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Research Article

Some new families of KP-digraphs

Germán Benítez-Bobadilla[1,*], Hortensia Galeana-Sánchez[2], César Hérnandez-Cruz [1,†]

[1] Facultad de Ciencias, Universidad Nacional Autónoma de México, Ciudad de México, México
[*] german@ciencias.unam.mx
[†] chc@ciencias.unam.mx

[2] Instituto de Matemáticas, Universidad Nacional Autónoma de México, Ciudad de México, México
hgaleana@matem.unam.mx

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Abstract: A kernel N of a digraph D is an independent set of vertices that is absorbent (for every vertex $u \in V(D) \setminus N$, there is a vertex $v \in N$ such that $(u, v) \in A(D)$). Let D be a digraph such that every proper induced subdigraph has a kernel. If D has a kernel, then D is a *kernel perfect digraph* (KP-digraph); otherwise, D is a *critical kernel imperfect digraph* (CKI-digraph). A digraph with the property P is a digraph such that whenever a vertex reaches two other vertices through asymmetrical arcs, then these two vertices have the same out-neighborhood. In particular, digraphs whose asymmetrical part is a disjoint union of cycles have the property P. In this work, KP-digraphs with the property P are characterized. As a consequence, KP-digraphs whose asymmetrical part is a Hamiltonian cycle are also characterized. For digraphs with a Hamiltonian cycle γ as their asymmetrical part and whose diagonals are symmetrical of length 2, two algorithms are presented; the first one determines whether a digraph is a KP-digraph or a CKI-digraph, and the second constructs the kernel of the original digraph if it is a KP-digraph. As a consequence, a characterization of all CKI-digraphs whose asymmetrical part is a Hamiltonian cycle and whose diagonals are symmetrical of length 2 is shown.

Keywords: kernel, kernel perfect digraph, circulant digraph, kernel imperfect digraph.

AMS Subject classification: 05C20, 05C69

1. Introduction

In this work, we will consider finite digraphs with neither multiple arcs nor loops. For general concepts, we refer the reader to [3, 4]. Let D be a digraph and (x, y) be an arc of D, we write $x \to_D y$. We say that (x, y) is *symmetrical* if (y, x) is an arc of D

[*] *Corresponding Author*

and it is denoted by $x \leftrightarrow_D y$, otherwise we say that (x, y) is *asymmetrical*, denoted by $x \mapsto_D y$. In cases where it is not necessary to specify the digraph in which we are working, the subscript will be omitted. The $Asym(D)$ is the spanning subdigraph of D such that the arcs of $Asym(D)$ are the asymmetrical arcs of D, and it is called the *asymmetrical part* of D.

Throughout this work, we only consider directed walks, directed paths, and directed cycles. The asymmetrical directed cycle with order n will be denoted by $\overrightarrow{C}_n$. Let $W = (v_0, v_1, \ldots, v_n)$ be a walk of a digraph D; we will denote by (v_i, W, v_j) the walk $(v_i, v_{i+1}, \ldots, v_j)$ contained in W. If W_1 is a walk from u to v and W_2 is a walk from v to w, then the union or the concatenation of W_1 with W_2 will be denoted by $W_1 \cup W_2$. If for every two different vertices u and v of a digraph D, there are a path from u to v and a path from v to u, then we say that D is *strongly connected* or *strong*. The family of Hamiltonian digraphs is highly studied, where one of the most famous problems is the directed traveling salesman problem (DTSP) [20], but there are different applications, for example in connectivity [15] and genomics [5].

Consider $\mathbb{Z}_m$ the cyclic group of integers modulo m, with $m \geq 2$. Let J be a nonempty subset of $\mathbb{Z}_m \setminus \{0\}$. The digraph $\overrightarrow{C}_m(J)$ is defined by $V(\overrightarrow{C}_m(J)) = \mathbb{Z}_m$ and $A(\overrightarrow{C}_m(J)) = \{(i, j) : \ i, j \in \mathbb{Z}_m, j - i \in J\}$, and it is called is *circulant (or rotational) digraph*. The family of circulant digraphs has been extensively studied due to their applications, in particular, sufficient conditions for the existence of Hamiltonian cycles can be found in [22]. In addition to the fact that circulant digraphs are a special case of Cayley digraphs, which are used to represent algebraic groups [9, 18]. A *directed n-antihole* is the digraph $\overrightarrow{A}_n$ such that $\overrightarrow{A}_n = \overrightarrow{C}_n(J)$ where $J = \{1, \ldots, n-2\}$, with $n \geq 3$. Observe that $\overrightarrow{A}_3 = \overrightarrow{C}_3$. The family of directed n-antihole, denoted by $\mathcal{A}$, is a special case of circulant digraphs.

Let D be a digraph and $\gamma = (x_0, x_1, \ldots, x_{n-1}, x_0)$ be a cycle of D. A *diagonal* of γ is an arc in $A(D) \setminus A(\gamma)$ whose ending vertices belong to $V(\gamma)$. If a diagonal of γ is a symmetrical arc of D, then we say that the diagonal is symmetrical. The length of a diagonal (x_i, x_j) is $|j - i|$, operations are taken modulo n. We say that a diagonal (x_i, x_j) *crosses* the diagonal (x_s, x_k) if and only if $i \in \{s+1, \ldots, k-1\}$ and $j \in \{k+1, \ldots, s-1\}$, or $i \in \{k+1, \ldots, s-1\}$ and $j \in \{s+1, \ldots, k-1\}$, operations are taken modulo n.

Let D be a digraph and S be a subset of $V(D)$. We say that S is absorbent if for every vertex $u \in V(D) \setminus S$ there is a vertex $v \in S$ such that $(u, v) \in A(D)$, and S is an independent set of D if for any pair of different vertices of S there is no arc between them. A subset N of vertices of D is a *kernel* of D if and only if it is both absorbent and independent. In [19], the concept of kernel was introduced by von Neumann and Morgenstern in the study of winning positions in 2-person games. Due to the multiple applications in different areas, kernels in digraphs have been studied from different contexts. For example, in logic, applications to counterexamples to the 0-1 laws in fragments of monadic second-order logic can be found in [16, 17].

Regarding computational complexity, Chvátal proved that recognizing digraphs that have a kernel is an **NP**-complete problem [8]. In addition, in [10], Fraekel proved that the kernel problem remains **NP**-complete even for planar digraphs D with degree constraints $\delta(x) \leq 3$, $\delta^+(x) \leq 2$ and $\delta^-(x) \leq 2$, for all vertices $x \in V(D)$, and in [13], Hell and Hernández-Cruz showed that the problem remains an **NP**-complete when the underlying graph is 3-colorable. Therefore, determining the existence or non-existence of a kernel in special classes of digraphs has been a very fruitful field of study, where there are still many interesting open problems. However, most of the results prove the existence of the kernel in a non-constructive way. Despite the above, it is known that kernel problem is polynomial time solvable for acyclic, quasi-transitive, and locally semicomplete digraphs [2, 13, 14]. Moreover, in [21], Szwarcfiter and Chaty proved that counting the number of distinct kernels of a digraph with no odd cycles is $\sharp$**P**-complete, even if the length of the longest cycle of the digraph is 2.

In 1980, Duchet conjectured that for each connected digraph without a kernel, which is not an odd cycle, there is an arc that can be removed and the obtained digraph remains without a kernel [7]. But Apartsin, Ferapontova and Gurvich found a counterexample to this conjecture, they proved that the circulant digraph $\overrightarrow{C}_{43}(1,7,8)$ has no kernel and after removing any arc in this digraph a kernel will appear. Moreover, they also proved that $\overrightarrow{C}_n(1,7,8)$ has a kernel if and only if $n \equiv 0 \pmod 3$ or $n \equiv 0 \pmod{29}$ [1]. Later, in [3], the authors proposed the problem of characterizing the circulant digraphs with kernel. This interesting problem remains open.

A *semikernel* of D is an independent set S such that if $(x, y) \in A(D)$ with $x \in S$, then there is $w \in S$ such that $(y, w) \in A(D)$, note that w can be x. Observe that every kernel is a semikernel but not every semikernel is a kernel.

Let D be a digraph such that every proper induced subdigraph has a kernel. If D has a kernel, then we say that D is a *kernel perfect digraph* or a *KP-digraph*, otherwise, we say that D is a *critical kernel imperfect digraph* or a *CKI-digraph*. By definition, CKI-digraphs are considered minimal obstructions for a digraph to have a kernel, they have been studied by different authors. In [6] it is proved that every CKI-digraph is strongly connected. Even more, Galeana-Sánchez and Neumann-Lara proved that the asymmetrical part of a CKI-digraph is also strongly connected [11]; in the same work, they proved the following result.

Theorem 1. *[11] If D is not a KP-digraph, then D contains an induced CKI-subdigraph.*

Notice that both asymmetrical odd cycles and directed n-antiholes, with $n \geq 3$, are CKI-digraphs, and even cycles are KP-digraphs. By Theorem 1, for every KP-digraph (CKI-digraph) D, the following properties hold, D has no proper induced CKI-subdigraph, and D has no proper induced subdigraph isomorphic to an odd cycle or a directed n-antihole, with $n \geq 3$.

In this work, we will focus on digraphs that satisfy the condition where, whenever a vertex reaches two others through asymmetrical arcs, then these two vertices have the same out-neighborhood in the digraph. The digraphs with this property will be said to have the property P. In particular, digraphs whose asymmetrical part is a Hamiltonian cycle have the property P. As previously mentioned, Hamiltonian digraphs, circulant digraphs, and kernels in digraphs have a large number of applications, so the characterization of the KP-digraphs whose asymmetrical part is a Hamiltonian cycle is of great importance. For example, one of these digraphs can model a route of places, in which the kernel is a set of places that can be visited from anywhere outside the set, but between two places in the kernel, they cannot be reached from one to another. Moreover, although not all CKI-digraphs have a Hamiltonian cycle as their asymmetrical part, the most intuitive ones, such as directed odd cycles and directed anti-holes, do. In [12], it was shown that there are CKI-digraphs whose asymmetrical part is not a Hamiltonian cycle, however, the construction shown is based on the directed odd cycles and directed n-antiholes.

