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Research Article

On subdivisions of oriented cycles in Hamiltonian digraphs with small chromatic number*

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Abstract: Cohen *et al.* conjectured that for each oriented cycle C, there is a smallest positive integer $f(C)$ such that every $f(C)$-chromatic strong digraph contains a subdivision of C. Let C be an oriented cycle on n vertices. For the class of Hamiltonian digraphs, El Joubbeh proved that $f(C) \leq 3n$. In this paper, we improve El Joubbeh's result by showing that $f(C) \leq 2n$ for the class of Hamiltonian digraphs.

Keywords: oriented cycle, Hamiltonian, chromatic number, subdivision.

AMS Subject classification: 05C15, 05C20

1. Introduction

In this paper, graphs are finite and simple, that is they have no loop nor multiple edges, while digraphs are oriented graphs. Let D be a digraph obtained by assigning to each edge $e = xy$ of G an orientation (x, y) or (y, x) but not both. G is called the underlying graph of D. If $e = xy$ is an edge of G, then we say that x and y are neighbors of each other and that they are adjacent. The degree of a vertex x is the number of its neighbors and it is denoted by $d_G(x)$. The neighborhood of a vertex x

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is the set of all its neighbors and it is denoted by $N_G(x)$. A path $P = x_1x_2\dots x_n$ is a graph on n vertices $x_1, x_2, \dots, x_n$ and whose edges are x_ix_{i+1}, for $1 \le i < n$. An oriented path $P = x_1x_2\dots x_n$ is an orientation of a path and it is a directed path if its arcs are (x_i, x_{i+1}), for $1 \le i < n$. A cycle $C = x_1x_2\dots x_nx_1$ is a graph on n vertices $x_1, x_2, \dots, x_n$ and whose edges are x_ix_{i+1}, for $1 \le i < n$ and the edge x_nx_1. An oriented cycle $C = x_1x_2\dots x_n$ is an orientation of a cycle and it is a directed cycle if its arcs are (x_i, x_{i+1}), for $1 \le i < n$ and (x_n, x_1). The length of a path or a cycle is the number of its edges. The length of an oriented path or oriented cycle is the number of its arcs. A block of a cycle C is a maximal directed path in C. Note that the number of blocks of a non-directed oriented cycle C is always even. Let $k_1, k_1', \dots, k_t, k_t'$ be the lengths of the blocks of an oriented cycle (which is non-directed) C, where $2t$ is the number of blocks of C. If the first block is of length k_1 and it is forward then we write $C = C^+(k_1, k_1', \dots, k_t, k_t')$, otherwise we write $C = C^-(k_1, k_1', \dots, k_t, k_t')$. Note that $C^+(k_1, k_1', \dots, k_t, k_t') = C^-(k_1', k_2, \dots, k_t, k_t', k_1)$ and $C^-(k_1, k_1', \dots, k_t, k_t') = C^+(k_1', k_2, \dots, k_t, k_t', k_1)$. A Hamiltonian path (resp. cycle) is a path (resp. cycle) passing through all the vertices of a graph G. A Hamiltonian directed path is a directed path passing through all the vertices of a digraph D. A Hamiltonian directed cycle is a directed cycle passing through all the vertices of a digraph D. A digraph is Hamiltonian if it has a Hamiltonian directed cycle. A digraph D is strongly connected (or strong) if for any two vertices x and y of D, there is a directed path in D from x to y. A subdivision D' of a digraph D can be obtained from D by replacing each arc (x, y) by a directed path from x to y, all these paths being internally disjoint (note that $D' = D$ if all these paths are of length 1). If A is a subset of the set of vertices of D, then $D[A]$ denotes the subdigraph of D induced by A. The chromatic number of a graph is the smallest integer n such that all the vertices can be colored using n colors in a way that any two neighbor vertices receive distinct colors. The chromatic number of a digraph D is that of its underlying graph and is denoted by $\chi(D)$. We say that D is n-chromatic if its chromatic number is n. A graph G is n-degenerate, if every subgraph of G has a vertex of degree at most n. It is well known that every n-degenerate graph has chromatic number at most $n+1$.

