Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-19
https://doi.org/10.22049/cco.2024.28772.1713

CCO
Commun. Comb. Optim.

Research Article

Independent transversal domination subdivision number of trees

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Received: 20 June 2023; Accepted: 31 July 2024
Published Online: 10 August 2024

Abstract: A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a *dominating* set if every vertex in $V \setminus S$ is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. The *domination subdivision number* $sd_\gamma(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the domination number. Sahul Hamid defined a dominating set which intersects every maximum independent set in G to be an *independent transversal dominating* set. The minimum cardinality of an independent transversal dominating set is called the *independent transversal domination number* of G and is denoted by $\gamma_{it}(G)$. We extend the idea of domination subdivision number to independent transversal domination. The independent transversal domination subdivision number of a graph G denoted by $sd_{\gamma_{it}}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the independent transversal domination number. In this paper we initiate a study of this parameter with respect to trees.

Keywords: dominating set, independent set, independent transversal dominating set, subdivision number.

AMS Subject classification: 05C69

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to the book by Chatrand and Lesniak [3]. All graphs in this paper are assumed to be connected. A set $S \subseteq V$

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of vertices in a graph $G = (V, E)$ is called a *dominating* set if every vertex in $V \setminus S$ is adjacent to a vertex in S and the *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set S of G with $|S| = \gamma(G)$ is called a γ-set of G. A comprehensive introduction to domination in graphs, has been given in the book by Haynes *et al.* [5]. A subset S of V is called an *independent set* of G if no two vertices of S are adjacent in G. The maximum cardinality of an independent set is called the *independence number* and is denoted by $\beta(G)$. A maximum independent set is called a β-*set* of G.

Sahul Hamid [4] introduced another basic domination parameter namely independent transversal dominating set as follows. A dominating set $S \subseteq V$ of a graph G is said to be an *independent transversal dominating set*$(ITDS)$ if S intersects every maximum independent set of G. The minimum cardinality of an independent transversal dominating set of G is called the *independent transversal domination number* of G and is denoted by $\gamma_{it}(G)$. An independent transversal dominating set S of G with $|S| = \gamma_{it}(G)$ is called a γ_{it}-set of G. One can observe that for any graph G, $\gamma(G) \leq \gamma_{it}(G)$. More work in independent transversal domination has been done in [1, 6–8, 10]. In real life scenarios, *independent transversal dominating sets* can give a solution to the facility location problem by identifying the minimum number of locations where facilities or critical services can be placed to service a group of vertices. By utilizing independent transversal dominating sets, one can strategically place monitoring devices to ensure network security. In social networks, the independent transversal dominating sets can represent influential individuals whose actions can impact a larger group, aiding in targeted marketing or information dissemination strategies.

An edge $uv \in E(G)$ is said to be *subdivided* if the edge uv is deleted, and a new vertex x is added, along with two new edges ux and xv. The vertex x is called the *subdivision vertex*. The *domination subdivision number* $sd_\gamma(G)$ is defined in [9], as the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the domination number. We extend this idea of domination subdivision number to independent transversal domination. We define the i*ndependent transversal domination subdivision number* $sd_{\gamma_{it}}(G)$ as the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the independent transversal domination number. In this paper we initiate a study of this parameter with respect to trees.

2. Notation

The *degree* of a vertex v in a graph G is the number of edges of G incident with v and it is denoted by $deg(v)$. A *leaf* is a vertex of degree one. An edge incident with a leaf vertex is called a *pendant edge*. A *support vertex* is a vertex adjacent to a leaf vertex. A support vertex is called a *strong support* if it is adjacent to at least two leaf vertices and a support vertex is called a *weak support* if it is adjacent to exactly one leaf. A *path* in a graph G, is an alternating sequence of vertices and edges beginning and ending with vertices, such that all the vertices are distinct. A path on n vertices is denoted by P_n. A graph G is *connected* if every pair of vertices are joined by a path. A connected acyclic graph is called a *tree*. The greatest distance between any two vertices in a graph G is called the *diameter* of G and it is denoted by *diam(G)*. A *diametral path* of a graph is a shortest path between a pair of vertices whose length is equal to the diameter of the graph.