The rest of the paper is organized as follows. Section 2 is devoted to digraphs with property P, which includes an operation that preserves the existence or non-existence of the property P and the existence or non-existence of a kernel, the characterization of the digraphs with the property P that have a kernel, whose asymmetrical part is strong, and the characterization of the KP-digraphs with property P. In Section 3 a characterization of KP-digraphs with a Hamiltonian cycle as their asymmetrical part and some consequences are provided. Section 4 is devoted to providing a construction that allows us to contract some subpaths of the asymmetrical cycle to obtain a smaller order digraph that preserves the existence or non-existence of a kernel. Section 5 focuses on the digraphs whose asymmetrical part is a Hamiltonian cycle and whose diagonals are symmetrical of length 2; through a construction that provides two algorithms, the first one determines whether the digraph is KP or CKI, and the second reconstructs the kernel for the KP-digraphs. Finally in Section 6 some consequences concerning circulant digraphs and digraphs with the property P whose asymmetric part is a disjoint union digraphs, are shown.

2. Property P and KP-digraphs

Let D be a digraph. We say that D has the property P if and only if for every $x \in V(D)$, we have that if $x \mapsto y$ and $x \mapsto w$, then $N^+(y) \setminus \{w\} = N^+(w) \setminus \{y\}$. Note that if D is a digraph such that $|N^+_{Asym(D)}(x)| \leq 1$, for every $x \in V(D)$, then D has the property P. It follows that digraphs whose asymmetrical part is a disjoint union of cycles have the property P. For example, digraphs with a Hamiltonian cycle as their asymmetrical part have the property P.

Let X be a subset of $V(D)$. We say that X is a *strong clique* of D if and only if $x \leftrightarrow_D y$ for every $x, y \in X$. Let D be a digraph and let v be a vertex of D. The digraph SD_v (CD_v) is the digraph obtained from D by blowing up the vertex v into an independent set (a strong clique); that is replace the vertex v by an independent set (a strong clique) X such that $N^+_{SD_v}(x) = N^+_D(v)$ and $N^-_{SD_v}(x) = N^-_D(v)$ ($N^+_{CD_v}(x) = N^+_D(v)$ and $N^-_{CD_v}(x) = N^-_D(v)$), for every $x \in X$. (see Figure 1). By definition, D is an induced subdigraph of SD_v, moreover, the proof of the following results are straightforward.

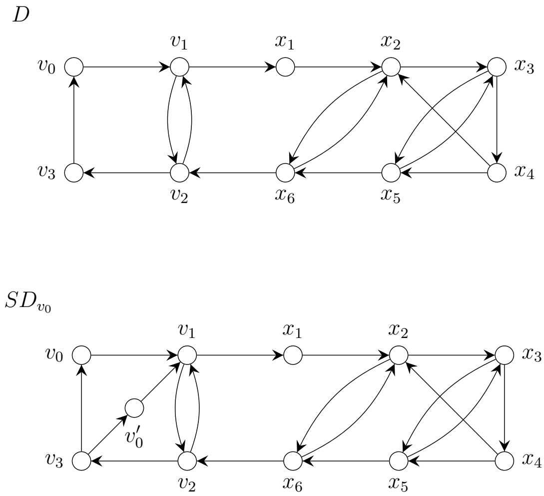


Figure 1. **Digraph D with property P and SD_{v_0}**

Lemma 1. *Let D be a digraph, and let v be a vertex of D. If SD_v is the digraph obtained from D by blowing up the vertex v into an independent set, then D has the property P if and only if SD_v has the property P.*

Lemma 2. *Let D be a digraph, and let u be a vertex of D. If CD_u is the digraph obtained from D by blowing up the vertex u into a strong clique, then D has the property P if and only if CD_u has the property P.*

As we already mentioned, a digraph with a Hamiltonian cycle as its asymmetrical part, has the property P. By Lemmas 1 and 2, it follows that SD_v (CD_v) has the property P, for every $v \in V(D)$. However, if the asymmetrical part of a digraph D is a Hamiltonian digraph, it does not imply that SD_v is a Hamiltonian digraph. (see Figure 1). Hence, there are an infinite number of digraphs with the property P whose asymmetrical part is strong but are not Hamiltonian. Despite this, the following result shows that SD_v preserves the existence or non-existence of a kernel of D.

Lemma 3. *Let D be a digraph, and let v be a vertex of D. If SD_v is the digraph obtained from D by blowing up the vertex v into an independent set, then D has a kernel if and only if SD_v has a kernel.*

Proof. Let D be a digraph, let v be a vertex of D, and let SD_v be the digraph obtained from D by blowing up the vertex v into an independent set X. First, suppose that N is a kernel of D. We have two possibilities.

Case 1. $v \in N$.
Consider $N' = (N \setminus \{v\}) \cup X$. Observe that $N \setminus \{v\}$ and X are independent sets in SD_v; moreover, there is no arc between any vertex of X and any vertex of N in SD_v, otherwise, there is an arc between v and some vertex of N in D, which is impossible. On the other hand, since N is absorbent in D, we have that N' is absorbent in SD_v. Hence, N' is a kernel of SD_v.

Case 2. $v \notin N$.
Note that N is also an independent set in SD_v, moreover, since $N^+_{SD_v}(x) = N^+_D(v)$, for every $x \in X$, it follows that N is an absorbent set in SD_v. Hence, N is a kernel of SD_v.

Therefore, SD_v has a kernel. Conversely, let $\widehat{N}$ be a kernel of SD_v. It is straightforward to prove that $\widehat{N}$ is a kernel of D if $\widehat{N} \cap X = \emptyset$, otherwise $\widehat{N} \cup \{v\}$ is a kernel of D. □

Analogously to Lemma 3, we have the following result.

Lemma 4. *Let D be a digraph, and let u be a vertex of D. If CD_u is the digraph obtained from D by blowing up the vertex u into a strong clique, then D has a kernel if and only if CD_u has a kernel.*

Proof. Let D be a digraph, let u be a vertex of D, and let CD_u be the digraph obtained from D by blowing up the vertex u into a strong clique X. First, suppose that N' is a kernel of CD_u. We have two cases.

Case 1. There is $z \in N' \cap X$.
We will prove that $N = (N' \setminus \{z\}) \cup \{u\}$ is a kernel of D. Let $x, y \in N$. Since $N' \setminus \{z\}$ is also an independent set of D, it follows that if x and y are not u, then there is no arc between x and y in D. Now, suppose that $x = u$ and $y \in N' \setminus \{z\}$. Observe that there is no arc between x and y in D, otherwise, there is an arc between z and y in CD_u which is impossible. Thus N is an independent set in D. Let $x \in V(D) \setminus N$. It follows that $x \in V(D) \setminus N'$. Since N' is a kernel of CD_u, there is $w \in N'$ such that $(x, w) \in A(CD_u)$. If $w \neq z$, then $w \in N$ and $(x, w) \in A(D)$. Otherwise, $w = z$, it implies that $(x, u) \in A(D)$. Hence, N is an absorbent set and a kernel of D.

Case 2. $N' \cap X = \emptyset$.
Note that N' is also an independent set in D, moreover, by definition of CD_u, we have that N' is an absorbent set in D. Hence, N' is a kernel of D.

Therefore, D has a kernel. Conversely, let N be a kernel of D. It is straightforward to prove that if $u \notin N$, then N is a kernel of CD_u, otherwise $N \cup \{z\}$ is a kernel of CD_u, with $z \in X$. □

The following results show that any arbitrary sequence of blow-ups into independent sets or into strong cliques, preserves whether property P holds or not, and the existence or non-existence of a kernel of D, respectively. Moreover, both results are an immediate consequence of recursively applying Lemmas 1 and 2, and Lemmas 3 and 4, respectively.

Theorem 2. *Let D be a digraph. If H is obtained from D after applying an arbitrary sequence blow-ups into independent sets or into strong cliques, then D has the property P if and only if H has the property P.*

Theorem 3. *Let D be a digraph. If H is obtained from D after applying an arbitrary sequence blow-ups into independent sets or into strong cliques, then D has a kernel if and only if H has a kernel.*

The following result characterizes the digraphs, with a strong asymmetrical part, which has a kernel.

Theorem 4. *Let D be a digraph with the property P, with order $n \geq 4$. If $Asym(D)$ is strong, then D has a kernel if and only if D contains an even cycle $(v_0, v_1, \ldots, v_{2k-1}, v_0)$ of length at least 4 such that:*

1. *for each i in $\{0, 2, 4, \ldots, 2(k-1)\}$, (v_i, v_{i+1}) is asymmetrical in D.*
2. *$\{v_0, v_2, v_4, \ldots, v_{2(k-1)}\}$ is an independent set in D.*

Proof. Let D be a digraph as in the hypothesis. First, suppose that D has a kernel, N. Since $Asym(D)$ is strong, for every $x \in V(D)$ there is $y \in V(D)$ such that $x \mapsto y$. Consider v_0 a vertex in N, it follows that there is v_1 in $V(D) \setminus N$, such that $v_0 \mapsto v_1$. Since N is a kernel, there is a vertex v_2 in D such that $v_1 \to v_2$, observe that $v_2 \neq v_0$. For v_2, there is $v_3 \in V(D)$ such that $v_2 \mapsto v_3$, note that since N is independent, $v_3 \in V(D) \setminus N$, moreover, $v_3 \neq v_1$. It follows that, there is $v_4 \in N$ such that $v_3 \to v_4$, note that $v_4 \neq v_2$. If $v_4 = v_0$, then $(v_0, v_1, v_2, v_3, v_0)$ is the desired even cycle, otherwise, there is $v_5 \in V(D) \setminus N$ such that $v_4 \mapsto v_5$; note that $v_5 \neq v_3$, and if $v_5 = v_1$, then $(v_2, v_3, v_4, v_1, v_2)$ is the desired cycle. Recursively, we can continue this procedure until finding the first repeating vertex v_k. If k is even, then $v_{2r} = v_k$ with $2r < k$. Consider the cycle $C = (v_{2r}, v_{2r+1}, \ldots, v_{k-1}, v_k = v_{2r})$. Otherwise, $v_{2s+1} = v_k$ with $2s + 1 < k$, and we consider the cycle $C = (v_{2s+2}, v_{2s+3}, \ldots, v_k = v_{2s+1}, v_{2s+2})$. It follows that C is an even cycle, which alternates vertices of N with vertices in $V(D) \setminus N$, hence the set of vertices with an even label is a subset of N, thus it is an independent set in D. Moreover, by construction (v_i, v_{i+1}) is asymmetrical in D, with i even.

