + t_3 + \cdots + t_r}{r}, \\ S_2 &= \frac{1}{\frac{n(n-1)}{2}} \Big(t_1 t_2 + t_1 t_3 + \cdots + t_1 t_n + t_2 t_3 + \cdots + t_{r-1} t_r \Big), \\ &\vdots \\ S_r &= t_1 t_2 \dots t_r. \end{aligned}$$

The Maclaurin symmetric mean inequality given below relates S_i's with themselves. For positive real numbers $t_1, t_2, t_3, \dots, t_r$, we have the following chain of inequalities

$$S_r^{\frac{1}{r}} \le S_{r-1}^{\frac{1}{r-1}} \le \cdots \le S_3^{\frac{1}{3}} \le S_2^{\frac{1}{2}} \le S_1. \tag{3.1}$$

Equalities hold iff $t_1 = t_2 = \cdots = t_r$.
The following couple of results give the energy for the cozero divisor graphs of $\mathbb{Z}_n$ for $n = q_1^{n_1} q_2$, where $q_1 < q_2$ are primes and n_1 is a positive integer.

Theorem 5. *Let $\Gamma'(\mathbb{Z}_n)$ be the cozero graph with $n = q_1^{n_1} q_2$. Then we have the following.*

(i)

$$\mathcal{E}(\Gamma'(\mathbb{Z}_n)) \ge 2 \Big(\sum_{j=1}^{n_1} \sum_{i=1}^{n_1-(j-1)} \Theta(q_1^i)\Theta(q_1^{j-1} q_2) + n_1(n_1-1) \Big(\prod_{i=1}^{n_1} \Theta(q_1^{n_i})\Theta(q_1^{n_i-1} q_2) \Big)^{\frac{1}{n_1}} \Big)^{\frac{1}{2}},$$

equality holds iff $n_1 = 1$.

(ii)

$$\mathcal{E}(\Gamma'(\mathbb{Z}_n)) \le 2 \left(n_1 \sum_{j=1}^{n_1} \sum_{i=1}^{n_1-(j-1)} \Theta(q_1^i)\Theta(q_1^{j-1} q_2) \right)^{\frac{1}{2}},$$

with equality iff $n_1 = 1$.

Proof. By Theorem 2, $\Gamma'(\mathbb{Z}_n)$ is a bipartite graph for $n = q_1^{n_1} q_2$. By definition, we have

$$\mathcal{E}(\Gamma'(\mathbb{Z}_n)) = 2 \Big(\lambda_1(M) + \lambda_2(M) + \cdots + \lambda_{n_1}(M) \Big), \tag{3.2}$$

where $\lambda_i(M)$ with $1 \le i \le n_1$ are the positive eigenvalues of the matrix M given in (2.3).
From Proposition 3, we have

$$tr(M^2) = \sum_{i=1}^{2n_1} \lambda_i^2(M) = 2 \sum_{j=1}^{n_1} \sum_{i=1}^{n_1-(j-1)} \Theta(q_1^i)\Theta(q_1^{j-1} q_2),$$

and thereby we get

$$\sum_{i=1}^{n_1} \lambda_i^2(M) = \sum_{j=1}^{n_1} \sum_{i=1}^{n_1-(j-1)} \Theta(q_1^i)\Theta(q_1^{j-1}q_2).$$

Also, by Proposition 2, we have

$$\prod_{i=1}^{2n_1} \lambda_i(M) = \det(M) = (-1)^{n_1} \prod_{i=1}^{n_1} \Theta(q_1^{n_i})\Theta(q_1^{n_i-1}q_2),$$

that is equivalent to

$$\prod_{i=1}^{n_1} \lambda_i(M) = \left(\prod_{i=1}^{n_1} \Theta(q_1^{n_i})\Theta(q_1^{n_i-1}q_2)\right)^{\frac{1}{2}}.$$

By Inequality 3.1, we have

$$\frac{1}{\frac{n_1(n_1-1)}{2}} \sum_{1\leq i<j\leq n_1} \lambda_i(\Gamma'(\mathbb{Z}_n))\lambda_j(\Gamma'(\mathbb{Z}_n)) \geq \left(\prod_{i=1}^{n_1} \lambda_i(\Gamma'(\mathbb{Z}_n))\right)^{\frac{2}{n_1}}, \tag{3.3}$$

with equality iff $\lambda_1(\Gamma'(\mathbb{Z}_n)) = \lambda_2(\Gamma'(\mathbb{Z}_n)) = \cdots = \lambda_{n_1}(\Gamma'(\mathbb{Z}_n))$. Using above information, we obtain

$$2\sum_{1\leq i<j\leq n_1} \lambda_i(\mathbb{Z}_n))\lambda_j(\mathbb{Z}_n)) \geq n_1(n_1-1)\left(\prod_{i=1}^{n_1} \Theta(q_1^{n_i})\Theta(q_1^{n_i-1}q_2)\right)^{\frac{1}{n_1}}.$$