A classical result of Gallai and Roy is the following:

Theorem 1. *(Roy-Galli [11]) Every digraph with chromatic number at least $n+1$ contains a directed path of length at least n.*

Now, the following question arises: Which digraphs are subdigraphs of all digraphs with large chromatic number?

Due to the famous theorem of Erdös [8] that asserts the existence of graphs with arbitrarily large chromatic number and arbitrarily large girth, we have that for every digraph H containing an oriented cycle C of length n, there is a digraph D with arbitrarily large chromatic number and girth greater than n. So H can not be a subdigraph of D. Thus, the only possibly candidates to generalize Theorem 1 are oriented trees. Burr [4] proved that every $(n-1)^2$-chromatic digraph contains every oriented tree of order n and conjectured that every every $2n-2$-chromatic digraph contains a copy of any oriented tree T on n vertices [4]. The best known result about Burr's conjecture is obtained by Addario-Berry et al. [1] who proved that every $(\frac{n^2}{2}-\frac{n}{2}+1)$-chromatic digraph contains a copy of any oriented tree T on n vertices. In the special case when T is an oriented path on n vertices and t blocks with $t \leq \sqrt{\frac{n-1}{2}}+1$, recently Ghazal and El Joubbeh [7] reached a better bound. However, the following celebrated theorem of Bondy shows that the story does not stop here.

Theorem 2. *(Bondy [3]) Every strongly connected digraph of chromatic number at least n contains a directed cycle of length at least n.*

A directed cycle of length at least n can be viewed as a subdivision of the directed cycle of length exactly n. Now the following question arises:
For a given non-directed cycle C, can we find a number $f(C)$ such that every digraph D with $\chi(D) \geq f(C)$ contains a subdivision of C? The answer is again negative due to the following result proved by Cohen et al. in 2018.

Theorem 3. *([5]) For any positive integers b and n, there is an acyclic digraph D with $\chi(D) > n$ and every oriented cycle of D has at least b blocks.*

However, for strong digraphs, Cohen et al. conjectured that the answer of the above question is positive.

Conjecture 1. (Cohen et al. [5]) For every oriented cycle C, there is a constant $f(C)$ such that every strong digraph with chromatic number at least $f(C)$, contains a subdivision of C.

If $C = C(k,l)$ is a cycle with two blocks Cohen *et al.* [5] proved that $f(C)$ is $O((k+l)^4)$ and this bound was improved by Kim *et al.* [9] to $O((k+l)^2)$. Recently, Ghazal *et al.* [10] proved that if D is a digraph (not necessarily strong) that has a Hamiltonian directed path and $\chi(D) > 3.max\{k,l\}$, then D contains a subdivision of $C(k,l)$.

Let D be a digraph. A set of vertices S of D is called a *stable set* if for any $x, y \in S$, neither (x,y) nor (y,x) is an arc of D. The stability of D, denoted by $\alpha(D)$, is the maximum size of a stable set in D, that is $\alpha(D)$ =max$\{|S|; S$ is a stable set of $D\}$. The following theorem shows that the vertices of a strong digraph can be covered by the vertices of at most $\alpha(D)$ directed cycles.

Theorem 4. *(Bessy et al. [2]) The vertex set of every strong digraph D is the union of the vertex sets of at most $\alpha(D)$ directed cycles of D.*

The above theorem, motivates us to study Cohen's conjecture restricted to Hamiltonian digraphs, because the subdigraph induced by the vertices of each of the above directed cycles is Hamiltonian. Note that Hamiltonian digraphs are particular cases of strong digraph. For Hamiltonian digraphs and for any oriented cycle, El Joubbeh obtained the following bound.

Theorem 5. *(El Joubbeh [6]) Every $3n$-chromatic Hamiltonian digraph contains a subdivision of any oriented cycle on n vertices.*

The previous theorem, shows that $f(C) \leq 3n$ for the class of Hamiltonian digraph, where n is the number of vertices of C. In this paper, we improve this bound and show that $f(C) \leq 2n$ for the class of Hamiltonian digraph.