Conversely, suppose that D contains an even cycle $(v_0, v_1, \ldots, v_{2k-1}, v_0)$ such that (v_i, v_{i+1}) is asymmetrical in D, for each $i \in \{0, 2, \ldots, 2(k-1)\}$, and $\{v_0, v_2, \ldots, v_{2(k-1)}\}$ is an independent set in D. We claim that $S_1 = \{v_0, v_2, v_4, \ldots, v_{2(k-1)}\}$ is a semikernel of D. By hypothesis, S_1 is an independent set. Let $v_i \in S_1$ such that $(v_i, y) \in A(D)$. Assume that $v_i \mapsto y$, otherwise $v_i \leftrightarrow y$ and v_i is the vertex sought. On the other hand, if $y = v_{i+1}$, then $(v_{i+1}, v_{i+2}) \in A(D)$ with $v_{i+2} \in S_1$ (subscripts modulo $2k$). Otherwise, $y \neq v_{i+1}$, by hypothesis $v_i \mapsto v_{i+1}$ and $v_i \mapsto y$, it follows that $N^+(v_{i+1}) = N^+(y)$. Thus $(y, v_{i+2}) \in A(D)$ with $v_{i+2} \in S_1$ (subscripts modulo $2k$). Hence, S_1 is a semikernel of D.

Let $A_1 = V(D) \setminus (S_1 \cup N^-(S_1))$. If $A_1 = \emptyset$, then S_1 is a kernel of D. Suppose that $A_1 \neq \emptyset$. Since $Asym(D)$ is strong, there is $x_1 \in A_1$ such that $x_1 \mapsto y$ for some $y \in S_1 \cup N^-(S_1)$, moreover, by definition of A_1, we have that $y \in N^-(S_1)$. We claim that if there is $w \in V(D)$ such that $x_1 \mapsto w$, then $w \in N^-(S_1)$. Let $w \in V(D)$ such that $x_1 \mapsto w$, by hypothesis, $N^-(w) = N^-(y)$, it follows that there is $v \in S_1$ such that $w \to v$.

Consider $S_2 = S_1 \cup \{x_1\}$. We claim that S_2 is a semikernel of D. Observe that S_1 and $\{x_1\}$ are independent sets of D, moreover, $x_1 \in A_1$ and S_1 is a semikernel of D. It follows that S_2 is an independent set of D. Since S_1 is a semikernel of D, it is enough to prove that if $x_1 \to z$, then there is $s \in S_2$ such that $z \to s$. Let $z \in V(D) \setminus S_2$ such that $x_1 \to z$. If $x_1 \leftrightarrow z$, then $s = x_1$, otherwise, we have that $z \in N^-(S_1)$, thus there is $s \in S_1 \subseteq S_2$ such that $z \to s$. Hence, S_2 is a semikernel of D.

Let $A_2 = V(D) \setminus (S_2 \cup N^-(S_2))$. If $A_2 = \emptyset$, then S_2 is a kernel of D. Suppose that $A_2 \neq \emptyset$. Since $Asym(D)$ is strong, there is $x_2 \in A_2$ such that $x_2 \mapsto y$ for some $y \in S_2 \cup N^-(S_2)$, moreover, by definition of A_2, we have that $y \in N^-(S_2)$. We claim that if $x_2 \mapsto w$, then $w \in N^-(S_2)$. Let $w \in V(D)$ such that $x_2 \mapsto w$, by hypothesis, $N^-(w) = N^-(y)$, it follows that there is $v \in S_2$ such that $w \to v$. Consider $S_3 = S_2 \cup \{x_2\}$, analogously to S_2, we have that S_3 is a semikernel of D. Let $A_3 = V(D) \setminus (S_3 \cup N^-(S_3))$. If $A_3 = \emptyset$, then S_3 is a kernel of D. Otherwise, we continue the procedure until find a semikernel S_k of D such that $A_k = \emptyset$. By construction, S_k is a kernel of D. Therefore, D has a kernel. □

Theorem 4 characterizes the digraphs with the property P, whose asymmetrical part is strong, with a kernel. However, if one of the proper induced subdigraphs, with at least two vertices, of D, whose asymmetrical part is strong, does not satisfy the property of Theorem 4, then D is not a KP-digraph. For example, the digraph D, in Figure 1 has a kernel $\{v_0, v_2, x_2, x_5\}$, but D is not a KP-digraph since $D[\{x_2, x_3, x_4\}]$ is an asymmetrical odd cycle, which is a CKI-digraph. Hence, the following result characterizes the KP-digraphs with the property P.

Theorem 5. *If D is a digraph with the property P, with order $n \geq 4$, then D is a KP-digraph if and only if for every induced subdigraph H of D with at least two vertices and a strong asymmetrical part, contains an even cycle $(v_0, v_1, \ldots, v_{2k-1}, v_0)$ of length at least 4 such that:*

1. *for each i in $\{0, 2, 4, \ldots, 2(k-1)\}$, (v_i, v_{i+1}) is asymmetrical in D.*
2. *$\{v_0, v_2, v_4, \ldots, v_{2(k-1)}\}$ is an independent set in D.*

Proof. Let D be a digraph as in the hypothesis. First, suppose that D is a KP-digraph. Let H be an induced subdigraph H of D with a strong asymmetrical part. Since D is a KP-digraph, D has no proper induced subdigraph isomorphic to an asymmetrical odd cycle. It follows that H has no proper induced subdigraph isomorphic to an asymmetrical odd cycle. Moreover, since $Asym(H)$ is strong and it is not an asymmetrical odd cycle, we have that $n \geq 4$. In addition, for every $x \in V(H)$

if $x \mapsto_H y$ and $x \mapsto_H w$, then $N_H^+(y) = N_H^+(w)$. By Theorem 4, we have that H contains an even cycle with the desired properties.

Conversely, suppose that for every induced subdigraph H of D with a strong asymmetrical part, H contains an even cycle $(v_0, v_1, \ldots, v_{2k-1}, v_0)$ of length at least 4 such that for each i in $\{0, 2, 4, \ldots, 2(k-1)\}$, (v_i, v_{i+1}) is asymmetrical in D, and $\{v_0, v_2, v_4, \ldots, v_{2(k-1)}\}$ is an independent set in D. We will prove that D is a KP-digraph. Let D_1 be an induced subdigraph of D. Note that, for every $x \in V(D_1)$ if $x \mapsto_{D_1} y$ and $x \mapsto_{D_1} w$, then $N_{D_1}^+(y) = N_{D_1}^+(w)$. If $Asym(D_1)$ is strong, then, by Theorem 4, we have that D_1 has a kernel. Assume that $Asym(D_1)$ is not strong. Proceeding by contradiction, suppose that D_1 has no kernel. By Theorem 1, D_1 contains an induced CKI-digraph H. It follows that $Asym(H)$ is strong, moreover, for every $x \in V(H)$ if $x \mapsto_H y$ and $x \mapsto_H w$, then $N_H^+(y) = N_H^+(w)$; by hypothesis, H contains an even cycle $(v_0, v_1, \ldots, v_{2k-1}, v_0)$ of length at least 4 such that for each i in $\{0, 2, 4, \ldots, 2(k-1)\}$, (v_i, v_{i+1}) is asymmetrical in D, and $\{v_0, v_2, v_4, \ldots, v_{2(k-1)}\}$ is an independent set in D. Hence, by Theorem 4, H has a kernel, which is a contradiction. We can conclude that D_1 has a kernel. Therefore, D is a KP-digraph. □

3. Digraphs whose asymmetrical part is a Hamiltonian cycle

In this section, KP-digraphs with a Hamiltonian cycle as their asymmetrical part are characterized, and some of their consequences are shown.

Let D be a digraph whose $Asym(D)$ is an asymmetrical Hamiltonian cycle $\gamma = (x_0, x_1, \ldots, x_{n-1}, x_0)$. As we already mentioned, D has property P. On the other hand, it is important to note that if H is a proper induced subdigraph of D, with at least two vertices, then the asymmetrical part of every induced subdigraph of H is not strong. It follows that H does not contain any induced CKI-subdigraph. By Theorem 1, H is a KP-digraph. We can conclude that, if D has a kernel, then D is a KP-digraph, otherwise, D is a CKI-digraph. Therefore, we have the following result.

Theorem 6. *If D is a digraph whose $Asym(D)$ is a Hamiltonian cycle, then D is a KP-digraph or D is a CKI-digraph.*

From now on, we consider that every digraph D is such that $Asym(D)$ is an asymmetrical Hamiltonian cycle $\gamma = (x_0, x_1, \ldots, x_{n-1}, x_0)$. Note that, every arc not in $A(\gamma)$ is a symmetrical diagonal of γ. Even more, by Theorem 6, if D has a kernel, then D is a KP-digraph, otherwise D is a CKI-digraph. Therefore, the results obtained in this work are based on determining the existence or non-existence of a kernel in considered digraphs. Combining this with Theorems 4 and 5, we have the following characterization of the KP-digraphs with a Hamiltonian cycle as their asymmetrical part.

Theorem 7. *Let D be a digraph. If $Asym(D)$ is a Hamiltonian cycle, then D is a KP-digraph if and only if D contains an even cycle $(v_0, v_1, \dots, v_{2k-1}, v_0)$ of length at least 4 such that:*

1. *for each i in $\{0, 2, 4, \dots, 2(k-1)\}$, (v_i, v_{i+1}) is asymmetrical in D.*
2. *$\{v_0, v_2, v_4, \dots, v_{2(k-1)}\}$ is an independent set in D.*

Corollary 1. *Let D be a digraph such that $Asym(D)$ is a Hamiltonian cycle γ. If the minimum length of a diagonal of γ is odd, then D is a KP-digraph.*

Proof. Let $\gamma = (x_0, x_1, \dots, x_{n-1}, x_0)$ be the asymmetrical Hamiltonian cycle of D. Suppose, without loss of generality, that (x_0, x_i) is a diagonal of γ with minimum length and its length is odd; it follows that $x_0 \leftrightarrow_D x_i$. Consider $C = (x_0, x_1, \dots, x_i, x_0)$. Note that C is an even cycle of D such that (x_j, x_{j+1}) is asymmetrical for each $j \in \{0, 1, \dots, i-1\}$. Moreover, since every diagonal of γ is symmetrical and (x_0, x_i) is a diagonal of γ with minimum length, we have that $\{x_0, x_2, \dots, x_{i-1}\}$ is an independent set of D. Hence, the hypothesis of Theorem 7 holds. Therefore, D is a KP-digraph. □

As a consequence of Corollary 1, we have that if D is a CKI-digraph with a Hamiltonian cycle γ as $Asym(D)$, then the minimum length of a diagonal of γ is even.

