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Research Article

Graceful coloring of some corona graphs - An algorithmic approach

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Abstract: A graceful k-coloring of a non-empty graph G is a proper vertex coloring with k colors that induces a proper edge coloring, where the color for an edge uv is the absolute difference between the colors assigned to the vertices u and v. The minimum k for which G admits a graceful k-coloring is called the graceful chromatic number of G ($\chi_g(G)$). The problem of determining the graceful chromatic number for some corona graphs with the related coloring algorithms are studied in this paper.

Keywords: graceful coloring, graceful chromatic number, corona graphs.

AMS Subject classification: 05C15, 05C78

1. Introduction

All graphs $G(V, E)$ examined in this paper are simple, connected and finite. The study of graph labeling was introduced by Alexander Rosa in 1967 [10], which is an assignment of integers to the vertices, edges (or both) of a graph G under certain conditions. There are several types of graph labeling in which β-labeling is one of the eminent labeling and are widely studied in the survey by Gallian [5]. It was referred as graceful labeling by Golomb [6]. A function f is a graceful labeling of G with q number of edges, if f is an injection from $V(G)$ to the set $\{0, 1, 2, \ldots, q\}$ such that the induced function f^* is bijective from $E(G)$ to the set $\{1, 2, \ldots, q\}$ defined as $f^*(xy) = |f(x) - f(y)|$, for every edge xy in G. A graph G possessing a graceful labeling is a graceful graph.

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With the origin of four color problem, the field of graph colorings has been developed into one of the most popular areas of graph theory. Vertex coloring and edge coloring received the most attention in the area of graph coloring. In such colorings, any two adjacent vertices or any two adjacent edges of G are assigned with distinct colors. The minimum number of colors needed in proper vertex coloring of the graph G is called the chromatic number of G ($\chi(G)$), while the minimum number of colors needed in proper edge coloring of the graph G is called the chromatic index of G ($\chi^{'}(G)$) [11].

The concept of graceful chromatic number was introduced by Gary Chartrand [2, 3], as an extension from graceful labeling. A graceful k-coloring of a non-empty graph $G(V, E)$ is a proper vertex coloring $f : V(G) \to \{1, 2, \ldots, k\}$, where $k \geq 2$, which induces a proper edge coloring $f^* : E(G) \to \{1, 2, \ldots, k-1\}$ defined by $f^*(uv) = |f(u) - f(v)|$, where $u, v \in V(G)$ [2]. The graceful chromatic number $\chi_g(G)$ of the graph G is the minimum k for which G has a graceful k-coloring. We define $[e, f] = \{e, e+1, \ldots, f-1, f\}$, for $\{e, f\} \subset \mathbb{Z}^+$ with $e < f$.
In the fundamental paper of the graceful coloring [2], the graceful chromatic number for some well known graphs were computed.

Theorem 1. *For a subgraph $G^{'}$ of G, $\chi_g(G^{'}) \leq \chi_g(G)$.*

Theorem 2. *Let $f : V(G) \to \{1, 2, \ldots, k\}, k \geq 2$ be a coloring of a nontrivial connected graph G. Then f is a graceful coloring of G if and only if*

(i) for each vertex v of G, the vertices in the closed neighborhood $N[v]$ are assigned distinct colors by f and

(ii) for each path (x, y, z) in G, $f(y) \neq \frac{f(x)+f(z)}{f(y)}$.

Theorem 3. *For a nontrivial connected graph G, $\chi_g(G) \geq \Delta + 1$, where Δ represents the maximum degree of G.*

Theorem 4. *For a cycle C_n, $n \geq 4$, $\chi_g(C_n) = \begin{cases} 4, & \text{if } n \neq 5 \\ 5, & \text{if } n = 5. \end{cases}$*

Theorem 5. *For a path P_n, $n \geq 5$, $\chi_g(P_n) = 4$.*

Theorem 6. *For a wheel graph W_n, $n \geq 6$, $\chi_g(W_n) = n$.*

Theorem 7. *If T is a tree with maximum degree Δ, then $\chi_g(T) \leq \lceil \frac{5\Delta}{3} \rceil$.*

Theorem 8. *If G is an r-regular graph, then $\chi_g(G) \geq r + 2$, where $r \geq 2$.*

The graceful chromatic number of caterpillars were investigated along with a characterization in [2]. The graceful chromatic number for some subclasses of the following graphs have been established in the literature: unicyclic graphs [1]; graphs with diameter at least 2 [2]; rooted trees $T_{\Delta,h}$ [4]. Also, the graceful chromatic number for few variants of ladder graphs [9] and for a subclass of trees [8] are discussed in the literature. The following result in [9] is useful in proving our main results.

Observation 9. ([9]) If $[1, \Delta + i]$, $i \in \mathbb{Z}^+$ colors are used in the graceful coloring of a graph G, then the possible colors for any vertex of maximum degree (Δ) are the first and last i colors from $[1, \Delta + i]$.

Graceful coloring for many graph products are not yet explored. So we concentrate in evaluating the graceful chromatic number for some corona product of graphs which was introduced by Roberto Frucht together with Frank Harary and was defined as follows:
The corona product of two graphs G and H is the graph $G \odot H$ obtained by taking one copy of G which has p_i vertices and p_i copies of H, and then joining the i^{th} vertex of G by an edge to every vertex in the i^{th} copy of H [7], an illustration shown in Figure 1.

