Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-15
https://doi.org/10.22049/cco.2024.29470.2019

CCO
Commun. Comb. Optim.

Research Article

On the inverse problem of some bond additive indices

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Received: 20 February 2024; Accepted: 17 July 2024
Published Online: 3 August 2024

Abstract: Inverse problem of topological indices deals with establishing whether or not a given number is a topological index of some graph. In this paper, we study the inverse topological index problem of some bond additive indices. In [3], it was conjectured that every positive integer except finitely many can be the Mostar index and edge Mostar index of some $c-$cyclic graph. We solve this conjecture for tricyclic graphs. We also study the inverse Albertson index problem and inverse sigma index problem for cacti and for cyclic graphs.

Keywords: inverse problem, mostar index, tricyclic graph, albertson index.

AMS Subject classification: 05C50, 05C69, 05C76

1. Introduction

The graphs discussed in this study are simple, connected and non-directional. For the basic graph theoretic terminologies and notations we refer [4, 8]. For a graph $G = (V, E)$, order and size are the cardinalities of the vertex set V and the edge set E respectively. The number of edges incident to a vertex v in G is its degree and is denoted as $d(v)$. The vertex with degree one is called the pendant vertex. Let $G = (V, E)$ be a graph, then the Albertson irregularity index [9] is defined as

$$Alb(G) = \sum_{uv \in E} |d(u) - d(v)|$$

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and the $\sigma-$ irregularity [7] index is defined as

$$\sigma(G) = \sum_{uv \in E} (d(u) - d(v))^2.$$

Similarly, the total irregularity index and total σ irregularity index [5] are defined as

$$Irr_t(G) = \frac{1}{2} \sum_{(u,v) \in V \times V} |d(u) - d(v)|,$$
$$\sigma_t(G) = \frac{1}{2} \sum_{(u,v) \in V \times V} (d(u) - d(v))^2.$$

For each edge $e = uv$ of a connected graph, let $n_u(e|G)$ denote the number of vertices closer to the vertex u than v. Analogously, let $m_u(e|G)$ denote the number of edges closer to the vertex u than v. Then the Mostar index [6] and the edge Mostar index [2] are defined as

$$Mo(G) = \sum_{uv \in E} |n_u(e|G) - n_v(e|G)|,$$
$$Mo_e(G) = \sum_{uv \in E} |m_u(e|G) - m_v(e|G)|.$$

For each edge $e = uv$, let $\phi(e|G)$ and $\mu(e|G)$ denote the contribution of the edge $e = uv$ for the Mostar and edge Mostar index respectively. The inverse problem of topological index is a realization problem which deals with establishing existence or non-existence of a graph G with given integer p as its topological index. In a previous study, Liju Alex *et al.* solved the inverse Mostar index problem [1] for trees and unicyclic graphs. In another work, I. Gutman *et al.* solved the inverse Mostar and edge Mostar index problem [3] for molecular graphs. They also conjectured that except for finitely many integers, all other positive integers can be the Mostar index of some $c-$ cyclic graphs for every $c \geq 3$. They also proposed a similar conjecture in the case of edge Mostar index as well [3]. In this paper, we settle this conjecture for the case of tricyclic graphs. Aysun Yuttras *et al.* [9] settled the inverse irregularity index problem for trees and unicyclic graphs and similar version of this problem for sigma index was studied by I. Gutman *et al.* [7]. Darko Dimitrov *et al.* [5] settled this problem for $c-$ cyclic graphs. In our work, we propose a alternate construction for the solutions of the inverse irregularity index and inverse sigma index problem for $c-$ cyclic graphs. We also settle this problem for the cacti graphs. Additionally, we also propose an alternate construction for the solution of the inverse problem of total irregularity and total sigma irregularity index for $c-$ cyclic graphs.

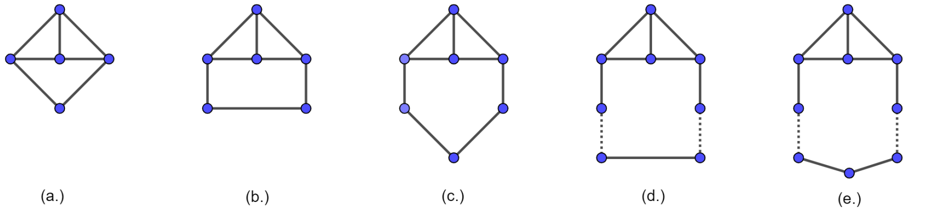

Figure 1. **Tricyclic Graphs described in Construction I**

2. Inverse Problem of Mostar Index

In this section, we settle the inverse Mostar index problem and inverse edge Mostar index problem for tricyclic graphs.

Construction I

Consider the graph $K_4 = G_0$, which is a tricyclic 3 regular graph. Choose $e = uv$ an edge in G_0. G_p is constructed by replacing the edge uv by a path $u = v_0, v_1, \ldots, v_{p+1} = v_0$. Clearly, G_p is a $p+4$ vertex tricyclic graph (see Figure 1).