In a graph $G = (V, E)$, the *open neighbourhood* of a vertex $v \in V$ is $N(v) = \{x \in V : vx \in E\}$, set of vertices adjacent to v. The *closed neighbourhood* is $N[v] = N(v) \cup \{v\}$. The *private neighbourhood* $pn(v, S)$ of $v \in S$ is defined by $pn(v, S) = N(v) - N(S - \{v\})$. Equivalently, $pn(v, S) = \{u \in V : N(u) \cap S = \{v\}\}$. Each vertex in $pn(v, S)$ is called a private neighbour of v. The *external private neighbourhood* $epn(v, S)$ of v with respect to S consist of those private neighbours of v in $V - S$. Thus, $epn(v, S) = pn(v, S) \cap (V - S)$.

3. Bounds

We state below a theorem which has been proved in [4].

Theorem 1. *[4] For any path P_n of order n, we have*

$$\gamma_{it}(P_n) = \begin{cases} 2, & if\ n = 2, 3 \\ 3, & if\ n = 6 \\ \lceil \frac{n}{3} \rceil, & otherwise \end{cases}$$

Subdividing one or more edges of a path P_n will result in a graph which is again a path. Further, the independent transversal domination number of P_2 and P_3 does not change when its edges are subdivided. Hence, by Theorem 1, the following theorem is immediate.

Theorem 2. *For paths P_n, $n \geq 4$,*

$$sd_{\gamma_{it}}(P_n) = \begin{cases} 1 & if\ n \equiv 0 (mod\ 3), n \neq 6 \\ 2 & if\ n \equiv 2 (mod\ 3)\ , n = 4 \\ 3 & if\ n \equiv 1 (mod\ 3)\ , n \neq 4 \\ 4 & if\ n = 6 \end{cases}$$

We state below two theorems proved in [4, 9].

Theorem 3. *[9] For any tree T of order $n \geq 3$, $sd_{\gamma}(T) \leq 3$*

Theorem 4. *[4] For any tree T, $\gamma_{it}(T)$ is either $\gamma(T)$ or $\gamma(T) + 1$.*

All trees T considered in the rest of the paper are of order $n \geq 4$. For convenience, throughout the proofs of the following theorems we denote by T', the resulting graph obtained by subdividing the edges in T in different contexts. First we prove that for any tree T with $\gamma(T) = \gamma_{it}(T)$, $sd_{\gamma_{it}}(T)$ is bounded above by 3.

Theorem 5. *For any tree T with $\gamma(T) = \gamma_{it}(T)$, $sd_{\gamma_{it}}(T) \leq 3$.*

Proof. Let T' be a tree obtained from T by subdividing $sd_{\gamma}(T)$ edges of T such that $\gamma(T') > \gamma(T)$. Then $\gamma_{it}(T') \geq \gamma(T') > \gamma(T) = \gamma_{it}(T)$. Thus, $\gamma_{it}(T') > \gamma_{it}(T)$ and hence, $sd_{\gamma_{it}}(T) \leq sd_{\gamma}(T)$. In view of Theorem 3, $sd_{\gamma_{it}}(T) \leq 3$. □

Next we prove that for any tree T with $\gamma_{it}(T) = \gamma(T) + 1$, $sd_{\gamma_{it}}(T)$ is bounded above by 4. We start with a simple lemma.