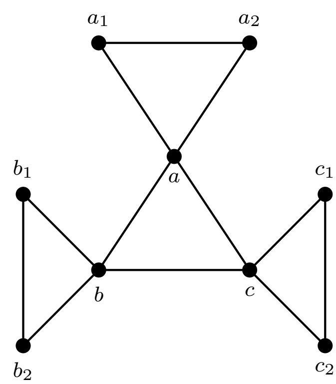


Figure 1. $K_3 \odot K_2$

The graceful chromatic number for corona product of graphs such as star ($K_{1,n}$), path (P_n), cycle (C_n), wheel (W_n) with path (P_m) are evaluated in this paper. Also, we present two main algorithms to assign the colors for the vertices of P_m which are used to run the graceful coloring algorithms for each case.

2. Main Results

Theorem 10. $\chi_g(K_{1,n} \odot P_m) = n + m + 1$, *for* $n \geq 2$, $m \geq 2$.

Proof. Let $u_i \in V(K_{1,n})$, for $0 \leq i \leq n$. By the definition of corona product of

$K_{1,n}$ and P_m, there are $(n+1)$ copies of P_m which are represented as P_m^i, $0 \le i \le n$ such that each u_i is adjacent to all the vertices of P_m^i. Let the vertices and edges of $K_{1,n} \odot P_m$ be
$V(K_{1,n} \odot P_m) = \{u_i : 0 \le i \le n\} \cup \{u_{i,j} : 0 \le i \le n, 1 \le j \le m\}$
$E(K_{1,n} \odot P_m) = \{(u_0 u_i) : 1 \le i \le n\} \cup \{(u_i u_{i,j}) : 0 \le i \le n, 1 \le j \le m\} \cup \{(u_{i,j} u_{i,j+1}) : 0 \le i \le n, 1 \le j \le m-1\}$.
The representation of $K_{1,n} \odot P_m$ is provided in Figure 2.

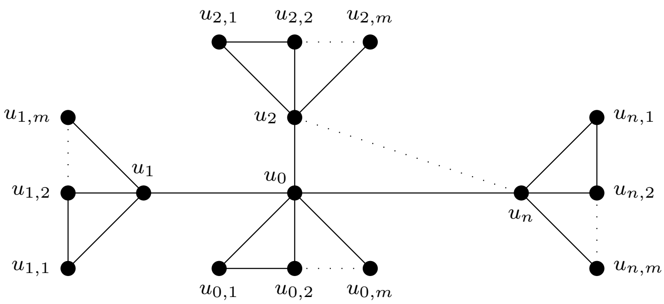


Figure 2. $K_{1,n} \odot P_m$

Note that, the maximum degree vertex in $K_{1,n} \odot P_m$ is u_0 with $d(u_0) = n+m$. Therefore, $\chi_g(K_{1,n} \odot P_m) \ge n+m+1$ (by Theorem 3). To prove $\chi_g(K_{1,n} \odot P_m) \le n+m+1$, we show a graceful coloring of $K_{1,n} \odot P_m$ with $n+m+1$ colors using Algorithm 1 which uses the two Algorithms 2 and 3. □

Algorithm 1 Graceful coloring of $G = K_{1,n} \odot P_m$

Input: $G = K_{1,n} \odot P_m$
Output: Graceful $(n+m+1)$-coloring of G

```
                    # colors for the vertices of K_{1,n}
f(u_0) ← n + m + 1
f(u_1) ← 1
f*(u_0, u_1) ← n + m
if  n == 2 and m == 3  then
    f(u_2) ← 2
    f*(u_0, u_2) ← 4
else
    for s_i ∈ N_i do
        f(u_i) ← m + i
        f*(u, u_i) ← n − i + 1
    end for
end if
                    # colors for u_{i,j}
if n == 2 and m == 3  then
    A = {4, 3, 5}
    for j = 1 to 3 do
```

$f(u_{0,j}) \leftarrow A[j-1]$
Update the corresponding edge colors and the edge colors between u_0 and $u_{o,j}(1 \leq j \leq m)$
end for
end if
if $n == 2$ and $m == 4$ **then**
$A = \{5, 4, 2, 3\}$
for $j = 1$ to 3 **do**
$f(u_{0,j}) \leftarrow A[j-1]$
Update the corresponding edge colors and the edge colors between u_0 and $u_{o,j}(1 \leq j \leq m)$
end for
else
for $i = 0$ to n **do**
AssignColors$\{u_{i,j}\}$
end for
end if

Algorithm 2 Colors$\{u_{i,j}\}$

Input: Integers from 1 to $\chi_g(G)$, i
Output: Colors for $u_{i,j} \in P_m^i$
$A \leftarrow \{1, 2, \ldots, \chi_g(G)\}$
$X \leftarrow \{$ set of vertices at distance 2 from $u_{i,j}\}$
$Y \leftarrow \{$ set of colors of the vertices at distance 2 from $u_{i,j}\}$
$B \leftarrow A \setminus (f(u_i) \cup Y)$
if $f(u_i) = 1$ or $f(u_i) = \chi_g(G)$ **then**
return B
else
Let $D \leftarrow \{$ set of edge colors between u_i and $X\}$
for every $a \in B$ **do**
if $|f(u_i) - a| \in D$ **then**
$C \leftarrow B \setminus \{a\}$
end if
end for
end if
if $\{f(u_i) + 1, f(u_i) - 1\} \subseteq C$ **then**
$E \leftarrow C \setminus (f(u_i) + 1)$
return E
else
return C
end if