Thereby with Equation (3.2), we have

$$\begin{aligned}
\mathcal{E}(\Gamma'(\mathbb{Z}_n)) &= 2\left(\Big(\sum_{i=1}^{n_1} \lambda_i(\Gamma'(\mathbb{Z}_n))\Big)^2\right)^{\frac{1}{2}} \\
&= 2\left(\sum_{i=1}^{n_1} \lambda_i(\Gamma'(\mathbb{Z}_n))^2 + 2\sum_{1\leq i<j\leq n_1} \lambda_i(\Gamma'(\mathbb{Z}_n))\lambda_j(\Gamma'(\mathbb{Z}_n))\right)^{\frac{1}{2}} &\quad (3.4)\\
&\geq 2\left(\sum_{j=1}^{n_1} \sum_{i=1}^{n_1-(j-1)} \Theta(q_1^i)\Theta(q_1^{j-1}q_2) + n_1(n_1-1)\left(\prod_{i=1}^{n_1} \Theta(q_1^{n_i})\Theta(q_1^{n_i-1}q_2)\right)^{\frac{1}{n_1}}\right)^{\frac{1}{2}}. &\quad (3.5)
\end{aligned}$$

For $n_1 = 1$, by (2.1), we have

$$\mathcal{E}(\Gamma'(\mathbb{Z}_n)) = 2(\Theta(q_1)\Theta(q_2))^{\frac{1}{2}},$$

and equality holds in this case.
Suppose equality holds in (3.5), then equality holds in (3.3) and in this case $\lambda_1(\Gamma'(\mathbb{Z}_n)) = \lambda_2(\Gamma'(\mathbb{Z}_n)) = \cdots = \measuredangle_{n_1}(\Gamma'(\mathbb{Z}_n))$. By Lemma 2, $(\Gamma'(\mathbb{Z}_n))$ is bipartite and it follows that $(\Gamma'(\mathbb{Z}_n))$ has one positive eigenvalue, one negative eigenvalue and by Theorem 3 other eigenvalues are zero. Thus $(\Gamma'(\mathbb{Z}_n))$ is bipartite and has three distinct eigenvalues. Therefore it follows that $n_1 = 1$ and $n = q_1 q_1$, that is, $(\Gamma'(\mathbb{Z}_n))$ is the complete bipartite graphs, besides it is well known that: if a graph is bipartite, then it has 3 distinct eigenvalues iff it is complete bipartite.
For the lower bound, from 3.1, we have

$$\left(\frac{1}{n_1}\sum_{i=1}^{n_1}\lambda_i(\Gamma'(\mathbb{Z}_n))\right)^2 \geq \frac{2}{n_1(n_1-1)}\sum_{1\leq i<j\leq n_1}\lambda_i(\Gamma'(\mathbb{Z}_n))\lambda_j(\Gamma'(\mathbb{Z}_n)), \tag{3.6}$$

or,

$$\begin{aligned} n_1(n_1-1)\left(\sum_{i=1}^{n_1}\lambda_i(\Gamma'(\mathbb{Z}_n))\right)^2 &\geq 2n_1\sum_{1\leq i<j\leq n_1}\lambda_i(\Gamma'(\mathbb{Z}_n))\lambda_j(\Gamma'(\mathbb{Z}_n)) \\ &= n_1\left(\left(\sum_{i=1}^{n_1}\lambda_i(\Gamma'(\mathbb{Z}_n))\right)^2 - \sum_{i=1}^{n_1}\lambda_i(\Gamma'(\mathbb{Z}_n))^2\right), \end{aligned}$$

which is thereby equal to

$$\left(\sum_{i=1}^{n_1}\lambda_i(\Gamma'(\mathbb{Z}_n))\right)^2 \leq n_1\sum_{i=1}^{n_1}\lambda_i(\Gamma'(\mathbb{Z}_n))^2.$$

Hence, we obtain

$$\begin{aligned} \mathcal{E}((\Gamma'(\mathbb{Z}_n))) &\leq 2\left(n_1\sum_{i=1}^{n_1}\lambda_i(\Gamma'(\mathbb{Z}_n))^2\right)^{\frac{1}{2}} \\ &= 2\left(n_1\sum_{j=1}^{n_1}\sum_{i=1}^{n_1-(j-1)}\Theta(q_1^i)\Theta(q_1^{j-1}q_2)\right)^{\frac{1}{2}}. \end{aligned}$$

The equality case is same as in part (i). □

The following result relating arithmetic-geometric mean inequality can be seen in [11].