Theorem 1. *For every even positive integer n, there exists a tricyclic graph G with Mostar index $Mo(G) = n$.*

Proof. G_0 is 3 regular tricyclic graph, since it is complete, for every edge $e = uv$, $\phi(e|G_0) = 0$, therefore, $Mo(G) = 0$. Let $e = uv$ be the edge which subdivided to form the graph G_1. Then the two subdivision edges of G_1 have contribution $\phi(e|G) = 1$ and all other edges has contribution zero. Thus, $Mo(G_1) = 2$. In G_2, the middle edge (the edge which is at equal distance from u and v) of the $u - v$ path of length 3 has contribution 0 and all other edges of the path has contribution $\phi(e|G) = 1$. The rest of the edges of the graph contribute zero. Therefore, $Mo(G_2) = 2$. Now, continuing like this, when p is odd, the $p+1$ edges in the $u - v$ path of G_p each have contribution $\phi(e|G_p) = 1$ and all the other edges have contribution zero. Therefore, $Mo(G_p) = p + 1$. When p is even, the middle edge of the longest $u - v$ path has contribution zero. All the other p edges in the $u - v$ path have contribution 1. The rest of the edges have contribution zero. Therefore, $Mo(G_p) = p$. Therefore, for every even number n, there exist an even order and odd order tricyclic graph G_n, G_{n+1} respectively such that $Mo(G_n) = Mo(G_{n+1}) = n$. □

Construction II

Let G_p, (where p is even) be the graph constructed as in Construction I. Attach a pendant edge on one of the end vertex of the middle edge of the $u - v$ path, call the resultant graph as $G_{p,1}$ (See Figure 2 (a.)). Similarly, consider G_p, p is odd as

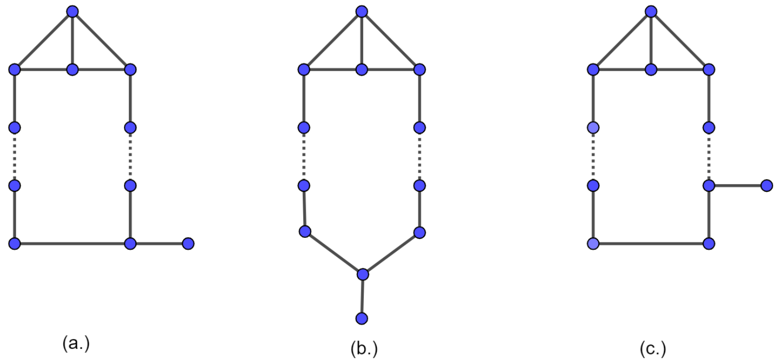

Figure 2. **The graphs (a.) $G_{p,1}$, (b.) $G_{p,2}$, (c.) $G_{p,3}$**

in Construction I. Attach a pendant edge on the middle vertex (the vertex which is at equal distance from both u and v) of the $u - v$ path, call the resultant graph as $G_{p,2}$ (See Figure 2 (b.)). Consider G_p, p odd of Construction I, let d be the distance between the vertex u (or v) to the middle edge e of the $u-v$ path. Let w be the vertex at distance $d - 1$ from u. Attach a pendant vertex at w, and denote the resultant graph by $G_{p,3}$ (See Figure 2 (c.)). If $d = 1$, attach the pendant edge at u or v.

Proposition 1. *For positive integers p,*

(a.) $Mo(G_{p,1}) = p + 6, p \geq 2$

(b.) $Mo(G_{p,2}) = p + 7, p \geq 1$

(c.) $Mo(G_{p,3}) = p + 11, p \geq 2$

Proof. The graph $G_{p,i}, i = 1, 2, 3$ has $p + 5$ vertices and $p + 7$ edges. Every pendant edge e has the contribution $\phi(e|G_{p_i}) = p+3$. In $G_{p,1}$, Every edge in the longest $u-v$ path except the middle edge has contribution 0 and the middle edge has contribution 1. The two other edges incident at u, contributes 1 each and the rest of the edges have contribution 0. Therefore, $Mo(G_{p,1}) = p+3+3 = p+6$. In the case of $G_{p,2}$, all the edges of the $u - v$ path must have contribution 0, the two other edges incident on u (similarly in v) each contributes 1 to the Mostar index. The remaining edges have contribution 0. Thus, $Mo(G_{p,2}) = p + 3 + 4 = p + 7$. In $G_{p,3}$, the middle edge, one edge incident on the pendant edge and one edge incident on v of the longest $u-v$ path has contribution 1, 2, 1 respectively. The two other edges incident on v (similarly on u) each has contribution 1 and all the other edges must have contribution 0. Thus, $Mo(G_{p,3}) = p + 3 + 8 = p + 11$. □

Using these constructions, we can solve the inverse Mostar index problem for tricyclic graphs.

Theorem 2. *For every positive integer $n \geq 13$, there exists a tricyclic graph G with Mostar index $Mo(G) = n$.*

Proof. By Theorem 1, every even integer can be the Mostar index of the tricyclic graph G_p. Also, every even number greater than or equal to 10 can be attained by the graph $G_{p,1}$ and $G_{p,2}$ as well. For odd integers, every odd number from 13 onwards can be attained by the graph $G_{p,3}$. □

We can also use some alternate constructions to settle the inverse Mostar index problem as well.