Algorithm 3 AssignColors$\{u_{i,j}\}$

Input: i, $f(u_i)$, m, Colors$\{u_{i,j}\}$

Output: Graceful coloring for the respective $u_{i,j}$

```
Set r as zero and X ←Colors{u_{i,j}}
for j = 1 to m increased by 2 do
    f(u_{i,j}) ← X[r]
    f*(u_i, u_{i,j}) ← |(f(u_i) − f(u_{i,j})|
    X ← X \ X[r]
end for
if m is even then
    k = m
else
    k = m − 1
end if
for  j = k to 2 decremented by 2  do
    SC ← ∅
    if  thenX ≠ ∅
        f(u_{i,j}) ← X[r]
        Update the corresponding edge colors
        X ← X \ X[r]
        if j ≠ 2 and adjacent edges of u_{i,j} receives same color  then
            SC ← SC ∪ f(u_{i.j})
            f(u_{i,j}) ← X[r]
            Update the corresponding edge colors
        end if
    else
        for t = 1 to |SC| do
            f(u_{i,j}) ← SC[t]
            Update the corresponding edge colors
            SC ← SC \ SC[t]
        end for
    end if
end for
if f*(u_{i,2}, u_i) == f*(u_{i,2}, u_{i,3}) then
    f(u_{i,2}) ← interchange the colors of u_{i,1} and u_{i,2}
    Update the corresponding edge colors
end if
if f*(u_{i,2}, u_i) == f*(u_{i,2}, u_{i,1}) then
    f(u_{i,2}) ← interchange the colors of u_{i,2} and u_{i,3}
    Update the corresponding edge colors
end if
for j = 1 to m − 2 do
```

if $f^*(u_{i,j}, u_{i,j+1}) == f^*(u_{i,j+1}, u_{i,j+2})$ **then**
  Interchange $f(u_{i,j+1})$ and $f(u_{i,j+2})$
  Update the corresponding edge colors
end if
end for

An illustration for graceful 6-coloring of $K_{1,3} \odot P_2$ is shown in Figure 3.

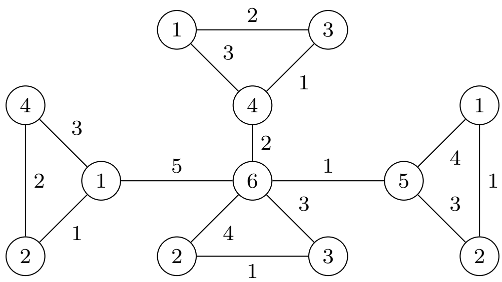


Figure 3. $\chi_g(K_{1,3} \odot P_2) = 6$

Corollary 1. For $n \geq 2$, $m \geq 2$, $\chi_g(K_{1,n} \odot mK_1) = n + m + 1$.

Next, we determine the graceful chromatic number of corona product of path P_n with another path P_m.

Theorem 11. *For $n \geq 2$, $m \geq 2$,* $\chi_g(P_n \odot P_m) = \begin{cases} \Delta + 1, & n = 2, 3, 4 \\ \Delta + 2, & n \geq 5. \end{cases}$

Proof. Let $u_i \in V(P_n)$, for $1 \leq i \leq n$ and $u_{i,j} \in V(P^i_m)$, for $1 \leq i \leq n$, $1 \leq j \leq m$. Thus
$V(P_n \odot P_m) = \{u_1, u_2, ..., u_n\} \cup \{u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$
$E(P_n \odot P_m) = \{(u_i u_{i+1}) : 1 \leq i \leq n-1\} \cup \{(u_i u_{i,j}) : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{(u_{i,j} u_{i,j+1}) : 1 \leq i \leq n, 1 \leq j \leq m-1\}$.
See Figure 4 for a representation of $P_n \odot P_m$.

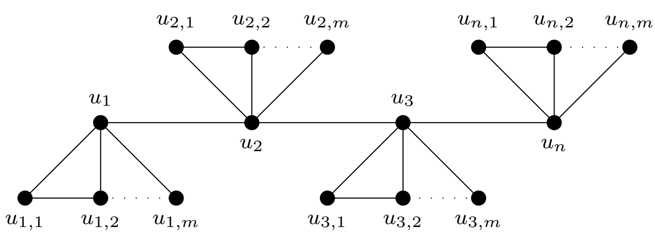


Figure 4. $P_n \odot P_m$

Case 1. $n = 2, 3, 4$.
Based on Theorem 3 , $\chi_g(P_n \odot P_m) \geq \Delta + 1$. To prove $\chi_g(P_n \odot P_m) \leq \Delta + 1$, we show a

graceful coloring of $P_n \odot P_m$ with $\Delta + 1$ colors using Algorithm 4.