Lemma 3. *Let $\varsigma_1, \varsigma_2, \ldots, \varsigma_\beta$ be non-negative numbers. Then*

$$\begin{aligned} \beta\left[\frac{1}{\beta}\sum_{j=1}^{\beta}\varsigma_j - \left(\prod_{j=1}^{\beta}\varsigma_j\right)^{\frac{1}{\beta}}\right] &\leq \beta\sum_{j=1}^{\beta}\varsigma_j - \left(\sum_{j=1}^{\beta}\sqrt{\varsigma_j}\right)^2 \\ &\leq \beta(\beta-1)\left[\frac{1}{\beta}\sum_{j=1}^{\beta}\varsigma_j - \left(\prod_{j=1}^{\beta}y_j\right)^{\frac{1}{\beta}}\right], \end{aligned}$$

with equality iff $\varsigma_1 = \varsigma_2 = \cdots = \varsigma_\beta$.

In the following result, we have bounds for th energy of $\Gamma'(\mathbb{Z}_n)$ for $n = q_1^{n_1} q_2$.

Theorem 6. *Let $\Gamma'(\mathbb{Z}_n)$ be the zero divisor graph with $n = q_1^{n_1} q_2$. Then*

$$\mathcal{E}(\Gamma'(\mathbb{Z}_n)) \geq \left(2 \sum_{j=1}^{n_1} \sum_{i=1}^{n_1-(j-1)} \Theta(q_1^i)\Theta(q_1^{j-1}q_2) + 2n_1(2n_1-1)\left(\prod_{i=1}^{n_1} \Theta(q_1^{n_i})\Theta(q_1^{n_i-1}q_2)\right)^{\frac{1}{2n_1}} \right)^{\frac{1}{2}},$$

and

$$\mathcal{E}(\Gamma'(\mathbb{Z}_n)) \leq \left((4n_1-2) \sum_{j=1}^{n_1} \sum_{i=1}^{n_1-(j-1)} \Theta(q_1^i)\Theta(q_1^{j-1}q_2) + 2n_1\left(\prod_{i=1}^{n_1} \Theta(q_1^{n_i})\Theta(q_1^{n_i-1}q_2)\right)^{\frac{1}{2n_1}} \right)^{\frac{1}{2}}.$$

Both inequalities are equalities iff $n_1 = 1$.

Proof. Since M has n_1 positive eigenvalues and n_2 positive eigenvalues and noting that $\sum_{i=1}^{n_1} \lambda_i^2(M) = \sum_{j=1}^{n_1} \sum_{i=1}^{n_1-(j-1)} \Theta(q_1^i)\Theta(q_1^{j-1}q_2)$ and $\prod_{i=1}^{n_1} \lambda_i(M) = \left(\prod_{i=1}^{n_1} \Theta(q_1^{n_i})\Theta(q_1^{n_i-1}q_2)\right)^{\frac{1}{2}}$. By putting $\varsigma_i = |\lambda_i^2(M)| = \lambda_i(M)^2$, for $1 \leq i \leq 2n_1$ in Lemma 3, we obtain

$$2n_1 \left(\frac{1}{2n_1} \sum_{i=1}^{2n_1} \lambda_i^2(M) - \left(\prod_{i=1}^{2n_1} \lambda_i^2(M) \right)^{\frac{1}{2n_1}} \right) \leq 2n_1 \sum_{i=1}^{2n_1} \lambda_i^2(M) - \left(\sum_{i=1}^{2n_1} \Big|\lambda_i(M)\Big| \right)^2,$$

and

$$2n_1 \sum_{i=1}^{2n_1} \lambda_i^2(M) - \left(\sum_{i=1}^{2n_1} |\lambda_i(M)| \right)^2 \leq 2n_1(2n_1-1) \left(\frac{1}{2n_1} \sum_{i=1}^{2n_1} \lambda_i^2(M) - \left(\prod_{i=1}^{2n_1} \lambda_i^2(M) \right)^{\frac{1}{2n_1}} \right).$$