Corollary 2. *Let $\overrightarrow{C}_n(1, \pm s_1, \pm s_2, \dots, \pm s_k)$ be a circulant digraph, with $n \geq 6$, such that $s_1 < s_j$ and $s_i \leq \lfloor \frac{n}{2} \rfloor$ for every $i, j \in \{1, \dots, k\}$. If s_1 is odd, then $\overrightarrow{C}_n(1, \pm s_1, \pm s_2, \dots, \pm s_k)$ is a KP-digraph.*

Consider $\overrightarrow{C}_n(1, \pm s_1, \pm s_2, \dots, \pm s_k)$ the circulant digraph, with $n \geq 6$. From Corollary 2, if n is even and every s_i is odd, then $\overrightarrow{C}_n(1, \pm s_1, \pm s_2, \dots, \pm s_k)$ is a KP-digraph. In particular, for every $k \geq 3$, $\overrightarrow{C}_{2k}(1, \pm s)$ is a KP-digraph if s is odd. In the other case, if n is odd, every s_i is odd and $s_i \leq \lfloor \frac{n}{2} \rfloor$, then the circulant digraph $\overrightarrow{C}_{2n+1}(1, \pm s_1, \pm s_2, \dots, \pm s_k)$ is a KP-digraph. In particular, for every $k \geq 3$, if s is odd and $s \leq k$, then $\overrightarrow{C}_{2k+1}(1, \pm s)$ is a KP-digraph. Observe that $\overrightarrow{C}_5(1, \pm 3) \cong \overrightarrow{C}_5(1, \pm 2) \cong \overrightarrow{A}_5$, which is a CKI-digraph. From above, we can conclude that for every $n \geq 6$, $\overrightarrow{C}_n(1, \pm 3)$ is a KP-digraph and the unique $\overrightarrow{C}_n(1, \pm 3)$ which is a CKI-digraph is $\overrightarrow{A}_5$.

The following results show a way to construct a cycle where the hypothesis of the Theorem 7 holds, thus obtaining sufficient conditions for a digraph whose asymmetrical part is a Hamiltonian cycle to be a KP-digraph.

Corollary 3. *Let D be a digraph with order n such that $Asym(D)$ is a Hamiltonian cycle γ, and let s be an integer greater than 1. If $n \equiv 0 \pmod{s+1}$, for every $i \in \{0, \dots, n-1\}$ the arc (x_i, x_{i+s}) is a symmetrical diagonal of γ and the length of any other symmetrical diagonal of γ is not congruent with 0 modulo $s+1$, then D is a KP-digraph*

Proof. Let D be a digraph with order n, with $\gamma = (x_0, x_1, \ldots, x_{n-1}, x_0)$ a Hamiltonian cycle as its asymmetrical part, and let s be an integer greater than 1. Since $n \equiv 0 \pmod{s+1}$, we have that $n = (s+1)j$ for some integer $j > 1$. Consider the even cycle

$$C = (x_0, x_1, x_{s+1}, x_{s+1+1}, x_{2(s+1)}, x_{2(s+1)+1}, \ldots x_{(j-1)(s+1)}, x_{(j-1)(s+1)+1}, x_0).$$

Note that C fulfills the hypotheses of Theorem 7, because $(x_{r(s+1)}, x_{r(s+1)+1})$ is asymmetrical in D, $|r(s+1) - t(s+1)| = l(s+1)$ and any other symmetrical diagonal of γ is not congruent with 0 modulo $s+1$, with $l \in \mathbb{Z}$ and for every $r, t \in \{0, 1, \ldots, j-1\}$. It follows that D is a KP-digraph. $\square$

In particular, when D is a circulant digraph and satisfies the hypotheses of Corollary 3 we have the following result.

Corollary 4. *Let $1 < s_1, s_2, \ldots, s_k, n$ be integers such that $s_i \leq \frac{n}{2}$. If there is $i \in \{1, \ldots, k\}$ such that $n \equiv 0 \pmod{s_i + 1}$ and $s_j \not\equiv 0 \pmod{s_i + 1}$ for every $j \in \{1, \ldots, k\}$, then $\overrightarrow{C}_n(1, \pm s_1, \pm s_2, \ldots, \pm s_k)$ is a KP-digraph.*

Note that, by Corollary 2, $\overrightarrow{C}_7(1, \pm 3)$ is a KP-digraph but the hypotheses of Corollary 4 do not hold.
The following result is not a characterization; however, it is a sufficient condition that is very useful to verify if a digraph is a KP-digraph.

Theorem 8. *Let D be a digraph such that $Asym(D)$ is a Hamiltonian cycle γ. If γ has a diagonal a such that no other diagonal crosses a, then D is a KP-digraph.*

Proof. Let $\gamma = (x_0, x_1, \ldots, x_{n-1}, x_0)$ be the asymmetrical Hamiltonian cycle of D and let a be a diagonal of γ such that no other diagonal crosses a. Suppose, without loss of generality, that $a = (x_0, x_i)$; it follows that $x_0 \leftrightarrow_D x_i$. Let D_1 be the digraph $D - x_0$. It follows that D_1 has a kernel, N_1. If there is $x_j \in N_1$ such that $x_0 \rightarrow_D x_j$, then N_1 is a kernel of D. Assume that $N^+(x_0) \cap N_1 = \emptyset$. On the other hand, if $N^-(x_0) \cap N_1 = \emptyset$, then $N_1 \cup \{x_0\}$ is a kernel of D. Hence, suppose that $N^-(x_0) \cap N_1 \neq \emptyset$; moreover, since $N^-(x_0) \setminus N^+(x_0) = \{x_{n-1}\}$, we have that $N^-(x_0) \cap N_1 = \{x_{n-1}\}$. Consider V_2 the set $\{x_j \in V(D) : \ (x_j, x_0) \notin A(D) \text{ and } j \in \{i, \ldots, n-1\}\}$. Let D_2 be the subdigraph of D induced by V_2, and N_2 be a kernel of D_2. Let N_1' be the set $N_1 \cap \{x_r \in V(D) : \ r \in \{0, \ldots, i\}\}$; we will prove that $N = N_2 \cup N_1' \cup \{x_0\}$ is a kernel of D. For the independence of N, we note that N_1', N_2 and $\{x_0\}$ are independent sets; moreover, there is neither arc from $\{x_0\}$ to some vertex in N_1' nor from N_1' to $\{x_0\}$. In addition, by definition of V_2, there is neither arc from $\{x_0\}$ to some vertex in N_2 nor from N_2 to $\{x_0\}$; since no other diagonal in γ crosses a, there is neither arc from N_2 to N_1' nor from N_1' to N_2. Hence N is independent in D.

Now, let $x_s \in V(D) \setminus N$. If $x_s \in N^-(x_0)$, then $x_s \to_D x_0$; otherwise $x_s \notin N^-(x_0)$ and $s \in \{1, \ldots, i-1\}$ or $s \in \{i, \ldots, n-1\}$. In case that $s \in \{1, \ldots, i-1\}$, then by definition of N_1 there is $w \in N_1$ such that $x_s \to_D w$. Even more, since no other diagonal of γ crosses a, we have that $w \in N_1'$. In case that $s \in \{i, \ldots, n-1\}$, it follows that $x_s \in V_2$ and there is $w \in N_2$ such that $x_s \to_D w$. Hence N is an absorbent set and a kernel of D. Therefore, D is a KP-digraph. □

Corollary 5. *If D is a CKI-digraph such that $Asym(D)$ is a Hamiltonian cycle γ, then for every diagonal a of γ there is a diagonal e such that e crosses a.*

4. Contraction of induced asymmetrical paths

In this section, we provide a construction that allows us to contract some subpaths of the asymmetrical cycle to obtain a smaller order digraph where the existence or non-existence of a kernel is preserved.

Construction 1. Let D be a digraph whose asymmetrical part is a Hamiltonian cycle $\gamma = (x_0, \ldots, x_{n-1}, x_0)$. Suppose that $C = (x_0, x_1, \ldots, x_k)$, with $k \geq 3$, is such that $\delta^+(x_0) > 1$, $\delta^-(x_0) > 1$, $\delta^+(x_k) > 1$, $\delta^-(x_k) > 1$, $\delta^+(x_i) = \delta^-(x_i) = 1$ for every $i \in \{1, \ldots, k-1\}$, and there is no arc between x_0 and x_k. Consider $\widehat{D}$ the digraph obtained from D the following modifications.

1. If k is odd, then:
   (a) Delete $V(C)$ and the arcs of D with at least one ending vertex in C from D.
   (b) Add two new vertices $\widehat{x}_0$ and $\widehat{x}_1$ and add the arcs $(x_{n-1}, \widehat{x}_0)$, $(\widehat{x}_0, \widehat{x}_1)$ and $(\widehat{x}_1, x_{k+1})$.
   (c) For each $i \in \{k+1, k+2, \ldots, n-1\}$, if $x_i \leftrightarrow_D x_0$, then add the arcs $(x_i, \widehat{x}_0)$ and $(\widehat{x}_0, x_i)$.
   (d) For each $i \in \{k+1, k+2, \ldots, n-1\}$, if $x_i \leftrightarrow_D x_k$, then add the arcs $(x_i, \widehat{x}_1)$ and $(\widehat{x}_1, x_i)$.
2. If k is even, then:
   (a) Delete $V(C)$ and the arcs of D with at least one ending vertex in C from D.
   (b) Add three new vertices $\widehat{x}_0$, $\widehat{x}_1$ and $\widehat{x}_2$ and add the arcs $(x_{n-1}, \widehat{x}_0)$, $(\widehat{x}_0, \widehat{x}_1)$, $(\widehat{x}_1, \widehat{x}_2)$ and $(\widehat{x}_2, x_{k+1})$.
   (c) For each $i \in \{k+1, k+2, \ldots, n-1\}$, if $x_i \leftrightarrow_D x_0$, then add the arcs $(x_i, \widehat{x}_0)$ and $(\widehat{x}_0, x_i)$.
   (d) For each $i \in \{k+1, k+2, \ldots, n-1\}$, if $x_i \leftrightarrow_D x_k$, then add the arcs $(x_i, \widehat{x}_2)$ and $(\widehat{x}_2, x_i)$.