2. Main Results

Suppose that $L = x_0x_1x_2 \dots x_N$ is a linear ordering of the vertices of a graph G. The interval $[x_i, x_j] = \{x_s; i \leq s \leq j\}$ and $[x_i, x_j[= \{x_s; i \leq s < j\}$. Let $e = x_lx_m$ and $e' = x_px_q$ be two edges of G and let k be a positive integer. We say that e and e' are secant edges of G with respect to L if $l < p < m < q$. In addition, if $p-l$, $m-p$ and $q-m$ are all at least k, then we say that they are k-secant edges of G with respect to L.

Ghazal *et al.* [10] proved that graphs with no secant edges with respect to a linear order has a bounded chromatic number.

Lemma 1. *([10]) If G has no secant edges with respect to some linear ordering of G, then $\chi(G) \leq 3$.*

Proof. Suppose that G has no secant edges with respect to some linear ordering L of G. We will prove that G is 2-degenerate and thus $\chi(G) \leq 3$. Let H be a subgraph of G and let $L' = x_1x_2 \dots x_N$ be the restriction of L to the vertices of H. Note that H has no secant edges with respect to L'. We will find a vertex v of H such that $d_H(v) \leq 2$. If H has no edge x_ix_j with $j - i > 1$, then $N_H(x_1) \subseteq \{x_2\}$ and hence $d_H(x_1) \leq 1 \leq 2$. In this case, we take $v = x_1$. Otherwise, let x_ix_j be an edge of H with $j - i > 1$ such that $j - i$ is minimum. Since H has no secant edges with respect to L', then $N_H(x_{i+1}) \subseteq \{x_i, x_{i+2}\}$. Whence $d(x_{i+1}) \leq 2$ and we can take $v = x_{i+1}$, in this case. $\square$

However, for graphs without k-secant edges, we have the following lemma:

Lemma 2. *If G has no k-secant edges with respect to some linear ordering of G, then $\chi(G) \leq 3k$.*

Proof. Suppose that G has no k-secant edges with respect to some linear ordering $L = x_0x_1x_2 \dots x_N$ of G. For each $0 \leq i \leq k-1$, let G_i be the subgraph of G induced by the set $V_i = \{x_{i+sk}; s \geq 0\}$ and let L_i be the restriction of L to V_i. If G_i has secant edges with respect to L_i, then these secant edges are k-secant edges of G with respect to L, which is a contradiction. So, each G_i has no secant edges and thus $\chi(G_i) \leq 3$. Note that the V_i's form a partition of the vertex set of G and therefore, $\chi(G) \leq \sum_{i=0}^{k-1} \chi(G_i) \leq \sum_{i=0}^{k-1} 3 = 3k$. □

Corollary 1. *If G is a graph with $\chi(G) \geq 3k+1$, then G has k-secant edges with respect to any linear ordering of G.*

Proof. Immediate consequence of Lemma 2. □

Theorem 6. *If D is a Hamiltonian digraph such that $\chi(D) \geq \sum_{i=1}^{t}(k_i + 3k'_i)$, then D contains a subdivision of $C^+(k_1, k'_1, \dots, k_t, k'_t)$.*

Proof. Suppose that D is a Hamiltonian digraph such that $\chi(D) \geq \sum_{i=1}^{t}(k_i + 3k'_i)$. Suppose that $C = 0, 1, 2, \dots, N, 0$ is a Hamiltonian directed cycle of D. We consider the linear ordering $L = 0, 1, 2, \dots, N$ of the vertices of D.

Set $b_1 = 0$ and define $D_1 = D[b_1, b_1 + k_1 - 1[$. Note that the length of the path $C[b_1, b_1 + k_1 - 1[$ is $k_1 - 1$ and the chromatic number of D_1 is at most $k_1 - 1$.

Let b_2 be maximum such that $D'_1 := D[b_1 + k_1 - 1, b_2[$ has no k'_1-secant edges. Note that b_2 exists, since the chromatic number of graphs without k'_1-secant edges is at most $3k'_1$ and since the chromatic number of D is large enough. By maximality of b_2, the digraph $D[b_1 + k_1 - 1, b_2]$ has k'_1-secant edges, say $e_1 = u_1v_1$ and $e'_1 = u'_1v'_1$ with $u_1 < u'_1 < v_1 < v'_1$. Thus $v'_1 = b_2$. Since D'_1 has no k'_1-secant edges, then $\chi(D'_1) \leq 3k'_1$.