Algorithm 4 Graceful $(\Delta + 1)$-coloring of $G = P_n \odot P_m$

Input: $G = P_n \odot P_m$, $n = 2, 3, 4$
Output: Graceful $(\Delta + 1)$-coloring of G

\# colors for the vertices in P_n
if $n == 2$ **then**
    $f(u_1) \leftarrow 1$
    $f(u_2) \leftarrow \Delta + 1$
    $f^*(u_1, u_2) \leftarrow \Delta$
end if
if $n == 3$ and $m == 3$ **then**
    $f(u_1) \leftarrow 5$
    $f(u_2) \leftarrow 1$
    $f(u_3) \leftarrow 6$
    Update the corresponding edge colors
end if
if $n == 4$ and $m == 3$ **then**
    $f(u_1) \leftarrow 5$
    $f(u_2) \leftarrow 1$
    $f(u_3) \leftarrow 6$
    $f(u_4) \leftarrow 2$
    Update the corresponding edge colors
else
    $fu_1) \leftarrow 2$
    $f(u_2) \leftarrow 1$

    $f(u_3) \leftarrow \Delta + 1$

    $f(u_4) \leftarrow \Delta$

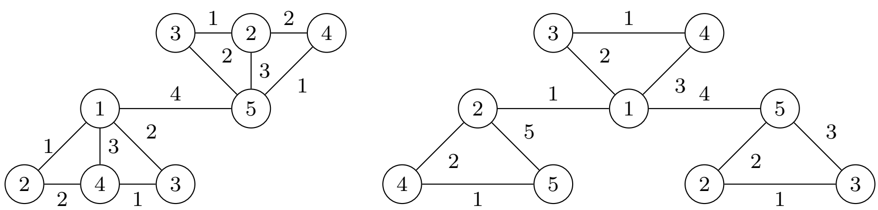
    Update the corresponding edge colors
end if

\# colors for $u_{i,j}$

if $n == 2$ and $m == 4$ **then**
    $A = \{5, 2, 4, 3\}$
    $B = \{4, 1, 3, 2\}$
    $C = \{3, 2, 1\}$
    for $j = 1$ to 4 **do**
        $f(u_{1,j}) \leftarrow A[j - 1]$
        $f^*(u_1, u_{1,j}) \leftarrow B[j - 1]$
    end for
    for $j = 1$ to 3 **do**
        $f^*(u_{1,j}, u_{1,j+1}) \leftarrow C[j - 1]$
    end for
end if
if $n == 2$ and $m == 5$ **then**
    $A = \{5, 2, 4, 3, 6\}$
    $B = \{4, 1, 3, 2, 5\}$

```
    C = {3, 2, 1, 3}
    for j = 1 to 5 do
        f(u_{1,j}) ← A[j − 1]
        f*(u_1, u_{1,j}) ← B[j − 1]
    end for
    for j = 1 to 4 do
        f*(u_{1,j}, u_{1,j+1}) ← C[j − 1]
    end for
end if
if n == 4 and m = 4 then
    A = {5, 2, 4, 3}
    B = {2, 5, 3, 4}
    C = {4, 1, 3, 2}
    D = {3, 2, 1}
    for j = 1 to 4 do
        f(u_{3,j}) ← A[j − 1]
        f*(u_3, u_{3,j}) ← B[j − 1]
        f(u_{4,j}) ← C[j − 1]
        f*(u_4, u_{4,j}) ← B[j − 1]
    end for
    for j = 1 to 3 do
        f*(u_{3,j}, u_{3,j+1}) ← D[j − 1]
        f*(u_{4,j}, u_{4,j+1}) ← D[j − 1]
    end for
else
    for i = 1 to n do
        AssignColors{u_{i,j}}
    end for
end if
```

A graceful 5-coloring of $P_2 \odot P_3$ and $P_3 \odot P_2$ are shown in Figure 5 (a) and (b) respectively.



(a) (b)

Figure 5. **Graceful 5-coloring of (a) $P_2 \odot P_3$ and (b) $P_3 \odot P_2$**

Case 2. $n \geq 5$.
Since $P_4 \odot P_m$ is a subgraph of $P_n \odot P_m$, then from Theorem 1, $\chi_g(P_n \odot P_m) \geq \chi_g(P_4 \odot P_m) = \Delta + 1$. We show that, it is not possible to use $\Delta + 1$ colors for the graceful coloring of $P_n \odot P_m$ for $n \geq 5$. Let S be the set of all maximum degree vertices in $P_n \odot P_m$. That is, $S = \{u_2, u_3, \ldots, u_{n-1}\}$. By Observation 9, all the vertices in S are to be colored only with $\{1, \Delta+1\}$. Since $n \geq 5$, we obtain a contradiction to proper edge coloring. Thus, at least one more color is needed for the graceful coloring of $P_n \odot P_m$. Hence, $\chi_g(P_n \odot P_m) \geq \Delta + 2$. To

prove $\chi_g(P_n \odot P_m) \leq \Delta+2$, we show a graceful $(\Delta+2)$-coloring of $P_n \odot P_m$ using Algorithm 5.

Algorithm 5 Graceful $(\Delta+2)$-coloring of $G = P_n \odot P_m$, $n \geq 5$

Input: $G = P_n \odot P_m$, $n \geq 5$
Output: Graceful $(\Delta+2)$-coloring of G

```
                                  # colors for u_i
for i = 1 to n incremented by 4 do
    f(u_i) ← 2
end for
for i = 2 to n incremented by 4 do
    f(u_i) ← Δ + 2
end for
for i = 3 to n incremented by 4 do
    f(u_i) ← 1
end for
for i = 4 to n incremented by 4 do
    f(u_i) ← Δ + 1
end for
for i = 1 to n − 1 do
    f*(u_i, u_{i+1}) ← |f(u_i) − f(u_{i+1})|
end for
                                  # colors for u_{i,j}
for i = 1 to n do
    AssignColors{u_{i,j}}
end for
```

□

A graceful 6-coloring of $P_5 \odot P_2$ is shown in Figure 6.