Construction III

Consider a cycle C_{n-1}(n is odd) with diametrically opposite vertices u and v. The graph $H_{n,1}$ is the tricyclic graph obtained by attaching an edge between the neighbours of u and connecting the neighbours of v by a path of length 2. In the cycle C_{n-1}(n is odd), if we connect the neighbours of u by an edge and the neighbours of v by an edge along with attaching a pendant edge at u or v. We denote the graph obtained as $H_{n,2}$. Consider the cycle C_{n-1} (n is odd) with diametrically opposite vertices u and v. Connect the neighbours of u by an edge and the neighbours of v by an edge and attach a pendant edge at any one of the neighbours of u or v and the resulting graph is denoted by $H_{n,3}$. Consider the cycle C_{n-3}(n odd) with diametrically opposite vertices u and v. The graph obtained by connecting the neighbours of u by a path of length 2 and the neighbours of v by a path of length 2 along with a pendant edge attached at one of the neighbouring vertex of u (or v) is denoted by $H_{n,4}$. In all four graphs, the neighbours of u in the cycle are denoted by u', u'' and those of v are denoted by v' and v'' (See Figure 3).

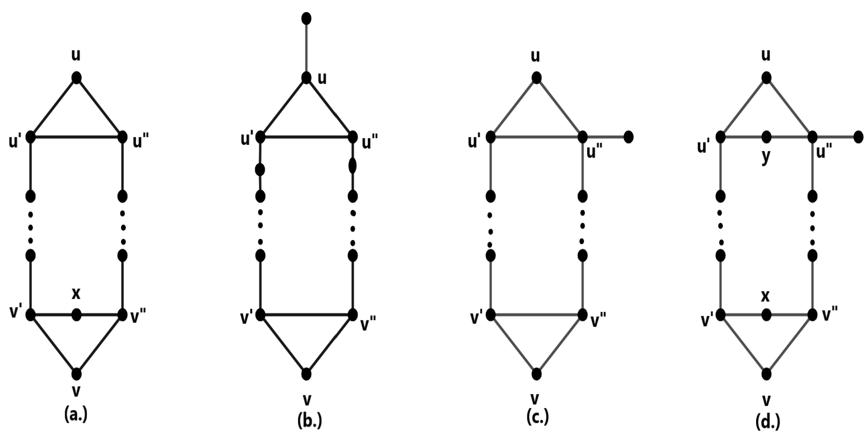

Figure 3. **Graphs (a.)** $H_{n,1}$ **(b.)** $H_{n,2}$ **(c.)** $H_{n,3}$ **(d.)** $H_{n,4}$

Proposition 2. *For the graphs $H_{n,1}, H_{n,2}, H_{n,3}$ and $H_{n,4}$,*

(a.) $Mo(H_{n,1}) = n + 7$.

(b.) $Mo(H_{n,2}) = 4n - 13$.

(c.) $Mo(H_{n,3}) = 4n - 9$.

(d.) $Mo(H_{n,4}) = 2n + 7$.

Proof. (a.) Each edge in the $u' - v'$ path and the $u'' - v''$ path has the contribution $\phi(e|H_{n,1}) = 0$. The edge $e = uu'$ and $e = uu''$ each have contribution $\phi(e|H_{n,1}) = \frac{n-1}{2}$. The four edges connecting the two paths of length 2 from v' to v'' each contributes $\phi(e|H_{n,1}) = 2$. Therefore, $Mo(G) = n - 1 + 8 = n - 7$.

(b.) Each pendant edge contributes $n - 2$ to the Mostar index. Every edge in the $u' - v'$ path and the $u'' - v''$ path has the contribution $\phi(e|H_{n,2}) = 1$. The edge $e = u'u''$ and $e = v'v''$ each have contribution $\phi(e|H_{n,2}) = 0$. The contribution of the edges $e = uu'$ and $e = uu''$ is $\phi(e|H_{n,2}) = \frac{n-5}{2}$. The edges $e = vv'$ and $e = vv''$ have the contribution $\phi(e|H_{n,2}) = \frac{n-1}{2}$. Therefore, $Mo(H_{n,2}) = n - 2 + n - 1 + n - 5 + n - 5 = 4n - 13$.

(c.) Each pendant edge contributes $n - 2$ to the Mostar index. Each edge in the $u' - v'$ path and the $u'' - v''$ path have the contribution $\phi(e|H_{n,3}) = 1$. The edges $u'u''$ and $v'v''$ contributes 1 to the Mostar index. Among the rest of the four edges, two of them contribute $\phi(e|H_{n,3}) = \frac{n-1}{2}$ and the other two contributes $\phi(e|H_{n,3}) = \frac{n-3}{2}$. Therefore, $Mo(G) = n-2+n-5+n-3+n-1+2 = 4n-9$.