Let $D_2 = D[b_2, b_2 + k_2[$ and let b_3 be maximum such that $D'_2 = D[b_2 + k_2, b_3[$ has no k'_2-secant edges. Note that b_3 exists, since the chromatic number of graphs without k'_2-secant edges is at most $3k'_2$ and since the chromatic number of D is large enough. By maximality of b_3, the digraph $D[b_1 + k_2, b_3]$ has k'_2-secant edges, say $e_2 = u_2v_2$ and $e'_2 = u'_2v'_2$ with $u_2 < u'_2 < v_2 < v'_2$. Thus $v'_2 = b_3$. Since D'_2 has no k'_2-secant edges, then $\chi(D'_2) \leq 3k'_2$.

Let $3 \leq i \leq t$ and suppose that $D_{i-1} = D[b_{i-1}, b_{i-1} + k_{i-1}[$, b_i and $D'_{i-1} = D[b_{i-1} + k_{i-1}, b_i[$ are found as in the above. We define $D_i = D[b_i, b_i + k_i[$ and let b_{i+1} be maximum such that $D'_i = D[b_i + k_i, b_{i+1}[$ has no k'_i-secant edges. Note that b_{i+1} exists, since the chromatic number of graphs without k'_i-secant edges is at most $3k'_i$ and

since the chromatic number of D is large enough. By maximality of b_{i+1}, the digraph $D[b_i+k_i, b_{i+1}]$ has k'_i-secant edges, say $e_i = u_iv_i$ and $e'_i = u'_iv'_i$ with $u_i < u'_i < v_i < v'_i$. Thus $v'_i = b_{i+1}$. Since D'_i has no k'_i-secant edges, then $\chi(D'_i) \leq 3k'_i$.
Note that for all $1 \leq i \leq t$, $\chi(D_i) \leq k_i$ and $\chi(D'_i) \leq 3k'_i$, except $\chi(D_1) \leq k_1 - 1$. Thus $\chi(D) \geq \sum_{i=1}^{t}(k_i + 3k'_i) > \sum_{i=1}^{t}\chi(D_i) + \sum_{i=1}^{t}\chi(D'_i)$.
To conclude, the arcs of $(\bigcup_{i=1}^{t} C[b_i, u_i] \cup e_i \cup C[u'_i, v_i] \cup e'_i) \bigcup (C[v_t, N] \cup (N, 0))$ form a subdivision of $C^+(k_1, k'_1, \ldots, k_t, k'_t)$. $\square$

Corollary 2. *If D is a Hamiltonian digraph such that $\chi(D) \geq \sum_{i=1}^{t}(k_i + 3k'_i)$, then D contains a subdivision of any oriented cycle whose blocks are of respective lengths $k_1, k'_1, \ldots, k_t, k'_t$.*

Proof. Suppose that D is a Hamiltonian digraph such that $\chi(D) \geq \sum_{i=1}^{i=t}(k_i + 3k'_i)$ and let C be any oriented cycle whose blocks are of respective lengths $k_1, k'_1, \ldots, k_t, k'_t$. Either $C = C^+(k_1, k'_1, \ldots, k_t, k'_t)$ or $C = C^-(k_1, k'_1, \ldots, k_t, k'_t)$. In the first case, Theorem 6 shows that D contains a subdivision of $C = C^+(k_1, k'_1, \ldots, k_t, k'_t)$. If $C = C^-(k_1, k'_1, \ldots, k_t, k'_t)$, then we consider the digraph D' obtained from D by reversing the orientation of each arc. Then D' is Hamiltonian and $\chi(D') = \chi(D) \geq \sum_{i=1}^{t}(k_i + 3k'_i)$. Thus by Theorem 6, D' contains a subdivision of $C = C^+(k_1, k'_1, \ldots, k_t, k'_t)$. Therefore, D contains a subdivision of $C = C^-(k_1, k'_1, \ldots, k_t, k'_t)$. $\square$