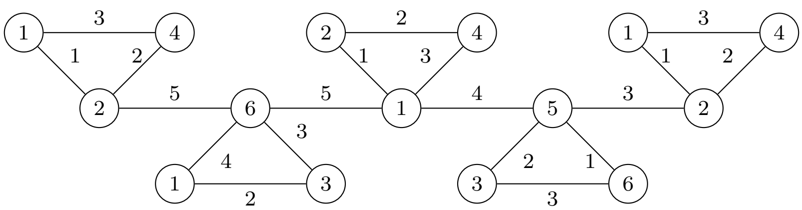


Figure 6. $\chi_g(P_5 \odot P_2) = 6$

Corollary 2. For $n \geq 3$, $m \geq 2$, $\chi_g(P_n \odot mK_1) = \begin{cases} \Delta+1, & n = 2,3,4 \\ \Delta+2, & n \geq 5. \end{cases}$

Consequently, we determine the graceful chromatic number of corona product of cycles C_n with path P_m.

Theorem 12. For $n \geq 3, m \geq 2$, $\chi_g(C_n \odot P_m) = \begin{cases} \Delta+2, & n \neq 5 \\ \Delta+3, & n = 5. \end{cases}$

Proof. Let the vertex set and the edge set of $C_n \odot P_m$ be
$V(C_n \odot P_m) = \{u_i : 1 \leq i \leq n\} \cup \{u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$
$E(C_n \odot P_m) = \{(u_1, u_n), (u_i, u_{i+1}) : 1 \leq i \leq n-1\} \cup \{(u_i, u_{i,j}) : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{(u_{i,j}, u_{i,j+1}) : 1 \leq i \leq n, 1 \leq j \leq m-1\}$.
Refer Figure 7 for the representation of $C_n \odot P_m$.

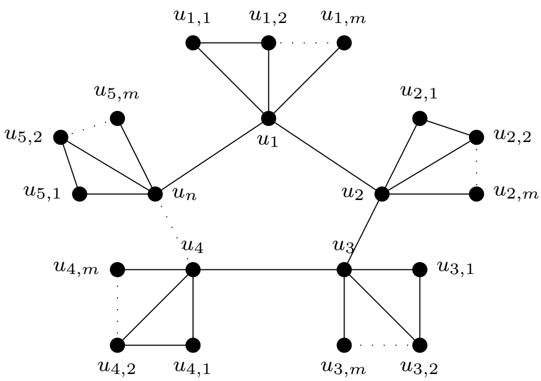


Figure 7. $C_n \odot P_m$

Case 1. $n \neq 5$.
By Theorem 3, $\chi_g(C_n \odot P_m) \geq \Delta + 1$. Assume that, there is a graceful $(\Delta+1)$-coloring f of $C_n \odot P_m$. The vertices $\{u_i\}$ which lies on C_n can be colored using only $\{1, \Delta+1\}$, by Observation 9. Without loss of generality, let $f(u_1) = 1$, so $f(u_2) = \Delta + 1$. Now, the vertex u_3 or u_n cannot be colored gracefully using the colors $\{1, \Delta+1\}$, which leads to a contradiction to our assumption. Hence, $\chi_g(C_n \odot P_m) \geq \Delta + 2$. It remains to define a graceful $(\Delta+2)$-coloring of $C_n \odot P_m$, which is attained in the following Algorithm 6.

Algorithm 6 Graceful $(\Delta+2)$-coloring of $G = C_n \odot P_m$, $n \neq 5$

Input: $G = C_n \odot P_m$, $n \neq 5$
Output: Graceful $(\Delta+2)$-coloring of G
colors for u_i

```
if m == 3 then
    if n is a multiple of 3 then
        for i = 1 to n incremented by 3 do
            f(u_i) ← 1
        end for
        for i = 2 to n incremented by 3 do
            f(u_i) ← 6
        end for
        for i = 3 to n incremented by 3 do
            f(u_i) ← 2
        end for
        Update the corresponding edge colors and the edge colors between u_i and u_{i+1}(1 ≤ i ≤ n − 1)
    else if
```

for **dothen**$t = 7$ to n incremented by 3
if $n == t$ **then**
$f(u_n) \leftarrow 7$
for $i = 1$ to $n-1$ incremented by 3 **do**
$f(u_i) \leftarrow 1$
end for
for $i = 2$ to n incremented by 3 **do**
$f(u_i) \leftarrow 6$
end for
for $i = 3$ to n incremented by 3 **do**
$f(u_i) \leftarrow 2$
end for
Update the corresponding edge colors and the edge colors between u_i and $u_{i+1}(1 \leq i \leq n-1)$
end if
end for
else if

for $t = 8$ to n incremented by 3 **do**
$A = \{1, 6, 2, 7\}$
if $n == t$ **then**
for $i = 1, 2, 3, 4$ **do**
$f(u_i) \leftarrow A[i-1]$
end for
$f(u_n) \leftarrow 7$
for $i = 5$ to $n-3$ incremented by 3 **do**
$f(u_i) \leftarrow 1$
end for
for $i = 7$ to $n-1$ incremented by 3 **do**
$f(u_i) \leftarrow 2$
end for
for $i = 6$ to $n-2$ incremented by 3 **do**
$f(u_i) \leftarrow 6$
end for
Update the corresponding edge colors and the edge colors between u_i and $u_{i+1}(1 \leq i \leq n-1)$
end if
end for
end if then
else if