Theorem 9. *Let D be the digraph whose asymmetrical part is a Hamiltonian cycle. If $\widehat{D}$ is the digraph obtained from D after applying Construction 1, then D is a KP-digraph if and only if $\widehat{D}$ is a KP-digraph.*

Proof. Let D be a digraph, and let C be as in the hypothesis of Construction 1. Consider $\widehat{D}$ the digraph obtained from D after applying Construction 1. Observe that the asymmetrical part of $\widehat{D}$ is a Hamiltonian cycle. Suppose that D is a KP-digraph. We will prove that $\widehat{D}$ is a KP-digraph. Note that it is enough to prove that $\widehat{D}$ has a kernel. Let N be a kernel of D. We have two cases.

Case 1. k is odd.
By Construction 1, we have three possibilities and their proofs are straightforward. If $x_0 \in N$, then $\widehat{N} = (N \cap V(\widehat{D})) \cup \{\widehat{x_0}\}$ is a kernel of $\widehat{D}$; if $x_k \in N$, then $\widehat{N} = (N \cap V(\widehat{D})) \cup \{\widehat{x_1}\}$ is a kernel of $\widehat{D}$; and if x_0 and x_k are not in N, then $\widehat{N} = N \cap V(\widehat{D})$ is a kernel of $\widehat{D}$.

Case 2. k is even.
By Construction 1, we have three possibilities and their proofs are straightforward. If $x_0 \in N$, then $\widehat{N} = (N \cap V(\widehat{D})) \cup \{\widehat{x_0}, \widehat{x_2}\}$ is a kernel of $\widehat{D}$; if $x_0 \notin N$ and $x_k \in N$, then $\widehat{N} = (N \cap V(\widehat{D})) \cup \{\widehat{x_2}\}$ is a kernel of $\widehat{D}$; and if x_0 and x_k are not in N, then $\widehat{N} = (N \cap V(\widehat{D})) \cup \{\widehat{x_1}\}$ is a kernel of $\widehat{D}$.

Conversely, we suppose that $\widehat{D}$ is a KP-digraph. To prove that D is a KP-digraph, it is enough to prove that D has a kernel. Let $\widehat{N}$ be a kernel of $\widehat{D}$. We have two cases.

Case 1. k is odd.
By Construction 1, we have three possibilities and their proofs are straightforward. If $\widehat{x_0} \in \widehat{N}$, then $(\widehat{N} \cap V(D)) \cup \{x_0, x_2, \ldots, x_{k-1}\}$ is a kernel of D; if $\widehat{x_1} \in \widehat{N}$, then $(\widehat{N} \cap V(D)) \cup \{x_1, x_3, \ldots, x_k\}$ is a kernel of D; and if $\widehat{x_0}$ and $\widehat{x_1}$ are not in $\widehat{N}$, then $(\widehat{N} \cap V(D)) \cup \{x_2, x_4, \ldots, x_{k-1}\}$ is a kernel of D.

Case 2. k is even.
By Construction 1, we have three possibilities and their proofs are straightforward. If $\widehat{x_1} \in \widehat{N}$, then $(\widehat{N} \cap V(D)) \cup \{x_1, x_3, \ldots, x_{k-1}\}$ is a kernel of D; if $\widehat{x_1} \notin \widehat{N}$ and $\widehat{x_0} \in \widehat{N}$, then $(\widehat{N} \cap V(D)) \cup \{x_0, x_2, \ldots, x_k\}$ is a kernel of D; and if $\widehat{x_0}$ and $\widehat{x_1}$ are not in $\widehat{N}$, then $(\widehat{N} \cap V(D)) \cup \{x_2, x_4, \ldots, x_k\}$ is a kernel of D. □

Observe that if D is a digraph such that $C = (x_0, x_1, \ldots, x_k)$ is such that $\delta^+(x_0) > 1$, $\delta^-(x_0) > 1$, $\delta^+(x_k) > 1$, $\delta^-(x_k) > 1$, $\delta^+(x_i) = \delta^-(x_i) = 1$ for every $i \in \{1, \ldots, k-1\}$, with $k \geq 3$, but there is an arc between x_0 and x_k, then $x_0 \leftrightarrow_D x_k$. Even more, by Theorem 8, D is a KP-digraph.

Note that if Construction 1 is applied recursively on a digraph D with a Hamiltonian cycle as its asymmetrical part, then we obtain a digraph with smaller order D', which have no two consecutive vertices whose in-degree and out-degree are 1. Moreover, the problem of determining whether D has a kernel or not, is equivalent to the problem of determining whether D' has a kernel or not. Hence, from now on, we will suppose that for every digraph D with a Hamiltonian cycle as their asymmetrical part, there are no two consecutive vertices whose in-degree and out-degree are 1.

5. Symmetrical diagonals of length 2

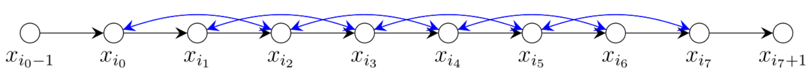
In this section, we consider the digraphs D whose asymmetrical part is a Hamiltonian cycle $\gamma = (x_0, x_1, \ldots, x_{n-1}, x_0)$ and if $x_i \leftrightarrow_D x_j$, then $j = i + 2$. In this sense, we say that if there is an arc in D which is not in γ, then is a symmetrical diagonal of γ of length two.

Theorem 10. *If n is a positive integer, then $\overrightarrow{C}_n(1, \pm 2)$ is a KP-digraph if and only if $n \equiv 0 \pmod 3$.*

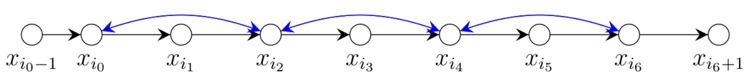
Proof. First, suppose that $\overrightarrow{C}_n(1, \pm 2)$ is a KP-digraph and $n = 3k + r$ for some $r \in \{0, 1, 2\}$. We will prove that $r = 0$. Let K be a kernel of D. By the symmetry of $\overrightarrow{C}_n(1, \pm 2)$, we suppose, without loss of generality, that $0 \in K$. It follows that $n-2$, $n-1$, 1 and 2 are not elements of K. Since $N^+(1) = \{n-1, 2, 3\}$, we have that $3 \in K$; this implies that $4, 5 \notin K$, moreover, 6 has to be an element of K and $7, 8 \notin K$. Continuing in a similar way, we have that $i \in K$ if $i \equiv 0 \pmod 3$, otherwise $i \notin K$, that is $K = \{0, 3, \ldots, 3k\}$. Observe that if $r \neq 0$, then there is an arc from $3k$ to 0, which is impossible because K is independent. Therefore, $r = 0$.

Conversely, if $n \equiv 0 \pmod 3$, then the result follows from Corollary 4. $\square$

Let D be a digraph with a Hamiltonian cycle $\gamma = (x_0, x_1, \ldots, x_{n-1}, x_0)$ as its asymmetrical part such that every diagonal of γ is symmetrical of length 2. Let $\sigma = (x_{i_0}, x_{i_1}, \ldots, x_{i_r})$ be a subpath of γ. We say that σ is a *complete chain of* 2-*diagonals*, of length r if and only if $x_{i_j} \leftrightarrow_D x_{i_{j+2}}$, for every $j \in \{0, \ldots, r-2\}$. See Figure 2. Analogously, we say that σ is an *alternating chain of* 2-*diagonals* of length r if and only if $x_{i_j} \leftrightarrow_D x_{i_{j+2}}$, for every $j \in \{0, 2, \ldots, r-2\}$, $x_{i_k} \leftrightarrow_{D^c} x_{i_{k+2}}$, for every $k \in \{1, 3, \ldots, r-3\}$, $x_{i_0 - 1} \leftrightarrow_{D^c} x_{i_1}$, and $x_{i_{r-1}} \leftrightarrow_{D^c} x_{i_r + 1}$. See Figure 2. Observe that if σ is a complete chain of 2-diagonals of length 2, then σ is an alternating chain of 2-diagonals. We say that a complete chain of 2-diagonals (alternating chain of 2-diagonals) is maximal if there is no complete chain of 2-diagonals (alternating chain of 2-diagonals) in which it is properly contained.



Complete chain of 2 diagonals of length 7



Alternating chain of 2-diagonals of length 6

Figure 2. **Complete chain of 2 diagonals and alternating chain of 2-diagonals**

Lemma 5. *Let D be a digraph such that $Asym(D)$ is a Hamiltonian cycle γ, and every diagonal of γ is a symmetrical diagonal of length two. If every complete chain of 2-diagonals of D has length 2 and D has at least one alternating chain of 2-diagonals, then D is a KP-digraph.*

Proof. Let $C_1, \ldots, C_k$ be the maximal alternating chains of 2-diagonals of D, according to the order in which they appear in γ. For each $i \in \{2, \ldots, k\}$, suppose that $C_i = (x_{0_i}, x_{1_i}, \ldots, x_{r_i})$, and without loss of generality, assume that $C_1 = (x_0, x_1, \ldots, x_{r_1})$. For each $i \in \{1, \ldots, k\}$, let P_i be the subpath $(x_{r_i}, \gamma, x_{0_{i+1}})$ of γ, and let l_i be the length of P_i. Since every arc not in γ is a symmetrical diagonal of γ of length two, the definition of C_i and by Theorem 9, we have that l_i is 1 or 2. Moreover, $\gamma = C_1 \cup P_1 \cup C_2 \cup \cdots \cup C_k \cup P_k$ and $l(\gamma) = r_1 + l_1 + \cdots + r_k + l_k$.
Let N be a subset of vertices of D such that for each $i \in \{1, \ldots, k\}$, $N \cap (V(C_i) \cup V(P_i))$ is equal to $\{x_{1_i}, x_{3_i}, \ldots, x_{(r_i)-1}\} \cup \{x_{(r_i)+1} = x_{(0_{i+1})-1}\}$ if $l_i = 2$; or $\{x_{r_i}\}$ if $l_i = 1$ and $r_i = 2$; or $\{x_{1_i}, x_{3_i}, \ldots, x_{(r_i)-3}\} \cup \{x_{r_i}\}$ if $l_i = 1$ and $r_i \geq 4$. The proof that N is a kernel of D is straightforward. □

Observe that Lemma 5 is a direct consequence of Theorem 8, however, the presented proof explicitly exhibits the kernel. The following construction allows us to contract the maximal complete chain of 2-diagonals of length $r > 2$ in D to obtain a digraph D' where the existence or non-existence of a kernel is preserved.