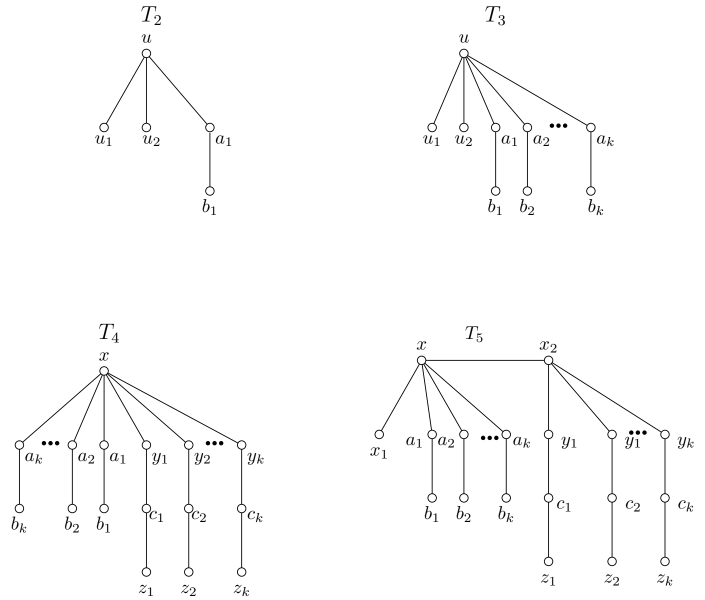

Figure 1. **Examples of trees with $sd_{\gamma_{it}}(T) = 3$**

Lemma 1. *If $T = T_4$, then $sd_{\gamma_{it}}(T) = 3$.*

Proof. Consider the graph T_4 in Figure 1. Let $x, a_1, a_2, \ldots, a_k$, $y_1, y_2, \ldots, y_k$, $z_1, z_2, \ldots, z_k$, $c_1, c_1, \ldots, c_k$, be as labelled in Figure 1 and $S = \{x, a_1, a_2, \ldots, a_k, c_1, c_1, \ldots, c_k\}$ is a γ-set of T. Clearly $\gamma_{it}(T) = \gamma(T)+1$. Now subdividing the edges xa_1 and xa_2 will not increase the value of $\gamma(T)$. Without loss of generality let the edge c_1y_1 be subdivided. Then, $(S \setminus \{c_1\}) \cup \{y_1, z_2\}$ is an $ITDS$ of T' as every maximum independent set of T' contains y_1 or z_2. Therefore, $\gamma_{it}(T') = \gamma(T)+1 = \gamma_{it}(T)$, which implies that $\gamma_{it}(T') = \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \geq 2$.

The following table (Table 1) gives the respective $ITDS$ of the resulting graph and the justification. We define w_i, $i = 1, 2$ as earlier.

Table 1. **Subdividing two edges**

Edges subdivided	$ITDS$	Every maximum independent set of T' contains
xa_1 and a_2b_2	$S \cup \{b_2\}$	b_2
xa_1 and a_1b_1 or a_1b_1 and a_2b_2	$S \cup \{b_1\}$	b_1
c_1y_1 and c_1z_2	$(S \setminus \{c_1\}) \cup \{w_1, z_2\}$	z_2
c_1y_1 and c_2y_2 or xy_1 and xy_2	$S \cup \{x\}$	x
a_1b_1 and xz_1	$(S \setminus \{a_1\}) \cup \{b_1, x\}$	x or b_1

Thus in all the cases, $\gamma_{it}(T') \leq \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \geq 3$.

Finally, if we subdivide two edges incident with c_1 and an edge incident with c_2, then we see that $\gamma_{it}(T') = \gamma(T) + 2 > \gamma(T) + 1 = \gamma_{it}(T)$. Therefore, $\gamma_{it}(T') > \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \leq 3$ which implies that $sd_{\gamma_{it}}(T) = 3$. □

Theorem 6. *For any tree T with $\gamma_{it}(T) = \gamma(T) + 1$ of order $n \geq 4$, $sd_{\gamma_{it}}(T) \leq 4$.*

Proof. Let S be a γ-set of T such that S does not contain leaf vertices. We deal with two cases.

Case 1. T has a strong support.

If there is a support say v adjacent to at least three leaf vertices, then subdividing two pendant edges incident at v, we see that $\gamma(T') = \gamma(T) + 2$. Then $\gamma_{it}(T') \geq \gamma(T')$ $> \gamma(T) + 1 = \gamma_{it}(T)$. Thus, $sd_{\gamma_{it}}(T) \leq 2$. If there are two strong supports say u,v adjacent to exactly two leaf vertices respectively, then subdividing a pendent edge incident at u and v respectively we see that $\gamma(T') = \gamma(T)+2$. Thus, $\gamma_{it}(T') > \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \leq 2$.