(d.) Each pendant edge contributes $n - 2$ to the Mostar index. Each edge in the $u' - v'$ path and the $u'' - v''$ path have the contribution $\phi(e|H_{n,4}) = 1$. Among the four edges in the path connecting $u' - u''$ two edges incident on u' contribute 3 each and the two other edges contributes 1 each. Similarly two edges incident on v' in the path connecting $v' - v''$ contributes 3 and the other two edges contributes 1 each. Therefore, $Mo(H_{n,4}) = n - 2 + n - 7 + 16 = 2n + 7$.

□

Theorem 3. *For every positive integer $n \geq 25$ there exists a tricyclic graph G such that $Mo(G) = n$.*

Proof. We divide the integers t into four different types.

Case 1. t is even.
We know $Mo(H_{n,1}) = n + 7$ where n is odd. When $t = 2k, k \geq 3$ choose $n = t + 1 = 2k + 1$, therefore $Mo(H_{n,1}) = n + 7 = 2k + 1 + 7 = 2k + 8$. Thus, every even integer from 14 onwards will be attained by the tricyclic graph $H_{n,1}$.

Case 2. $t = 8k + 7$.
Choose the tricyclic graph $H_{n,2}$ of order n(odd). We know $Mo(H_{n,2}) = 4n - 13$ where

n is odd. Choose $n = 2k+5, k \geq 1$, then $Mo(H_{n,2}) = 4n-13 = 4(2k+5)-13 = 8k+7$. Thus, every odd integer of the form $8k + 7, k \geq 1$ will be attained by the tricyclic graph $H_{n,2}$.

Case 3. $t = 8k + 3$.
We know $Mo(H_{n,3}) = 4n - 9$ where n is odd. Choose $n = 2k + 3, k \geq 2$, then $Mo(H_{n,3}) = 4n - 9 = 4(2k + 3) - 9 = 8k + 3$. Thus, every odd integer of the form $8k + 3, k \geq 2$ will be attained by the tricyclic graph $H_{n,3}$.

Case 4. $t = 4k + 1$.
We know $Mo(H_{n,4}) = 2n + 7$ where n is odd. Choose $n = 2k - 3, k \geq 6$, then $Mo(H_{n,4}) = 2n + 7 = 2(2k - 3) + 7 = 4k + 1$. Thus, every odd integer of the form $4k + 1, k \geq 6$ will be attained by the tricyclic graph $H_{n,4}$.

□

3. Inverse Problem of edge Mostar Index

In this section, we settle the edge Mostar index inverse problem for tricyclic graphs.

Construction IV

For every even number $n \geq 4$, consider the cycle C_{n-1} and let u and v be two vertices of the cycle at a distance $d(u, v) = \frac{n-2}{2}$. Connect the neighbours of u (u',u'') by an edge and the neighbours of v (v',v'') by a path of length 2. We denote the graph as $H_{n,5}$ (see Figure 4).

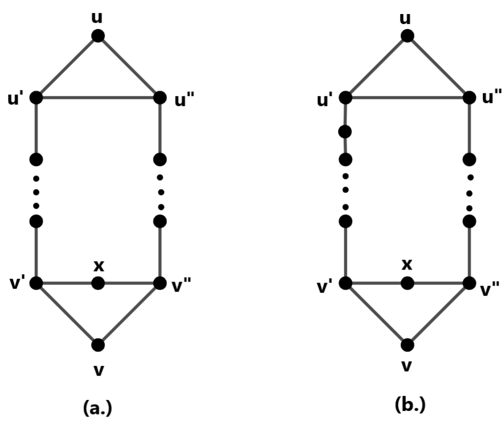

Figure 4. **Graphs (a.)** $H_{n,1}$ **(b.)** $H_{n,5}$

Proposition 3. *For the graphs* $H_{n,1}, H_{n,5}$

(a.) $Mo_e(H_{n,1}) = n + 7$.

(b.) $Mo_e(H_{n,5}) = n + 11$.

Proof. (a.) Each edge in the $u' - v'$ path and the $u'' - v''$ path has the contribution $\mu(e|H_{n,1}) = 0$. The edge $e = uu'$ and $e = uu''$ each have contribution $\mu(e|H_{n,1}) = \frac{n-1}{2}$. The four edges connecting the two paths of length 2 from v' to v'' each contributes $\mu(e|H_{n,1}) = 2$. Therefore, $Mo(H_{n,1}) = n - 1 + 8 = n - 7$.

(b.) Each edge in the $u'' - v''$ path has the contribution $\mu(e|H_{n,5}) = 0$, also every edge in the $u' - v'$ path except the edge incident on u' has the contribution $\mu(e|H_{n,5}) = 0$. For the edge $u'z$ in the $u' - v'$ path, the contribution is 1. The edge $e = uu'$ has contribution $\mu(e|H_{n,5}) = \frac{n}{2}$ and the edge $e = uu''$ contribute $\mu(e|H_{n,5}) = \frac{n+2}{2}$ to the edge Mostar index. The edge $u'u''$ contribute 1 to the edge Mostar index. Among the remaining four edges connecting the two paths of length 2 from v' to v'', two edge incident on v' contribute 1 each and two edges incident on v'' contribute $\mu(e|H_{n,5}) = 3$. Therefore, $Mo_e(H_{n,5}) = \frac{n}{2} + \frac{n+2}{2} + 2 + 2 + 6 = n + 11$.