Construction 2. Let D be a digraph whose asymmetrical part is a Hamiltonian cycle $\gamma = (x_0, \ldots, x_{n-1}, x_0)$, and let σ be a maximal complete chain of 2-diagonals of length $r \geq 3$. Suppose, without loss of generality, that $\sigma = (x_0, x_1, \ldots, x_r)$. Construct the digraph D', obtained from D after the following modifications.

1. If $r \equiv 0 \pmod 3$, then:
   (a) Delete $V(\sigma)$ and the arcs of D with at least one ending vertex in σ from D.
   (b) Add a new vertex x_{σ_0} and add the arcs (x_{n-1}, x_{σ_0}) and (x_{σ_0}, x_{r+1}).
   (c) If $x_{n-2} \leftrightarrow_D x_0$, then add the arcs (x_{n-2}, x_{σ_0}) and (x_{σ_0}, x_{n-2}).
   (d) If $x_r \leftrightarrow_D x_{r+2}$, then add the arcs (x_{σ_0}, x_{r+2}) and (x_{r+2}, x_{σ_0}).
2. If $r \equiv 1 \pmod 3$, then:
   (a) Delete $V(\sigma)$ and the arcs of D with at least one ending vertex in σ from D.
   (b) Add two new vertices x_{σ_0} and x_{σ_1}, and add the arcs (x_{n-1}, x_{σ_0}), $(x_{\sigma_0}, x_{\sigma_1})$ and (x_{σ_1}, x_{r+1}).
   (c) If $x_{n-2} \leftrightarrow_D x_0$, then add the arcs (x_{n-2}, x_{σ_0}) and (x_{σ_0}, x_{n-2}).
   (d) If $x_r \leftrightarrow_D x_{r+2}$, then add the arcs (x_{σ_1}, x_{r+2}) and (x_{r+2}, x_{σ_1}).
3. If $r \equiv 2 \pmod 3$, then:
   (a) Delete $V(\sigma)$ and the arcs of D with at least one ending vertex in σ from D.
   (b) Add three new vertices x_{σ_0}, x_{σ_1} and x_{σ_2} and add the arcs (x_{n-1}, x_{σ_0}), $(x_{\sigma_0}, x_{\sigma_1})$, $(x_{\sigma_1}, x_{\sigma_2})$, $(x_{\sigma_0}, x_{\sigma_2})$, $(x_{\sigma_2}, x_{\sigma_0})$ and (x_{σ_2}, x_{r+1}).

(c) If $x_{n-2} \leftrightarrow_D x_0$, then add the arcs (x_{n-2}, x_{σ_0}) and (x_{σ_0}, x_{n-2}).

(d) If $x_r \leftrightarrow_D x_{r+2}$, then add the arcs (x_{σ_2}, x_{r+2}) and (x_{r+2}, x_{σ_2}).

Lemma 6. *Let D be a digraph whose asymmetrical part is a Hamiltonian cycle $\gamma = (x_0, \ldots, x_{n-1}, x_0)$, and let $\sigma = (x_0, x_1, \ldots, x_r)$ be a maximal complete chain of 2-diagonals, of length $r \leq n-1$, of D. If D has a cycle $C = (v_0, v_1, \ldots, v_{2k-1}, v_0)$ of length at least 4, for each $i \in \{0, 2, \ldots, 2(k-1)\}$ the arc (v_i, v_{i+1}) is asymmetrical in D, $\{v_0, v_2, \ldots, v_{2(k-1)}\}$ is an independent set of D and $V(C) \cap V(\sigma) \neq \emptyset$, then x_0 is a vertex of $V(C) \cap V(\sigma)$ and one of the following statement hold.*

1. *If $x_0 \in V(C)$ and its subscript in C is even, then $V(C) \cap V(\sigma) = \{x_i : i \equiv 0 \pmod 3$ or $i \equiv 1 \pmod 3$ and $0 \leq i \leq r\}$; moreover, x_i has subscript even in C if $i \equiv 0 \pmod 3$, otherwise x_i has subscript odd in C.*

2. *If $x_0 \in V(C)$ and its subscript in C is odd, then*

   (a) *$V(C) \cap V(\sigma) = \{x_0\} \cup \{x_i : i \equiv 1 \pmod 3$ or $i \equiv 2 \pmod 3$ and $0 \leq i \leq r\}$; moreover, x_i has subscript even in C if $i \equiv 1 \pmod 3$, otherwise x_i has subscript odd in C.*

   (b) *$V(C) \cap V(\sigma) = \{x_i : i \equiv 0 \pmod 3$ or $i \equiv 2 \pmod 3$ and $0 \leq i \leq r\}$; moreover, x_i has subscript even in C if $i \equiv 2 \pmod 3$, otherwise x_i has subscript odd in C.*

Proof. Let D, γ, σ and C be as in the hypotheses of Lemma 6. By hypothesis, $V(C) \cap V(\sigma) \neq \emptyset$; we will prove that $x_0 \in V(C)$. Let x_j be the first vertex of σ in C; proceeding by contradiction suppose that $x_j \neq x_0$. Observe that $N^-(x_j) \subseteq \{x_{j-2}, x_{j-1}, x_{j+2}\}$. By choice of x_j, we have that x_{j-2} and x_{j-1} are not vertices of C, it follows that the predecessor of x_j in C is x_{j+2}. Since $x_j \leftrightarrow_D x_{j+2}$, we have that x_{j+2} has odd subscript in C, in consequence, x_j has even subscript in C and x_{j+1} is the successor of x_j in C. Observe that $N^+(x_{j+1}) \subseteq \{x_{j-1}, x_{j+2}, x_{j+3}\}$; by choice of x_j, we have that x_{j-1} is not a vertex of C. In addition, if x_{j+2} is the successor of x_{j+1} in C, then $C = (x_{j+2}, x_j, x_{j+1}, x_{j+2})$ which is impossible because C has even length. Thus x_{j+3} is the successor of x_{j+1} in C and x_{j+4} is the successor of x_{j+3} in C. On the other hand, we have that $N^-(x_{j+2}) = \{x_j, x_{j+1}, x_{j+4}\}$, the successor of x_{j+2} in C is x_j, the successor of x_j in C is x_{j+1} and the successor of x_{j+1} in C is x_{j+3}. It follows that, the predecessor of x_{j+2} in C is x_{j+4}. Hence, $C = (x_{j+2}, x_j, x_{j+1}, x_{j+3}, x_{j+4}, x_{j+2})$ which is a contradiction to the length of C. Therefore, $x_0 \in V(C) \cap V(\sigma)$.

First, we suppose that x_0 has even subscript in C. We will prove that $V(C) \cap V(\sigma) = \{x_i : i \equiv 0 \pmod 3$ or $i \equiv 1 \pmod 3$ and $0 \leq i \leq r\}$. Since $x_0 \in V(C)$ and its subscript in C is even, we suppose, without loss of generality, that $v_0 = x_0$. By definition of C, we have that $x_1 = v_1$, moreover, since $N^+(x_1) = \{x_2, x_3\}$ and $x_0 \leftrightarrow_D x_2$, it follows that $x_3 = v_2$ and $x_4 = v_3$. Similarly, since $N^+(x_4) = \{x_2, x_5, x_6\}$, $x_0 \leftrightarrow_D x_2$ and $x_3 \leftrightarrow_D x_5$, it follows that $x_6 = v_4$ and $x_7 = v_5$. Continuing with this procedure, we have that $x_i \in V(C)$ with $i \equiv 0 \pmod 3$ or $i \equiv 1 \pmod 3$, for every $0 \leq i \leq r\}$; even more, if $i \equiv 0 \pmod 3$ then x_i has subscript even in C, otherwise x_i has subscript odd in C. It is important to note that, by construction, x_j is not a successor of any vertex in C, with $j \equiv 2 \pmod 3$ and $2 \leq j \leq r$, hence x_j is not a vertex of C. Therefore, $V(C) \cap V(\sigma) = \{x_i : \ i \equiv 0 \pmod 3$ or $i \equiv 1 \pmod 3$ and $0 \leq i \leq r\}$.

Now, we suppose that $x_0 \in V(C)$ and its subscript in C is odd. Suppose, without loss of generality, that $x_0 = v_1$. Since (v_0, v_1) is asymmetrical, $x_{n-1} = v_0$. Note that $N^+(x_0) \subseteq \{x_{n-2}, x_1, x_2\}$. If $x_{n-2} = v_2$, then $x_{n-1} = v_3$; even more, $C = (x_{n-1}, x_0, x_{n-2}, x_{n-1})$, which is impossible because the length of C is even. Hence, we have two cases.

Case 1. $x_1 = v_2$.

Thus $x_2 = v_3$. Note that $N^+(x_2) = \{x_0, x_3, x_4\}$ but $x_0 = v_1$ and $x_1 \leftrightarrow_D x_3$, this implies that $x_4 = v_4$ and $x_5 = v_5$. Since $N^+(x_5) = \{x_3, x_6, x_7\}$ and $x_4 \leftrightarrow_D x_6$, we have that $v_6 = x_3$ or $v_6 = x_6$. If $v_6 = x_3$, then $x_4 = v_7$, which is impossible because $x_4 = v_4$. Hence, $v_6 = x_6$ and $v_7 = x_7$. Continuing with this procedure, we have that $x_0 \in V(C)$ and $x_i \in V(C)$ with $i \equiv 1 \pmod 3$ or $i \equiv 2 \pmod 3$, for every $1 \leq i \leq r$; even more, if $i \equiv 1 \pmod 3$ then x_i has subscript even in C, otherwise x_i has subscript odd in C.

Case 2. $x_2 = v_2$.

Thus $x_3 = v_3$. Since $N^+(x_3) = \{x_1, x_4, x_5\}$ and $x_2 \leftrightarrow_D x_4$, we have that $v_4 = x_1$ or $v_4 = x_5$. If $v_4 = x_1$, then $v_5 = x_2$, which is impossible because $x_2 = v_2$. Hence, $v_4 = x_5$ and $v_5 = x_6$. Since $N^+(x_6) = \{x_4, x_7, x_8\}$, $x_2 \leftrightarrow_D x_4$ and $x_5 \leftrightarrow_D x_7$, it follows that $v_6 = x_8$ and $v_7 = x_9$. With this procedure, we have that $x_i \in V(C)$ with $i \equiv 0 \pmod 3$ or $i \equiv 0 \pmod 3$, for every $1 \leq i \leq r$; even more, if $i \equiv 2 \pmod 3$ then x_i has subscript even in C, otherwise x_i has subscript odd in C. □

Lemma 7. *Let D be a digraph whose asymmetrical part is a Hamiltonian cycle $\gamma = (x_0, \ldots, x_{n-1}, x_0)$, and let σ be a complete chain of 2-diagonals of length $r \leq n-1$. If D' is the digraph obtained from D after applying Construction 2, then D is a KP-digraph if and only if D' is a KP-digraph.*

Proof. Let D be a digraph, let σ be a complete chain of 2-diagonals of length $r \leq n-1$, and let D' be the digraph obtained from D after Construction 2. Suppose, without loss of generality, that $\sigma = (x_0, x_1, \ldots, x_r)$.