if n is a multiple of 4 **then**
for $i = 1$ to $n-3$ incremented by 4 **do**
$f(u_i) \leftarrow 1$
end for
for $i = 2$ to $n-2$ incremented by 4 **do**
$f(u_i) \leftarrow \Delta + 2$
end for
for $i = 3$ to $n-1$ incremented by 4 **do**
$f(u_i) \leftarrow 2$
end for

for $i = 4$ to n incremented by 4 **do**
$f(u_i) \leftarrow \Delta + 1$
end for
Update the corresponding edge colors and the edge colors between u_i and $u_{i+1}(1 \leq i \leq n-1)$ **then**
else if
if $n == 9$ **then**
for $i = 1$ to $n - 2$ incremented by 3 **do**
$f(u_i) \leftarrow 1$
end for
for $i = 2$ to $n - 1$ incremented by 3 **do**
$f(u_i) \leftarrow \Delta + 2$
end for
for $i = 3$ to n incremented by 3 **do**
$f(u_i) \leftarrow 2$
end for
Update the corresponding edge colors and the edge colors between u_i and $u_{i+1}(1 \leq i \leq n-1)$
end if then
else if
for $t = 13$ to n incremented by 4 **do**
if $n == t$ **then**
for $i = n-2, n-5, n-8$ **do**
$f(u_i) \leftarrow 1$
end for
for $i = n-1, n-4, n-7$ **do**
$f(u_i) \leftarrow 2$
end for
for $i = n, n-3, n-6$ **do**
$f(u_i) \leftarrow \Delta + 2$
end for
for $i = 1$ to $n - 12$ incremented by 4 **do**
$f(u_i) \leftarrow 1$
end for
for $i = 2$ to $n - 11$ incremented by 4 **do**
$f(u_i) \leftarrow 2$
end for
for $i = 3$ to $n - 10$ incremented by 4 **do**
$f(u_i) \leftarrow \Delta + 2$
end for
for $i = 4$ to $n - 9$ incremented by 4 **do**
$f(u_i) \leftarrow \Delta + 1$
end for
end if
end for
Update the corresponding edge colors and the edge colors between u_i and $u_{i+1}(1 \leq i \leq n-1)$ **then**
else if

```
        if n == 6 then
            for i = 1 to n − 2 incremented by 3 do
                f(u_i) ← 1
            end for
            for i = 2 to n − 1 incremented by 3 do
                f(u_i) ← Δ + 2
            end for
            for i = 3 to n incremented by 3 do
                f(u_i) ← Δ + 1
            end for
            Update the corresponding edge colors and the edge colors between u_i and u_{i+1}(1 ≤ i ≤ n − 1)
        end if then
    else if

        for t = 10 to n incremented by 4 do
            if n == t then
                for i = n − 2, n − 5 do
                    f(u_i) ← 1
                end for
                for i = n − 1, n − 4 do
                    f(u_i) ← 2
                end for
                for i = n, n − 3 do
                    f(u_i) ← Δ + 2
                end for
                for i = 1 to n − 9 incremented by 4 do
                    f(u_i) ← 1
                end for
                for i = 2 to n − 8 incremented by 4 do
                    f(u_i) ← 2
                end for
                for i = 3 to n − 7 incremented by 4 do
                    f(u_i) ← Δ + 2
                end for
                for i = 4 to n − 6 incremented by 4 do
                    f(u_i) ← Δ + 1
                end for
                Update the corresponding edge colors and the edge colors between u_i and u_{i+1}(1 ≤ i ≤ n − 1)
            end if
        end for then
    else
        for t = 3 to n incremented by 4 do
            if n == t then
                for i = 1 to n − 2 incremented by 4 do
                    f(u_i) ← 1
                end for
                for i = 2 to n − 1 incremented by 4 do
                    f(u_i) ← 2
                end for
```

for $i = 3$ to n incremented by 4 **do**
$f(u_i) \leftarrow \Delta + 2$
end for
for $i = 4$ to $n - 3$ incremented by 4 **do**
$f(u_i) \leftarrow \Delta + 1$
end for
Update the corresponding edge colors and the edge colors between u_i and $u_{i+1} (1 \leq i \leq n - 1)$
end if
end for
end if
end if

\# colors for $u_{i,j}$

for $i = 1$ to n **do**
AssignColors $\{u_{i,j}\}$
end for

Case 2. $n = 5$.
Since $P_n \odot P_m$ is a subgraph of $C_n \odot P_m$, $\chi_g(C_5 \odot P_m) \geq \chi_g(P_5 \odot P_m) = \Delta + 2$ (from Theorem 1 and Theorem 11). We claim that $\chi_g(C_5 \odot P_m) \neq \Delta + 2$. Assume $(C_5 \odot P_m)$ is graceful $(\Delta + 2)$-colorable. Then the possible colors for the vertices of C_5 are $\{1, 2, \Delta + 1, \Delta + 2\}$ (by Observation 9) which is not possible for the graceful coloring of C_5. Hence, $\chi_g(C_5 \odot P_m) \geq \Delta + 3$. To show, $\chi_g(C_5 \odot P_m) = \Delta + 3$, we define a graceful $(\Delta + 3)$-coloring of $C_5 \odot P_m$, which is attained using Algorithm 7.