□

Using the construction defined above we will solve the inverse edge Mostar index problems for tricyclic graphs.

Theorem 4. *For every positive integer greater than $n \geq 19$, there exists a tricyclic graph G with $Mo_e(G) = n$.*

Proof. For every even integer $t = 2k, k \geq 7$, consider the graph $H_{n,1}$ of order $n = 2k - 7, k \geq 7$. We know, $Mo_e(H_{n,1}) = n + 7 = 2k - 7 + 7 = 2k$. Therefore, every even integer greater than 14 can be attained using the graph $H_{n,1}$. Now, in the case of odd integers $n = 2k + 1, k \geq 9$, consider the graph $H_{n,5}$ of order $n = 2k - 10, k \geq 9$. We know, $Mo_e(H_{n.5}) = n + 11 = 2k - 10 + 11 = 2k + 1$. Therefore, every odd integer greater than 19 can attained by the graph $H_{n,5}$. □

Remark 1. Theorem 4 does not imply that only positive integers greater than or equal to 19 can be realized as the edge Mostar index of tricyclic graphs. On the other hand,there are positive integers between 1 and 19 that are the edge Mostar index of tricyclic graphs, and some corresponding graphs have been plotted in the Figure 5.

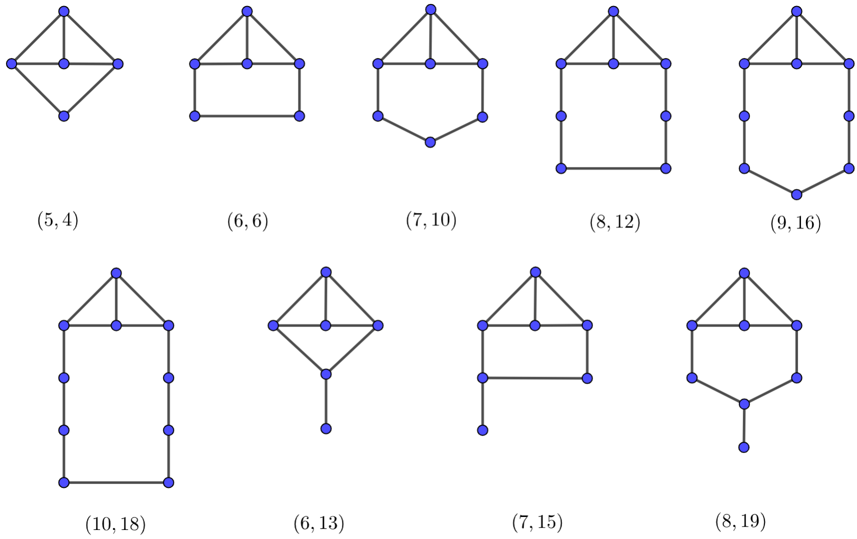

Figure 5. In the pair (a, b), a denote the order of the graph and b denote edge Mostar index of the graph.

4. Inverse Problem of Albertson index and Sigma index

In this section, we provide an alternate construction for the inverse Albertson index problem and inverse sigma index problem for $c-$ cyclic graphs, $c \geq 2$. In [7], I. Gutman *et al.* studied the inverse sigma index problem for trees, unicyclic graphs, bicyclic graphs and connected graphs. Darko Dimitrov*et al.* extended this study on to the class of arbitrary $c-$ cyclic graphs and settled the inverse problem for both the irregularity indices [5]. Although they proved that for every even number $n \geq 4$, there exist infinitely many $c-$ cyclic graph with sigma index $\sigma(G) = n$ and $Alb(G) = n$. Their construction does not give an infinitely many graphs for $n = 2$. In our study we prove that for every even number $n \geq 2$, there exist infinitely many $c-$cyclic graphs $c \geq 3$, such that $\sigma(G) = n$ and $Alb(G) = n$. We also, settle the inverse problem for cacti graphs. An edge $e = uv$ of a graph G is called an (a, b) edge if $d(u) = a$ and $d(v) = b$ (or vice versa) with $a \geq b$.

Construction V

Consider the bicyclic graph G with two distinct cycles C_a and C_b connected by a bridge $e = uv$. Subdivide the the edge $e = uv$ and the resultant graph be denoted by G'. Attach the path of length 2 (or p) on the new subdivision vertex, call the resultant graph as G'' (see Figure 6). Rename the graph G'' as G and repeat this process by taking a $(3, 3)$ edge in the graph G, subdivide the edge, attach a path on the new vertex.