First, suppose that D is a KP-digraph. We will prove that D' is a KP-digraph. By Theorem 7, D has an even cycle $C = (v_0, v_1, \ldots, v_{2k-1}, v_0)$ such that (v_i, v_{i+1}) is asymmetrical in D, for every $i \in \{0, 2, \ldots, 2(k-1)\}$, and $S = \{v_0, v_2, \ldots, v_{2(k-1)}\}$ is independent in D. Moreover, there is a kernel N of D such that $N = S \cup S_1$, where S_1 is a kernel of $D[V(D) \setminus (S \cup N^-(S))]$. Note that, if $V(C) \cap V(\sigma) = \emptyset$, then C is a cycle in D', with the same properties. Hence, D' is a KP-digraph; even more, $N' = S \cup S_1'$ is a kernel of D', where S_1' is a kernel of $D'[V(D') \setminus (S \cup N^-(S))]$. Now assume that $V(C) \cap V(\sigma) \neq \emptyset$. By Lemma 6, x_0 is a vertex of C and we have three possibilities.

If $r \equiv 0 \pmod 3$, then consider the following cases.

Case 1. x_0 has even subscript in C.
By Lemma 6, $V(C) \cap V(\sigma) = \{x_i : \ i \equiv 0 \pmod 3 \text{ or } i \equiv 1 \pmod 3 \text{ and } 0 \leq i \leq r\}$; moreover, x_i has subscript even in C if $i \equiv 0 \pmod 3$, otherwise x_i has subscript odd in C. It follows that $(v, x_0, x_1, x_3, x_4, \ldots, x_r, x_{r+1})$ and (x_{r+1}, C, v) are two subpaths of C with even length, where v is the predecessor of x_0 in C. By Construction 2, we have that $C' = (x_{\sigma_0}, x_{r+1}) \cup (x_{r+1}, C, v) \cup (v, x_{\sigma_0})$ is a cycle of D' with even length. For the convenience of the proof, we rewrite C' as follows $C' = (w_0, w_1, \ldots, w_{2l-1}, w_0)$ such that, w_i has even subscript in C if and only if w_i has even subscript in C'. By Construction 2 and by the properties of C, (w_i, w_{i+1}) is asymmetrical in D', for every $i \in \{0, 2, \ldots, 2(l-1)\}$; even more, since $\{v_0, v_2, \ldots, v_{2(l-1)}\}$ is independent in D and by Construction 2, we have that $\{w_0, w_2, \ldots, w_{2(l-1)}\}$ is an independent set in D'. By Theorem 7, D' is a KP-digraph.

Case 2. x_0 has odd subscript in C.
By Lemma 6 we have two subcases:

Case 2.1. $V(C) \cap V(\sigma) = \{x_0\} \cup \{x_i : \ i \equiv 1 \pmod 3 \text{ or } i \equiv 2 \pmod 3 \text{ and } 0 \leq i \leq r\}$ such that x_i has subscript even in C if $i \equiv 1 \pmod 3$, otherwise x_i has subscript odd in C. In particular, x_{r-2} has even subscript in C and x_{r-1} has odd subscript in C. In addition, x_{r-3} and x_r are not vertices of C. Since $N^+(x_{r-1}) = \{x_{r-3}, x_r\}$, it follows that x_{r-1} has no successor in C, which is impossible. Therefore, this subcase is impossible.

Case 2.2. $V(C) \cap V(\sigma) = \{x_i : \ i \equiv 0 \pmod 3 \text{ or } i \equiv 2 \pmod 3 \text{ and } 0 \leq i \leq r\}$ such that x_i has subscript even in C if $i \equiv 2 \pmod 3$, otherwise x_i has subscript odd in C. Since x_0 has odd subscript in C, we have that x_{n-1} is the predecessor of x_0 in C. It follows that (v, C, x_{n-1}) and $(x_{n-1}, x_0, x_2, x_3, x_5, x_6, \ldots, x_{k-1}, x_k, v)$ are two subpaths of C with even length, where v is the successor vertex of x_k in C. By Construction 2, $C' = (x_{n-1}, x_{\sigma_0}, v) \cup (v, C, x_{n-1})$ is an even cycle of D'. For the convenience, we rewrite C' as follows $C' = (w_0, w_1, \ldots, w_{2l-1}, w_0)$ such that, w_i has even subscript in C if and only if w_i has even subscript in C. By Construction 2 and by the properties of C, (w_i, w_{i+1}) is asymmetrical in D', for every $i \in \{0, 2, \ldots, 2(l-1)\}$; even more, since $\{v_0, v_2, \ldots, v_{2(l-1)}\}$ is independent in D and by Construction 2, we have that $\{w_0, w_2, \ldots, w_{2(l-1)}\}$ is an independent set in D'. By Theorem 7, D' is a KP-digraph.
It is important to note that if $V(C) \cap V(\sigma) \neq \emptyset$, then we exhibit an even cycle $C' = (w_0, w_1, \ldots, w_{2l-1}, w_0)$ in D' such that $N' = S' \cup S'_1$ is a kernel of D', where $S' = \{w_0, w_2, \ldots, w_{2(l-1)}\}$ and S'_1 is a kernel of $D'[V(D') \setminus (S' \cup N^-(S'))]$.
The case $r \equiv 1 \pmod 3$ is similar to the case $r \equiv 0 \pmod 3$. By Lemma 6, the even cycle of D' is $(x_{\sigma_0}, x_{\sigma_1}, v_l) \cup (v_l, C, v_{2k-1})$ if x_0 has even subscript in C, where v_l is the successor vertex of x_r in C. Otherwise, the even cycle of D' is $(x_{n-1}, x_{\sigma_0}, x_{\sigma_1}, x_{r+1}) \cup (x_{r+1}, C, x_{n-1})$.
Also, the case $r \equiv 2 \pmod 3$ is similar to the case $r \equiv 0 \pmod 3$. In this case, x_0 has odd subscript in C and, by Lemma 6, the even cycle of D' is $(x_{n-1}, x_{\sigma_0}, x_{\sigma_2}, x_{r+1}) \cup (x_{r+1}, C, x_{n-1})$ or $(x_{n-1}, x_{\sigma_0}, x_{\sigma_1}, x_{\sigma_2}, v_l) \cup (v_l, C, x_{n-1})$, where v_l is the successor vertex of x_r in C.

For the converse, suppose that D' is a KP-digraph. We will prove that D is a KP-digraph. By Theorem 7, there is an even cycle $C' = (w_0, w_1, \dots, w_{2k-1}, w_0)$ such that (w_i, w_{i+1}) is asymmetrical in D', for every $i \in \{0, 2, \dots, 2(k-1)\}$, and $S' = \{w_0, w_2, \dots, w_{2(k-1)}\}$ is independent in D'. Moreover, $N' = S' \cup S'_1$ is a kernel of D', where S'_1 is a kernel of $D'[V(D') \setminus (S' \cup N^-(S'))]$. Observe that if $V(\sigma) \cap V(C') = \emptyset$, then C' is a cycle in D with the same properties. Hence, D is a KP-digraph. Now assume that $V(\sigma) \cap V(C') \neq \emptyset$, and by Lemma 6, we have three possibilities.

When $r \equiv 0 \pmod 3$, we can assume that $x_{\sigma_0} \in V(C')$. We have the following cases.

Case 1. x_{σ_0} has even subscript in C'.

Suppose, without loss of generality, that $w_0 = x_{\sigma_0}$; it follows that $w_1 = x_{r+1}$. Consider

$$C = (x_0, x_1, x_3, x_4, \dots, x_{r-3}, x_{r-1}, x_r, x_{r+1}) \cup (w_1, C', w_{2k-1}) \cup (w_{2k-1}, x_0).$$

Since (w_{2k-1}, x_{σ_0}) and (x_{σ_0}, x_{r+1}) are arcs of D', by Construction 2, (w_{2k-1}, x_0) and (x_r, x_{r+1}) are arcs of D. It follows that C is a cycle of D. Note that the length of $(x_0, x_1, x_3, x_4, \dots, x_{r-3}, x_{r-2}, x_r, x_{r+1})$ is odd and the length of $(x_{r+1} = w_1, C', w_{2k-1}) \cup (w_{2k-1}, x_0)$ is odd, it implies that C has even length. By the choice of C', we have that (w_i, w_{i+1}) is asymmetrical in D, for every $i \in \{2, \dots, 2(k-1)\}$ and by Construction 2 and the structure of D, (x_j, x_{j+1}) is asymmetrical in D, for every $j \equiv 0 \pmod 3$ with $0 \le j \le r$. On the other hand, since S' is independent in D', by Construction 2 and the structure of D, we have that $S = \{x_0, x_3, \dots, x_r\} \cup \{w_2, w_4, \dots, w_{2(k-1)}\}$ is independent in D. By Theorem 7, D is a KP-digraph.

Case 2. x_{σ_0} has odd subscript in C'.

Suppose, without loss of generality, that $w_1 = x_{\sigma_0}$; it follows that $w_0 = x_{n-1}$. Note that $w_2 \neq x_{n-2}$, otherwise $C' = (x_{n-1}, x_{\sigma_0}, x_{n-2}, x_{n-1})$ which is impossible. It follows that $w_2 = x_{r+1}$, or $w_2 = x_{r+2}$. Consider

$$C = (x_{n-1}, x_0, x_2, x_3, x_5, x_6, \dots, x_{r-1}, x_r, w_2) \cup (w_2, C', w_0 = x_{n-1}).$$

Since (x_{n-1}, x_{σ_0}) and (x_{σ_0}, w_2) are arcs of D', by Construction 2, (x_{n-1}, x_0) and (x_r, w_2) are arcs of D. It follows that C is a cycle of D. Note that both $(x_{n-1}, x_0, x_2, x_3, x_5, x_6, \dots, x_{r-1}, x_r, w_2)$ and (w_2, C', w_0) have even length, it implies that C has even length. By the choice of C', we have that (w_i, w_{i+1}) is asymmetrical in D, for every $i \in \{2, \dots, 2(k-1)\}$; by Construction 2 and the structure of D, (x_j, x_{j+1}) is asymmetrical in D, for every $j \equiv 2 \pmod 3$ with $2 \le j \le r$. Even more, since S' is independent in D', by Construction 2 and the structure of D, we have that $S = \{x_0, x_3, \dots, x_r\} \cup \{w_2, w_4, \dots, w_{2(k-1)}\}$ is independent in D. By Theorem 7, D is a KP-digraph.