Algorithm 7 Graceful $(\Delta + 3)$-coloring of $G = C_n \odot P_m$, $n = 5$

Input: $G = C_n \odot P_m$, $n = 5$
Output: Graceful $(\Delta + 3)$-coloring of G

\# colors for u_i

$A = \{1, \Delta + 2, \Delta + 3, 2, 3\}$
$B = \{\Delta + 1, 1, \Delta + 1, 1\}$
for $i = 1$ to 5 **do**
$f(u_i) \leftarrow A[i - 1]$
end for
for $i = 1$ to 4 **do**
$f^*(u_i, u_{i+1}) \leftarrow B[i - 1]$
$f^*(u_5, u_1) \leftarrow 2$
end for

\# colors for $u_{i,j}$

for $i = 1$ to n **do**
AssignColors$\{u_{i,j}\}$
end for

□

A graceful $(\Delta + 2)$-coloring of $C_4 \odot P_3$ is shown in Figure 8.

Corollary 3. For $n \geq 3, m \geq 2$, $\chi_g(C_n \odot mK_1) = \begin{cases} \Delta + 2, & n \neq 5 \\ \Delta + 3, & n = 5. \end{cases}$

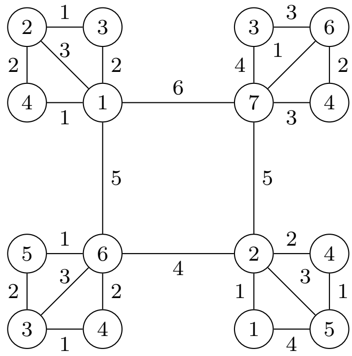


Figure 8. $\chi_g(C_4 \odot P_3) = 7$

The graceful chromatic number of corona product of wheel graph W_n having n vertices with path P_m is evaluated as follows:

Theorem 13. For $n \geq 4, m \geq 4$, $\chi_g(W_n \odot P_m) = \begin{cases} \Delta + 2, & n = 4, 5 \\ \Delta + 1, & n \geq 6. \end{cases}$

Proof. A wheel graph W_n on n vertices is obtained by joining a central vertex to all the vertices of the cycle C_{n-1}. Let u_1 be the central vertex of W_n such that the vertices $\{u_i : 2 \leq i \leq n\}$ are adjacent to u_1. For $1 \leq i \leq n$ and $1 \leq j \leq m$, $\{u_{i,j}\} \in V(P_m)$ such that each $\{u_{i,j}\}$ is adjacent to u_i. Thus,
$V(W_n \odot P_m) = \{u_i \cup u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$
$E(W_n \odot P_m) = \{(u_1, u_i) : 2 \leq i \leq n\} \cup \{(u_2, u_n), (u_i, u_{i+1}) : 2 \leq i \leq n-1\} \cup \{(u_i, u_{i,j}) : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{(u_{i,j}, u_{i,j+1}) : 1 \leq i \leq n, 1 \leq j \leq m-1\}$.
Refer Figure 9 for the representation of $W_n \odot P_m$.

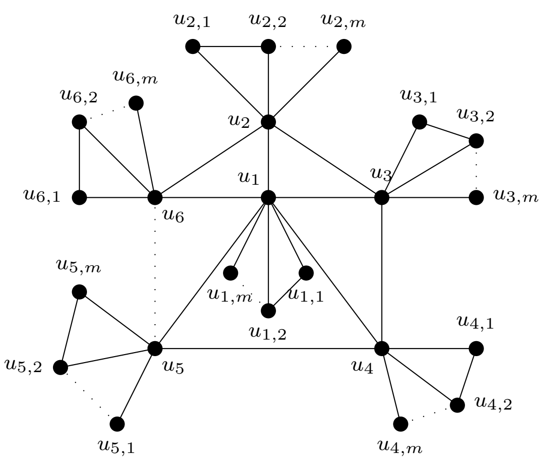


Figure 9. $W_n \odot P_m$

Case 1. $n = 4, 5$.
When $n = 4$, the maximum degree of $W_4 \odot P_m$ is $3 + m$ which is attained at the vertices $\{u_i : 1 \leq 1 \leq 4\}$. By Theorem 3 $\chi_g(W_4 \odot P_m) \geq \Delta + 1$. We now show that, $\chi_g(W_4 \odot P_m) \neq \Delta + 1$. Suppose on a contrary, if $\Delta + 1$ colors are used in the graceful coloring of $W_4 \odot P_m$, then by Observation 9, $f(u_i) \in \{1, \Delta + 1\}, 1 \leq i \leq 4$. This is not possible, since the vertices u_i are mutually adjacent. Hence, $\chi_g(W_4 \odot P_m) \geq \Delta + 1$.
When $n = 5$, the central vertex u_1 is of maximum degree and the vertices u_i, $2 \leq i \leq 5$ are of degree $\Delta - 1$. From Theorem 3, $\chi_g(W_5 \odot P_m) \geq \Delta + 1$. Suppose if $\Delta + 1$ colors are used in the graceful coloring of $W_5 \odot P_m$, then for $2 \leq i \leq 5$, $f(u_i) \in \{1, 2, \Delta, \Delta + 1\}$ (preserve edge coloring). Without loss of generality, let $f(u_1) = 1$ (by Observation 9). Then the available colors for u_2, u_3, u_4, u_5 are $\{2, \Delta, \Delta + 1\}$ which are not sufficient for the graceful coloring, since the vertices u_i, $1 \leq i \leq n$ are at distance 2 from each other. Hence, at least $\Delta + 2$ colors are required for the graceful coloring of $W_5 \odot P_m$. To claim, $\chi_g(W_n \odot P_m) \leq \Delta + 2$, $n = 4, 5$, we define a graceful $(\Delta + 2)$-coloring of $W_n \odot P_m$ in Algorithm 8. Hence, $\chi_g(W_n \odot P_m) = \Delta + 2$, $n = 4, 5$.