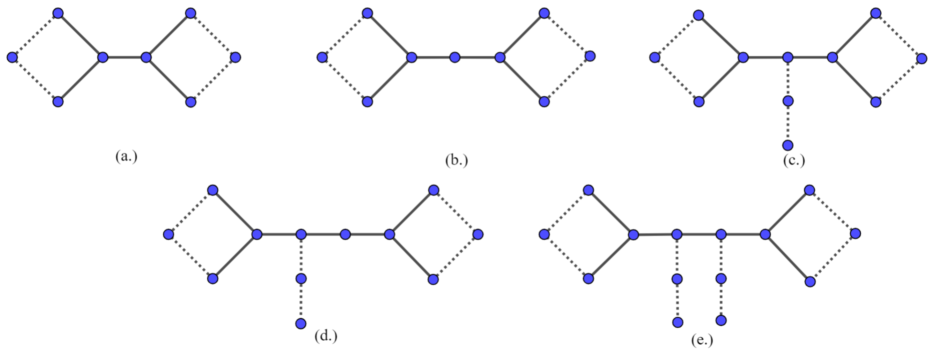

Figure 6. **Graphs in Construction V.**

Theorem 5. *For every even number $n = 2k, k \geq 2$, there exists infinitely many bicyclic graph G with $Alb(G) = n$ and $\sigma(G) = n$.*

Proof. Consider the Construction V. In G, the two edges each incident on the bridge uv contributes 1 each to the Albertson and sigma index and the rest of the edge contribute 0. Since we can consider any cycles for C_a and C_b, there are infinitely many bicyclic graphs with $Alb(G) = 4$ and $\sigma(G) = 4$. In G' along with the original edges, the two new edges formed by the subdivision contributes 1 each to both the index, therefore $Alb(G') = 6$ and $\sigma(G') = 6$. Now when we attach a path at the new subdivision vertex, the edge of the path incident at the new subdivision vertex and the pendant edge contributes 1 each, the rest of the edges in the path have contribution 0. Also, the contribution of the two subdivided edges becomes zero. Therefore, there exist infinitely many bicyclic graphs with $Alb(G') = 6$ and $\sigma(G') = 6$. Now, when we repeat this process, one $(3, 3)$ edge in G becomes two $(3, 2)$ edges, and consequently the total contribution will be increased by 2, thus $Alb(G') = Alb(G) + 2$ and $\sigma(G') = \sigma(G) + 2$. When we attach a path to the new subdivision vertex in G', both the $(3, 2)$ subdivision edges becomes $(3, 3)$ edges, and in the path there will be one $(3, 2)$ edge, one $(2, 1)$ edge and several $(2, 2)$ edges. Therefore, the total contribution becomes zero. Thus, $Alb(G'') = Alb(G')$ and $\sigma(G'') = \sigma(G')$. Thus, the repeated application of transformation gives, infinitely many bicyclic graphs with $Alb(G) = n = 2k$ and $\sigma(G) = n = 2k, k \geq 2$. □

Construction VI

Start with a cycle C_n with vertices $v_1, v_2, \ldots, v_n, n \geq 3$. Take another cycle C_n with vertices $u_1, u_2, \ldots, u_n$. Join the vertices $v_i u_i, i = 1, 2, \ldots, n$ the resultant is a 3 regular $n + 1$ cyclic graph denoted by G. Let G' be the graph obtained by subdividing any

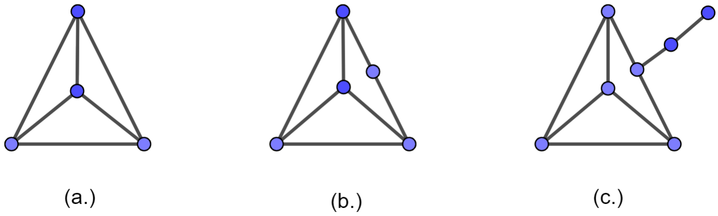

Figure 7. **Construction VI for Tricyclic Graphs.**

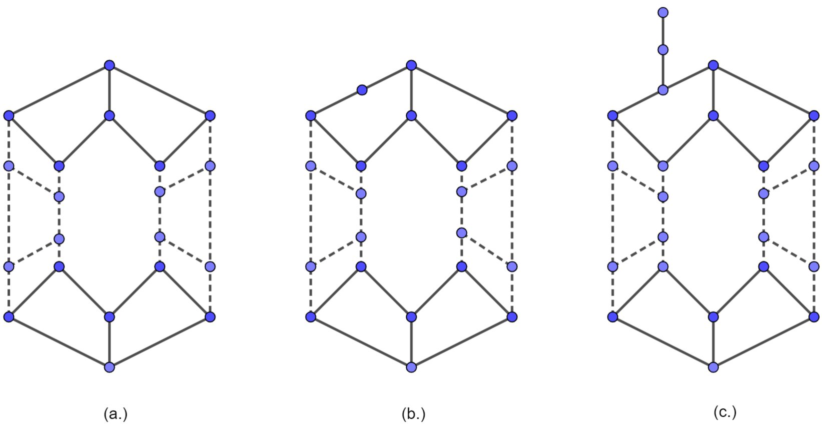

Figure 8. **Construction VI for $c-$ cyclic Graphs, $c \geq 4$.**

(3.3) edge of the graph and G'' be the graph obtained by attaching a path onto the new subdivision vertex of G' (see Figures 7 and 8). In the case of tricyclic graph, start with the complete graph $G = K_4$. Let G' and G'' be obtained as in the previous case.