The above inequalities can be further modified as

$$4n_1 \sum_{i=1}^{n_1} \lambda_i^2(M) - \Big(\mathcal{E}(\Gamma'(\mathbb{Z}_n))\Big)^2 \geq 2n_1 \left(\frac{1}{n_1} \sum_{i=1}^{n_1} \lambda_i^2(M) - \left(\prod_{i=1}^{n_1} \lambda_i^2(M) \right)^{\frac{1}{n_1}} \right),$$

and

$$4n_1 \sum_{i=1}^{n_1} \lambda_i^2(M) - \Big(\mathcal{E}(\Gamma'(\mathbb{Z}_n))\Big)^2 \leq 2n_1(2n_1-1) \left(\frac{1}{n_1} \sum_{i=1}^{n_1} \lambda_i^2(M) - \left(\prod_{i=1}^{n_1} \lambda_i^2(M) \right)^{\frac{1}{n_1}} \right).$$

Therefore, we have

$$\mathcal{E}(\Gamma'(\mathbb{Z}_n)) \leq \left((4n_1-2)\sum_{i=1}^{n_1}\lambda_i^2(M) + 2n_1\left(\prod_{i=1}^{n_1}\lambda_i^2(M)\right)^{\frac{1}{n_1}}\right)^{\frac{1}{2}}$$
$$= \left((4n_1-2)\sum_{j=1}^{n_1}\sum_{i=1}^{n_1-(j-1)}\Theta(q_1^i)\Theta(q_1^{j-1}q_2) + 2n_1\left(\prod_{i=1}^{n_1}\Theta(q_1^{n_i})\Theta(q_1^{n_i-1}q_2)\right)^{\frac{1}{2n_1}}\right)^{\frac{1}{2}},$$

and

$$\mathcal{E}(\Gamma'(\mathbb{Z}_n)) \geq \left(2\sum_{i=1}^{n_1}\lambda_i^2(M) + 2n_1(2n_1-1)\left(\prod_{i=1}^{n_1}\lambda_i^2(M)\right)^{\frac{1}{n_1}}\right)^{\frac{1}{2}}$$
$$= \left(2\sum_{j=1}^{n_1}\sum_{i=1}^{n_1-(j-1)}\Theta(q_1^i)\Theta(q_1^{j-1}q_2) + 2n_1(2n_1-1)\left(\prod_{i=1}^{n_1}\Theta(q_1^{n_i})\Theta(q_1^{n_i-1}q_2)\right)^{\frac{1}{2n_1}}\right)^{\frac{1}{2}}.$$

That proves both the inequalities.
Equalities hold iff equality occurs in Lemma 3, which is equivalent to $\lambda_1(M) = \lambda_2(M) = \cdots = \lambda_{n_1}(M)$ and $-\lambda_1(M) = -\lambda_2(M) = \cdots = -\lambda_{n_1}(M)$. By Theorem 3, 0 is another eigenvalue of $\Gamma'(\mathbb{Z}_n)$, it follows that it has three distinct eigenvalues. Which in turn implies that $\Gamma'(\mathbb{Z}_n)$ is the complete bipartite graph and we get $n_1 = 1$ and $n = q_1q_2$. Conversely, it is easy to see that equality holds for $n = q_1^{n_1}q_2$ along with $n_1 = 1$. Similarly, equality holds for second part. □

4. Conclusion

This article gives the spectrum and the energy of the cozero divisor graphs. In particular a deep analysis is carried for $\Gamma'(\mathbb{Z}_n)$ with $n = q_1^{n_1}q_2$, its spectrum is discussed, bound for the largest and the smallest eigenvalues are obtained, bound on spread is presented and inertia is given.
The findings highlight the bipartite nature of $\Gamma'(\mathbb{Z}_n)$ for $n = q_1^{n_1}q_2$, which significantly influences its spectral properties. The analysis of the quotient matrix M, derived from the adjacency matrix of $\Gamma'(\mathbb{Z}_n)$, proves crucial in obtaining bounds for the eigenvalues and energy of the graph.
The determinant, inverse and rank of the quotient matrix is obtained, trace of square of the quotient matrix is given. The bounds for the energy of $A(\Gamma'(\mathbb{Z}_n))$ is obtained and the values for which bounds are sharp are characterized. However, the spectral analysis for other values of n are yet to be discussed and the structure of $\Gamma'(\mathbb{Z}_n)$ for general n seems complicated. These problems may be considered in future work.
Our results provide a comprehensive understanding of the spectral properties of $\Gamma'(\mathbb{Z}_n)$ for a wide range of values of n, which is crucial in understanding the structure