The case $r \equiv 1 \pmod 3$ is similar to the case $r \equiv 0 \pmod 3$. By Lemma 6, $(x_0, x_1, x_3, x_4, \dots, x_{r-4}, x_{r-3}, x_{r-1}, x_r, w_2) \cup (w_2, C', w_{2k-1}) \cup (w_{2k-1}, x_0)$ is the even cycle of D, if x_{σ_0} has even subscript in C. Otherwise, the even cycle of D is $(x_{n-1}, x_0, x_1, x_2, x_4, x_5, \dots, x_{r-3}, x_{r-2}, x_r, x_{r+1}) \cup (x_{r+1}, C', x_{n-1})$.

Also, the case $r \equiv 2 \pmod 3$ is similar to the case $r \equiv 0 \pmod 3$. In this case, x_{σ_0} has odd subscript in C' and, by Lemma 6, the even cycle of D' has two possibilities: $(x_{n-1}, x_0, x_2, x_3, \dots, x_{r-3}, x_{r-2}, x_r, x_{r+1}) \cup (x_{r+1}, C', x_{n-1})$ or $(x_{n-1}, x_0, x_1, x_2, \dots, x_{r-1}, x_r, w_2) \cup (w_2, C, x_{n-1})$ □

Theorem 11. *If D is a digraph, then there is a digraph $\hat{D}$ where every complete chain of 2-diagonals has length 2, such that D has a kernel if and only if $\hat{D}$ has a kernel.*

The digraph $\hat{D}$ sought in the proof of Theorem 11 is obtained from D by recursively applying Construction 2. It is important to note that if $\hat{D}$ has at least one maximal complete chain of 2-diagonals, then, by Theorem 8, $\hat{D}$ has a kernel, and by Theorem 11, D has a kernel. Hence D is a KP-digraph. In the other case, if $\hat{D}$ has no maximal complete chain of 2-diagonals, then $\hat{D}$ is an asymmetrical cycle. If $\hat{D}$ is an even cycle, then $\hat{D}$ has a kernel, and by Theorem 11, D has a kernel. Thus, D is a KP-digraph. But if $\hat{D}$ is an odd cycle, then $\hat{D}$ has no kernel and, by Theorem 11, D has no kernel and D is a CKI-digraph. Therefore, we have the following result.

Theorem 12. *Let D be a digraph with a Hamiltonian cycle as its asymmetrical part. If $\hat{D}$ is a digraph obtained from D after replacing every maximal complete chain of 2-diagonals as in Construction 2, then D is a CKI-digraph if and only if $\hat{D}$ is an asymmetrical odd cycle.*

From Construction 2, Theorem 11 and Lemmas 5 and 7, we obtain 2 algorithms. The first has a digraph D with a Hamiltonian cycle γ as its asymmetrical part, where any other arc of D is a symmetrical diagonal of γ of length two, as input. The algorithm finds the maximal complete chains of 2-diagonals of length greater than 2 and contracts them using Construction 2, preserving the vertex labels of the contractions performed. By contracting all the maximal complete chains of 2-diagonals of length greater than 2, then three possible results are obtained. In case of obtaining an odd cycle, the output of the algorithm says that D is a CKI-digraph, otherwise D is a KP-digraph.

The second algorithm takes the labels obtained in each step and the digraph $\hat{D}$ obtained from the first algorithm as input. In case that $\hat{D}$ is not an odd cycle, then $\hat{D}$ is an even cycle, or every complete chain of 2-diagonals of $\hat{D}$ has length 2, and has at least one alternating chain of 2-diagonals. By Lemma 5, in both cases there is a kernel that is easy to find. The algorithm takes a kernel of $\hat{D}$ and constructs a kernel in the digraph obtained in the previous step of the first algorithm; continuing this procedure recursively, we obtain a kernel of D.

To conclude, by Theorem 12, there exists an infinite family of CKI-digraphs whose asymmetrical part is a Hamiltonian cycle and every symmetrical diagonal has length 2. Furthermore, all of these digraphs can be obtained by replacing vertices or arcs with a complete chain of 2-diagonals of length congruent with 0 or 1 modulo 3, respectively.

6. Consequences and conclusions

In this section, some consequences of Theorems 5 and 7 are provided.

Corollary 6. *If $n \geq 8$, then $\overrightarrow{C}_n(1, \pm 2, \pm 3)$ is a KP-digraph if and only if $n \equiv 0 \pmod 4$.*

Proof. First, suppose that $\overrightarrow{C}_n(1, \pm 2, \pm 3)$ is a KP-dgiraph and $n = 4k + r$ with $r \in \{0, 1, 2, 3\}$ and $k \geq 2$. Let N be a kernel of $\overrightarrow{C}_n(1, \pm 2, \pm 3)$. Suppose, without loss of generality, that $0 \in N$. It follows that $1, 2, 3, n-1, n-2, n-3$ are not in N. Since $N^+(1) = \{2, 3, 4, n-1, n-2\}$, $4 \in N$, it implies that $5, 6, 7$ are not in N. Since $N^+(5) = \{6, 7, 8, 3, 2\}$, it follows that $8 \in N$. Continuing with this procedure, we have that $4j \in N$ for every $j \in \{0, 1, \ldots, k\}$. Observe that if $r \neq 0$, then $(4k, 0)$ is an arc of $\overrightarrow{C}_n(1, \pm 2, \pm 3)$, which is impossible. Therefore $r = 0$.
Now, suppose that $n = 4k$. Notice that $(0, 1, 4, 5, \ldots, 4(k-1), 4(k-1)+1, 0)$ is an even cycle where the hypotheses of Theorem 7 are fulfilled. □

Corollary 7. *If $n \geq 8$, then $\overrightarrow{C}_n(1, \pm 2, \pm 4)$ is a KP-digraph.*

Proof. Note that $(0, 1, 3, 4, 0)$ is an even cycle where the hypotheses of Theorem 7 are fulfilled. □

Corollary 8. *Let $n \geq 8$ be an integer. If $n \equiv 0 \pmod 3$ or $n \equiv 0 \pmod 4$, then $\overrightarrow{C}_n(1, \pm 2, \pm 5)$ is a KP-digraph.*

Proof. First, suppose that $n \equiv 0 \pmod 3$. It follows that $n = 3k$ for some $k \geq 3$. We have two cases.

Case 1. $k = 2r$.
Hence, $n = 6r$. Note that $(0, 1, 6, 7, 12, 13, \ldots, 6(r-1), 6(r-1)+1, 0)$ is an even cycle of $\overrightarrow{C}_n(1, \pm 2, \pm 5)$ where the hypotheses of Theorem 7 are fulfilled.

Case 2. $k = 2r + 1$.
Hence, $n = 6r + 3$. Note that $(0, 1, 6, 7, \ldots, 6r, 6r+1, 0)$ is an even cycle of $\overrightarrow{C}_n(1, \pm 2, \pm 5)$ where the hypotheses of Theorem 7 are fulfilled. Thus $\overrightarrow{C}_n(1, \pm 2, \pm 5)$ is a KP-digraph.
When $n \equiv 0 \pmod 4$ it is enough to note that $(0, 1, n-4, n-3, n-8, n-7, \ldots, 4, 5, 0)$ is an even cycle of $\overrightarrow{C}_n(1, \pm 2, \pm 5)$ where the hypotheses of Theorem 7 are fulfilled. Therefore, $\overrightarrow{C}_n(1, \pm 2, \pm 5)$ is a KP-digraph. □

Corollary 9. *If $n \geq 12$ is an integer, then $\overrightarrow{C}_n(1, \pm 4, \pm 5)$ is a KP-digraph.*

Proof. Observe that $(0, 1, 6, 7, 3, 4, 0)$ is an even cycle of $\overrightarrow{C}_n(1, \pm 4, \pm 5)$ where the hypotheses of Theorem 7 are fulfilled. □

The following results are consequences of Theorem 5 and the results in the preceding sections.

Corollary 10. *Let D be a digraph. If $Asym(D)$ is a disjoint union of asymmetrical cycles $\overrightarrow{C}_{n_1}, \ldots, \overrightarrow{C}_{n_k}$, then D is a KP-digraph if and only if $D[V(\overrightarrow{C}_{n_i})]$ has a kernel, for every $i \in \{1, \ldots, k\}$.*

Observe that the circulant digraph $\overrightarrow{C}_{jk}(j)$, with $k \geq 3$, is the disjoint union of k asymmetrical cycles with order j. If $\gamma_1, \ldots, \gamma_k$ are those cycles, then we have the following result.

Corollary 11. *Let D be a digraph. If $Asym(D)$ is the circulant digraph $\overrightarrow{C}_{jk}(j)$ with $k \geq 3$, then D is a KP-digraph if and only if $D[V(\gamma_i)]$ has a kernel.*

Proceeding similarly to Corollary 10, we have the following result.

Corollary 12. *Let D be a digraph with property P, with $n \geq 4$. If $Asym(D)$ is the disjoint union of digraphs $D_1, \ldots, D_k$, then D is a KP-digraph if and only if $D[V(D_i)]$ is a KP-digraph.*

From Corollary 12, we can conclude that if the asymmetrical part of a digraph D with property P is the disjoint union of digraphs, then the problem of verifying whether a digraph is a KP-digraph is equivalent to the problem of verifying that each of the subdigraphs induced by the vertices forming the digraphs in the asymmetrical part is a KP-digraph, instead of verifying each of the proper induced subdigraphs of D. In particular, by Corollary 10, if the asymmetrical part is the disjoint union of cycles, then it suffices to verify that each of the subdigraphs induced by the vertices of each cycle forming the asymmetrical part has a kernel. For this purpose, the results from this section and the preceding sections prove to be very helpful.

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