Theorem 6. *For every even number $n = 2k, k \geq 2$, there exists infinitely many c-cyclic graph G with $Alb(G) = n$ and $\sigma(G) = n$ where $c \geq 3$.*

Proof. Consider the Construction VI. Since G is a regular graph $Alb(G) = \sigma(G) = 0$. In G', one $(3, 3)$ edge becomes a two $(3, 2)$ edge, and no change happens in the remaining edges. Thus the total contribution is increased by 2. Therefore, $Alb(G') = Alb(G) + 2$ and $\sigma(G') = \sigma(G) + 2$. In G'', the two $(3, 2)$ edges of G' becomes $(3, 3)$ edges, resulting in a decrease of the total contribution by 2. But in the path there is a $(3, 2)$ edge and a $(2, 1)$ edge and they two contribute 2 to the total sum. The rest of the edges does not contribute anything. Thus, $Alb(G'') = Alb(G')$ and $\sigma(G'') = \sigma(G')$.

Since, we can add infinitely many distinct paths in each cases, there are infinitely many c-cyclic graphs with $Alb(G) = n = 2k$ and $\sigma(G) = n = 2k, k \geq 1$. □

Now we settle the inverse problem for irregularity indices on cacti graphs. From the Construction I, we can have the following result.

Theorem 7. *For every even number $n = 2k, k \geq 2$, there exists infinitely many cacti graph G with $Alb(G) = n$ and $\sigma(G) = n$.*

Proof. Since all the graphs constructed in Construction I are cacti graphs, we have the result. □

Next we restrict our class of graphs as cacti having fixed number of cycles. On this class we have the following construction.

Construction VII

Start with a cycle C_3 (Level 1) and attach two bridges on two different vertices of the cycle C_3 and attach C_3 in each of the bridges(Level 2). Continuing like this on each cycle C_3, i.e, attach a bridge on each of the vertex of degree 2 in the cycle and attach a C_3 on each of these bridges. If we want a cacti with k (odd) distinct cycles, continue this process will continue until we get k cycles in the graph. Now to get a cacti with $k+1$ cycles, connect the $k+1$-th C_3 on the first level C_3 by a bridge on to the vertex of degree 2 in the first C_3(Level 0). Thus we get a cacti having k distinct cycles, for each k. Let the graph be denoted by G. Take a $(3, 3)$ edge in G, subdivide the edge and let the resultant graph be denoted by G'. Take G', attach a path on to the new subdivision vertex, call the resultant graph as G''. Continue this process (See Figure 9).

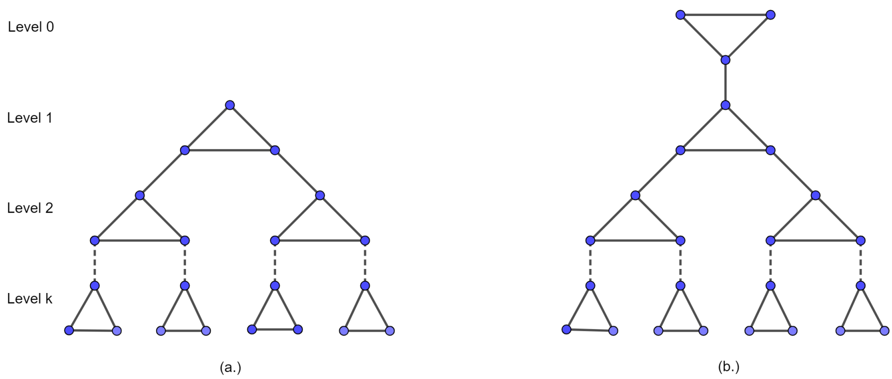

Figure 9. **Cacti's in Construction VII**

Theorem 8. *(a.) For every even number $n \geq c+3$, where c is odd and $c \geq 3$, there exist infinitely many cacti graph G with $c-$ cycles such that $Alb(G) = n$ and $\sigma(G) = n$.*

(b.) For every even number $n \geq c+2$, where c is even and $c \geq 4$, there exist infinitely many cacti graph G with $c-$ cycles such that $Alb(G) = n$ and $\sigma(G) = n$.

Proof. Consider the Construction VII. Let G be the cacti graph in the construction with k cycles. When k is odd, there are two $(3,2)$ edges in Level 1 and $k+1$ other $(3,2)$ pair edges in the graph. All the other edges are $(3,3)$ pair. Therefore, $Alb(G) = k+3$ and $\sigma(G) = k+3, k \geq 3, k$ is odd. Let the graph be denoted as G, in G' one $(3,3)$ edge becomes two $(3,2)$ edge. All the other edges have the same contribution as G. Thus, $Alb(G') = Alb(G)+2$ and $\sigma(G') = \sigma(G)+2$. Now in G'', two subdivision $(3,2)$ edge becomes $(3,3)$ edge and consequently the contribution is decreased by 2. But in the new path attached, one new $(3,2)$ edge and one $(2,1)$ contributes a sum of 2. Therefore, $Alb(G'') = Alb(G')$ and $\sigma(G'') = \sigma(G')$. Therefore, there are infinitely many cacti graph with k cycles having $Alb(G) = 2m$ and $\sigma(G) = 2m, m \geq \frac{k+3}{2}$. When k is even, there are two $(3,2)$ edges in Level 0 and k other $(3,2)$ pair edges in the graph. All the other edges are $(3,3)$ pair. Therefore, $Alb(G) = k+2$ and $\sigma(G) = k+2, k \geq 4$, As in the previous argument, there are infinitely many cacti graphs with k cycles having $Alb(G) = 2m$ and $\sigma(G) = 2m, m \geq \frac{k+2}{2}$. □