Algorithm 8 Graceful $(\Delta + 3)$-coloring of $G = C_n \odot P_m$, $n = 5$

Input: $G = W_n \odot P_m$, $n = 4, 5$
Output: Graceful $(\Delta + 2)$- coloring of G

\# colors for u_i

$f(u_1) \leftarrow 1$
if $n == 4$ **then**
  $A = \{2, \Delta + 1, \Delta + 2\}$
  $B = \{1, \Delta, \Delta + 1\}$
  $C = \{\Delta, 1\}$
  for $i = 2, 3, 4$ **do**
    $f(u_i) \leftarrow A[i - 2]$
    $f^*(u_1, u_i) \leftarrow B[i - 2]$
  end for
  for $i = 2, 3$ **do**
    $f^*(u_i, u_{i+1}) \leftarrow C[i - 2]$
  end for
  $f^*(u_2, u_4) \leftarrow \Delta - 1$
end if
if $n == 5$ **then**
  $A = \{2, \Delta + 1, 3, \Delta + 2\}$
  $B = \{1, \Delta, 2, \Delta + 1\}$
  $C = \{\Delta - 1, \Delta - 2, \Delta - 1\}$
  for $i = 2, 3, 4, 5$ **do**
    $f(u_i) \leftarrow A[i - 2]$
  end for
  $f^*(u_1, u_i) \leftarrow B[i - 2]$
end if
for $i = 2, 3, 4$ **do**
  $f^*(u_i, u_{i+1}) \leftarrow C[i - 2]$
end for
$f^*(u_2, u_5) \leftarrow \Delta$

\# colors for $u_{i,j}$

for $i = 1$ to n **do**
  AssignColors$\{u_{i,j}\}$

end for

Case 2. $n \geq 6$.
$\chi_g(W_n \odot P_m) \geq \Delta+1$, for $n \geq 6$ (from Theorem 3). To show $\chi_g(W_n \odot P_m) \leq \Delta+1$, we define a graceful $(\Delta+1)$-coloring of $W_n \odot P_m$ using Algorithm 9. Hence, $\chi_g(W_n \odot P_m) = \Delta+1$, for $n \geq 6$. □

A depiction of a graceful $(\Delta+2)$-coloring for $W_4 \odot P_4$ can be observed in Figure 10.

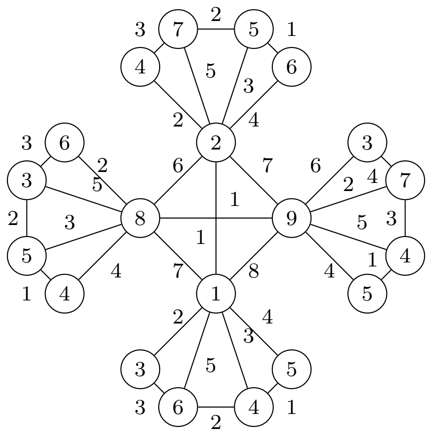


Figure 10. $\chi_g(W_4 \odot P_4) = 9$

Corollary 4. For $n \geq 4, m \geq 4$, $\chi_g(W_n \odot mK_1) = \begin{cases} \Delta+2, & n = 4,5 \\ \Delta+1, & n \geq 6. \end{cases}$

Algorithm 9 Graceful $(\Delta+1)$-coloring of $G = W_n \odot P_m$, $n \geq 6$

Input: $G = W_n \odot P_m$, $n \geq 6$
Output: Graceful $(\Delta+1)$-coloring of G
\# colors for the vertices of u_i

$f(u_1) \leftarrow 1$
Set t as zero
$X = \{2, 3, 4, ..., \Delta+1\}$
for $i = 2$ to n incremented by 3 **do**
    $f(u_i) \leftarrow X[t]$
    $X \leftarrow X \setminus X[t]$
end for
Set s as zero
for $i = 3$ to n incremented by 3 **do**
    $f(u_i) \leftarrow \Delta + 1 - s$
    Increment s by 2

end for
Set r as zero
for $i = 4$ to n incremented by 3 **do**
    $f(u_i) \leftarrow \Delta - r$
    Increment r by 2
    Update the corresponding edge colors
end for
#colors for $u_{i,j}$
for $i = 1$ to n **do**
    AssignColors$\{u_{i,j}\}$
end for

3. Conclusion

There has not been much research done on the concept of the graceful coloring of graph products. In this paper, we analyze the graceful chromatic number of corona product of graphs. Also we provide suitable algorithms for the graceful coloring process. Working on the following open problems stated below are interesting and a challenging one.

Problem 1. Characterize the graphs for which (i) $\chi_g(G) = \chi(G) + 1$ (ii) $\chi_g(G) = \chi^{'}(G)$ (iii) $\chi_g(G) = \chi^{'}(G) + 1$.

Problem 2. Evaluate the graceful chromatic number of some general graphs like bipartite graphs, co-bipartite graphs, regular graphs, and split graphs.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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