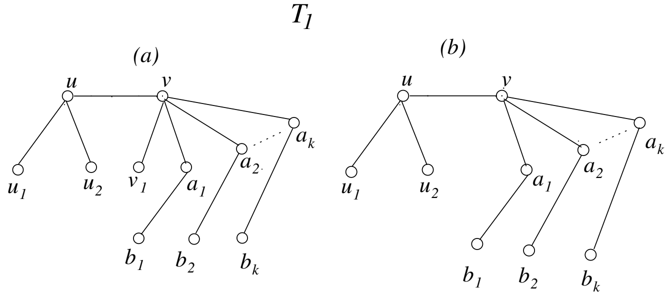

Figure 2. **Examples of trees with** $sd_{\gamma_{it}}(T) = 1$

Suppose that there is exactly one strong support say u adjacent to exactly two leaf vertices. Let $\text{diam}(T) = 3$ or 4. If $\text{diam}(T) = 3$, then $T = T_2$. If $\text{diam}(T) = 4$, then $T = T_1(a)$ or $T_1(b)$ or T_3. If $T = T_1(a)$, then $(S \setminus \{a_1, v\}) \cup \{b_1, v_1\}$ is an $ITDS$ of T, where $v, v_{1,}, u, u_1, u_2, a_1, a_2, \ldots, a_k, b_1, \ldots, b_k$ are as labelled in Figure 2. Hence, $\gamma_{it}(T) = \gamma(T)$. If $T = T_1(b)$, then $(S \setminus \{a_1\}) \cup \{b_1\}$ is an $ITDS$ of T, where $u, u_1, u_2, v, a_1, \ldots, a_k, b_1, \ldots, b_k$ are as labelled in Figure 2. Hence, $\gamma_{it}(T) = \gamma(T)$. Thus, $sd_{\gamma_{it}}(T) \leq 3$. If $T = T_2$ or T_3, then subdividing a pendant edge incident at u and the two edges incident at a weak support, we see that $\gamma(T') = \gamma(T) + 2$. Hence, $\gamma_{it}(T') > \gamma(T)$. Thus, $sd_{\gamma_{it}}(T) \leq 3$.

Let $\text{diam}(T) \geq 5$. Let $P = (v_1, v_2, \ldots, v_{k-1}, v_k)$ be a diametral path in T where $k = \text{diam}(T) + 1$. Without loss of generality let $v_{k-1} \neq u$. Suppose that v_{k-2} is a weak support. If $|epn(v_{k-2}, S)| = 1$, then $(S \setminus \{v_{k-2}, v_{k-1}\}) \cup \{v_k, z\}$, where z is the leaf vertex adjacent to v_{k-2} is an $ITDS$ of T. Hence, $\gamma_{it}(T) = \gamma(T)$, which is not the case. If $|epn(v_{k-2}, S)| = 2$, then subdividing two pendant edges one incident at u and the other incident at v_{k-2}, we see that $\gamma(T') = \gamma(T) + 2$. Thus, $\gamma_{it}(T') \geq \gamma(T') = \gamma(T) + 2 > \gamma(T) + 1 = \gamma_{it}(T)$. Therefore, $sd_{\gamma_{it}}(T) \leq 2$. Suppose that v_{k-2} is not a weak support and $\deg(v_{k-2}) \geq 3$. If $v_{k-2} \notin S$, then $(S \setminus \{v_{k-1}\}) \cup \{v_k\}$ is an $ITDS$ of T. Thus, $\gamma_{it}(T) = \gamma(T)$, which is not the case. If $v_{k-2} \in S$, then clearly $|epn(v_{k-2}, S)| = 1$ and $v_{k-3} \in epn(v_{k-2}, S)$. Now $(S \setminus \{v_{k-2}, v_{k-1}\}) \cup \{v_{k-3}, v_k\}$ is an $ITDS$ of T. Hence, $\gamma_{it}(T) = \gamma(T)$, which is not the case. If $\deg(v_{k-2}) = 2$ and $v_{k-3} \in S$, then $(S \setminus \{v_{k-1}\}) \cup \{v_k\}$ is an $ITDS$ of T. Hence, $\gamma_{it}(T) = \gamma(T)$, which is not the case. If $\deg(v_{k-2}) = 2$ and $v_{k-3} \notin S$, then $|epn(v_{k-1}, S)| = 2$ and $v_{k-2} \in epn(v_{k-1}, S)$. Now subdividing a pendant edge incident at u and the edge $v_{k-1}v_k$, we see that $\gamma(T') = \gamma(T) + 2$. As discussed earlier, $sd_{\gamma_{it}}(T) \leq 2$.

Case 2. T has no strong support.
If $\text{diam}(T) = 3$, then $T = P_4$. If $\text{diam}(T) = 4$, then T is a star K_n, $n \geq 2$, with at least two edges subdivided and for the said graphs, $\gamma_{it}(T) = \gamma(T)$, which is not the case.
Let $P = (v_1, v_2, v_3, \ldots, v_k)$, where $k = \text{diam}(T) + 1$ be a diametral path in T. Now consider the following subcases.