5. Inverse problem of total irregularity index

In this section, we settle the inverse total irregularity index problem for $c-$ graphs, $c \geq 3$. In [5], Dimitrov *et al.* established that every even number greater than or equal to $2(c-1)i$ can be the σ_t irregularity index of some $c-$ graphs. We give an alternate construction to complete this work by proving that every even integer greater than $2(c-1)i$ can be the σ_t irregularity index of some $c-$ cyclic graph. A pair (u,v) of vertices is called an (a,b) pair if the degree $d(u) = a$ and $d(v) = b$ or vice versa $(a \geq b)$.

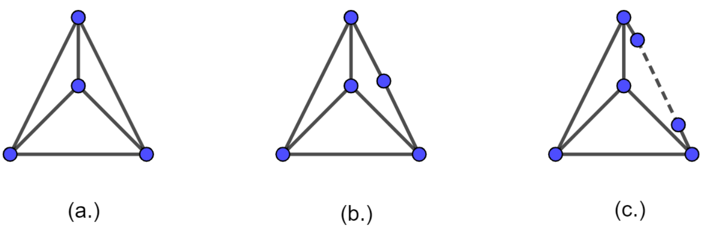

Figure 10. **Construction VIII for tricyclic graphs.**

Construction VIII

Start with a cycle C_n with vertices $v_1, v_2, \ldots, v_n, n \geq 3$, take another cycle C_n with vertices $u_1, u_2, \ldots, u_n$. Join the vertices $v_i u_i, i = 1, 2, \ldots, n$ the resultant graph is a 3

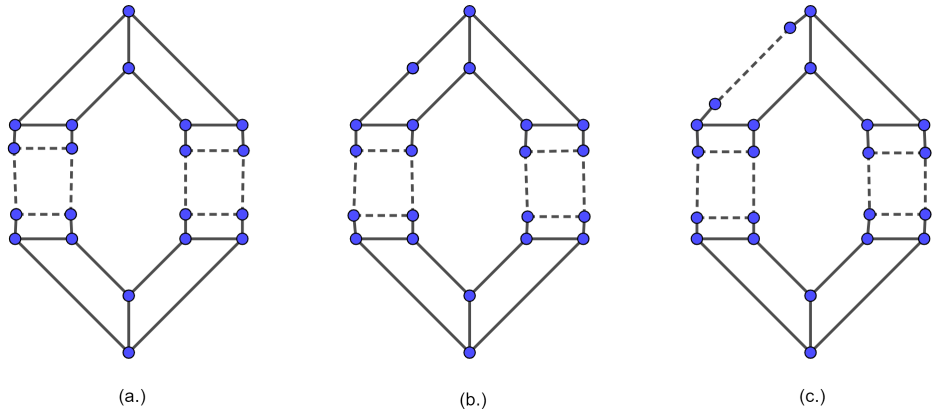

Figure 11. **Construction VIII for $c-$ cyclic graphs.**

regular $n+1$ cyclic graph denoted by G. Let G' be the graph obtained by subdividing any (3.3) edge of the graph. Continue this subdivision on any other edge of the graph (see Figures 10 and 11). In the case of tricyclic graph, start with a graph $G = K_4$ and continue the subdivision process as before.

Theorem 9. *For every even integer $n \geq 2(c-1)i, i = 1, 2, \ldots, c \geq 3$, there exists c-cyclic graph G with $irr_t(G) = n, \sigma_t(G) = n$.*

Proof. Let G be the $c-$ cyclic graph constructed as in Construction VIII. G has $2(c-1)$ vertices of degree 3 and therefore $irr_t(G) = 0, \sigma_t(G) = 0$. When we subdivide an edge in G, we get $2(c-1)$ pairs of $(3, 2)$ vertices in graph. Therefore, $irr_t(G) = 2(c-1), \sigma_t(G) = 2(c-1)$. If we continue this process i times there will be $2(c-1)i$ pairs of $(3, 2)$ vertices in the graph, therefore, $irr_t(G) = \sigma_t(G) = 2(c-1)i$. □

Conclusion. In this study, we have explored 'topological index inverse problems' for cyclic graphs. There are several variant of these problems which are yet to be solved. The 'inverse Mostar and edge Mostar index problems' for $c-$ cyclic graphs, $c \geq 4$ is an open problem for further studies.

Acknowledgements: We are grateful to the anonymous reviewers for their insightful comments that significantly strengthened this paper.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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