Corollary 3. *If D is a Hamiltonian digraph such that $\chi(D) \geq \sum_{i=1}^{t}(3k_i + k'_i)$, then D contains a subdivision of any oriented cycle whose blocks are of respective lengths $k_1, k'_1, \ldots, k_t, k'_t$.*

Proof. Enough to remark that $C^+(k_1, k'_1, \ldots, k_t, k'_t) = C^-(k'_1, k_2, \ldots, k_t, k'_t, k_1)$ and $C^-(k_1, k'_1, \ldots, k_t, k'_t) = C^+(k'_1, k_2, \ldots, k_t, k'_t, k_1)$ and then apply Corollary 2. $\square$

Theorem 7. *Every $2n$-chromatic Hamiltonian digraph contains a subdivision of any oriented cycle on n vertices.*

Proof. Let C be any oriented cycle on n vertices. Let $k_1, k'_1, \ldots, k_t, k'_t$ be the lengths of its blocks. Since $\sum_{i=1}^{t}(k_i + k'_i) = n$, then either $\sum_{i=1}^{t} k_i \leq \frac{n}{2}$ or $\sum_{i=1}^{t} k'_i \leq \frac{n}{2}$. Due to Corollaries 2 and 3, we may assume without loss of generality that $\sum_{i=1}^{t} k'_i \leq \frac{n}{2}$. Therefore, $\sum_{i=1}^{t}(k_i + 3k'_i) = \sum_{i=1}^{t}(k_i + k'_i) + 2\sum_{i=1}^{t} k'_i = n + 2\sum_{i=1}^{t} k'_i \leq n + 2(\frac{n}{2}) = 2n = \chi(D)$. Then by Corollary 2, the result follows. $\square$

Due to Theorem 4, we obtain the following result.

Corollary 4. *Let C be a cycle on n vertices and let α be a positive number. Let D be a strong digraph with $\alpha(D) \leq \alpha$. If $\chi(D) \geq 2\alpha n$, then D contains a subdivision of C.*

Proof. Let C, n, α and D be as in the statement of the theorem. By Theorem 4, the vertex set of D is the union of the vertex sets of directed cycles $C_1, \ldots, C_k$ of D, with $k \leq \alpha(D) \leq \alpha$. Since $\chi(D) \geq 2\alpha n$, then there is $1 \leq j \leq k$ such that the subdigraph D_j, induced by the vertex set of C_j satisfies $\chi(D_j) \geq 2n$. To conclude, we apply Theorem 7 to D_j and C. □

Thus, for a given $\alpha > 0$ and oriented cycle C on n vertices, we have proved that $f(C) \leq 2\alpha n$ for the strong digraphs with $\alpha(D) \leq \alpha$.

Let $h(n)$ be the smallest positive integer such that every Hamiltonian $h(n)$-chromatic digraph contains a subdivision of any oriented cycle on n vertices. We have proved that $h(n) \leq 2n$. Since the digraph with vertices a, b and c and arcs $(a, b), (b, c)$ and (c, a) does not contain any subdivision of $C^+(2, 1)$, then we deduce that $h(n) \geq n+1$. Note that in the proof of Theorem 6, we had the secant edges $e_i = u_i v_i$ and $e'_i = u'_i v'_i$ with $u_i < u'_i < v_i < v'_i$. From the path $C[u_i, v'_i]$, we have just used the portion $C[u'_i, v_i]$ in constructing our cycle and we needed that this portion should be of length at least k'_i. This was guaranteed by the fact that $v_i - u'_i \geq k'_i$. We have used neither the portion $C[u_i, u'_i]$, nor the portion $C[v_i, v'_i]$, nor the fact that $u'_i - u_i \geq k'_i$ and $v'_i - v_i \geq k'_i$. So, we could have altered the definition of k-secant edges (without effecting the construction of the cycle) as follows: Two edges $e = x_l x_m$ and $e' = x_p x_q$ of G are k-secant edges of G with respect to L if $l < p$, $m - p \geq k$ and $m < q$. Using this new definition, the bound in Lemma 2 perhaps would decrease, and consequently we could reach a better bound of $h(n)$.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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