and behavior of this graph. The results also have implications for future research on cozero divisor graphs and their applications in algebra and graph theory.
The study of cozero divisor graphs is an active area of research, and there are many open questions and conjectures that remain to be resolved. Future work could involve extending our results to other types of rings, such as local rings or rings with a specific structure, or studying the properties of cozero divisor graphs in more depth. Additionally, exploring the connections between cozero divisor graphs and other areas of mathematics, such as algebraic geometry or combinatorial optimization, could lead to new insights and applications.

Acknowledgements: Z. R. is partially supported by the University of Sharjah Research Grant No. 23021440148 and MASEP Research Group. Also M. G. Partially supported by Shahid Rajaee Teacher Training University under grant number 27043854.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] M. Afkhami and K. Khashyarmanesh, *The cozero divisor graph of a commutative ring*, Southeast Asian Bull. Math. **35** (2011), no. 5, 753–762.

[2] ———, *On the cozero-divisor graphs of commutative rings and their complements*, Bull. Malays. Math. Sci. Soc. **35** (2012), no. 4, 935–944.

[3] ———, *Planar, outerplanar, and ring graph of the cozero-divisor graph of a finite commutative ring*, J. Algebra Appl. **11** (2012), no. 6, Article ID: 1250103. https://doi.org/10.1142/S0219498812501034.

[4] ———, *On the cozero-divisor graphs and comaximal graphs of commutative rings*, J. Algebra Appl. **12** (2013), no. 3, Article ID: 1250173. https://doi.org/10.1142/S0219498812501733.

[5] S. Akbari, F. Alizadeh, and S. Khojasteh, *Some results on cozero-divisor graph of a commutative ring*, J. Algebra Appl. **13** (2014), no. 3, Article ID: 1350113. https://doi.org/10.1142/S0219498813501132.

[6] S. Akbari and S. Khojasteh, *Commutative rings whose cozero-divisor graphs are unicyclic or of bounded degree*, Comm. Algebra **42** (2014), no. 4, 1594–1605.

[7] M. Bakhtyiari, R. Nikandish, and M.J. Nikmehr, *Coloring of cozero-divisor graphs of commutative von Neumann regular rings*, Proc. Math. Sci. **130** (2020), no. 1, Article ID: 49.
https://doi.org/10.1007/s12044-020-00569-5.

[8] D.M. Cvetković, P. Rowlinson, and S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge University Press, Cambridge, 2009.

[9] K.C. Das and P. Kumar, *Some new bounds on the spectral radius of graphs*, Discrete Math. **281** (2004), no. 1-3, 149–161.
https://doi.org/10.1016/j.disc.2003.08.005.

[10] I. Gutman, *The energy of a graph*, Ber. Math. Statist. Sekt. Forsch. Graz **103** (1978), 1–22.

[11] H. Kober, *On the arithmetic and geometric means and on Hölder's inequality*, Proceedings of the American Mathematical Society **9** (1958), no. 3, 452–459.
https://doi.org/10.2307/2033003.

[12] X. Li, Y. Shi, and I. Gutman, *Graph Energy*, Springer, New York, 2012.

[13] P. Mathil, B. Baloda, and J. Kumar, *On the cozero-divisor graphs associated to rings*, AKCE Int. J. Graphs Comb. **19** (2022), no. 3, 238–248.
https://doi.org/10.1080/09728600.2022.2111241.

[14] R. Nikandish, M.J. Nikmehr, and M. Bakhtyiari, *Metric and strong metric dimension in cozero-divisor graphs*, Mediterr. J. Math. **18** (2021), no. 3, Article ID: 112.
https://doi.org/10.1007/s00009-021-01772-y.

[15] S. Shen, W. Liu, and W. Jin, *Laplacian eigenvalues of the unit graph of the ring* $\mathbb{Z}_n$, Appl. Math. Comput. **459** (2023), Article ID: 128268.
https://doi.org/10.1016/j.amc.2023.128268.

[16] M. Young, *Adjacency matrices of zero-divisor graphs of integers modulo* n, Involve **8** (2015), no. 5, 753–761.
https://doi.org/10.2140/involve.2015.8.753.