Subcase 2.1. $\text{diam}(T) = 5$.
Suppose that v_3 is a support. Clearly $v_2, v_3 \in S$ and $(S \setminus \{v_3, v_2\}) \cup \{v_1, x\}$, where x is the leaf vertex adjacent to v_3 is an $ITDS$ of T. Hence, $\gamma_{it}(T) = \gamma(T)$, which is not the case. A similar discussion holds when v_4 is a support. Suppose that v_3 and v_4 are not supports. If $\deg(v_3) \geq 3$ and $\deg(v_4) = 2$, then subdividing the edges v_1v_2, v_2v_3, v_3v_4 and v_4v_5, we see that $\gamma_{it}(T') \geq \gamma(T') = \gamma(T) + 2 > \gamma(T) + 1 = \gamma_{it}(T)$. Therefore, $sd_{\gamma_{it}(T)} \leq 4$. If $\deg(v_3), \deg(v_4) \geq 3$, then $(S \setminus \{v_2, v_5\}) \cup \{v_1, v_6\}$ is an $ITDS$ of T. Hence, $\gamma_{it}(T') = \gamma(T)$, which is not the case. If $\deg(v_3) = 2 = \deg(v_4)$, then $T = P_6$ and by Theorem 2, $sd_{\gamma_{it}}(P_6) = 4$.

Subcase 2.2. $\text{diam}(T) = 6$.
Suppose that v_3 is a support and $deg(v_3) \geq 4$. Let y be the leaf neighbour of v_3. If $|epn(v_3, S)| = 2$, then subdividing the edges v_3y, v_3v_4, v_4v_5 and v_5v_6, we see that $\gamma_{it}(T') \geq \gamma(T') > \gamma(T) + 2 > \gamma(T) + 1 = \gamma_{it}(T)$. Therefore, $sd_{\gamma_{it}}(T) \leq 4$. If

$|epn(v_3, S)| = 1$, then $(S \setminus \{v_3, v_2\}) \cup \{v_1, y\}$ is an $ITDS$ of T which implies that $\gamma_{it}(T) = \gamma(T)$. A similar discussion holds when v_5 is a support. Suppose that v_3 is not a support. If $\deg(v_3) \geq 3$, then $(S \setminus \{v_2\}) \cup \{v_1\}$ is an $ITDS$ of T, which implies that $\gamma_{it}(T) = \gamma(T)$. A similar discussion holds if v_5 is not a support and $\deg(v_5) \geq 3$. Suppose that $\deg(v_3) = \deg(v_5) = 2$. If v_4 is a support, then $(S \setminus \{v_2\}) \cup \{v_1\}$ is an $ITDS$ of T which implies that $\gamma_{it}(T) = \gamma(T)$. If v_4 is not a support and $\deg(v_4) = 2$, then $T = P_7$ and $sd_{\gamma_{it}}(P_7) = 3$. Suppose that v_4 is not a support and $\deg(v_4) \geq 3$. If v_4 is adjacent to a support, then $T = T_4$. By Lemma 1, $sd_{\gamma_{it}}(T) = 3$. If v_4 is not adjacent to a support, then choose S such that $v_4 \in S$. Then $(S \setminus \{v_2\}) \cup \{v_1\}$ is an $ITDS$ of T which implies that $\gamma_{it}(T) = \gamma(T)$. Thus as discussed earlier, $sd_{\gamma_{it}}(T) \leq 3$.

Subcase 2.3. $\mathrm{diam}(T) = 7$.

Suppose that v_3 is a support and x be its leaf neighbour. Here $|epn(v_3, S)| \leq 2$. If $|epn(v_3, S)| = 1$, then $(S \setminus \{v_3, v_2\}) \cup \{v_1, x\}$, is an $ITDS$ of T which implies that $\gamma_{it}(T) = \gamma(T)$. If $|epn(v_3, S)| = 2$ and $\deg(v_3) \geq 3$, then subdividing the edges v_1v_2, v_2v_3, v_3v_4 and v_3x, we see that $\gamma_{it}(T') \geq \gamma(T') = \gamma(T) + 2 > \gamma(T) + 1 = \gamma_{it}(T)$. Therefore, $sd_{\gamma_{it}}(T) \leq 4$. Suppose that v_3 is not a support and $\deg(v_3) \geq 3$. Then $(S \setminus \{v_2\}) \cup \{v_1\}$ is an $ITDS$ of T, which implies that $\gamma_{it}(T) = \gamma(T)$. A similar argument holds for v_6. Now consider the case $\deg(v_3) = \deg(v_6) = 2$. If v_4 is a support, then $(S \setminus \{v_2\}) \cup \{v_1\}$ is an $ITDS$ of T. If v_5 is a support, then $(S \setminus \{v_7\}) \cup \{v_8\}$ is an $ITDS$ of T. Hence, if either v_4 or v_5 is a support, then $\gamma_{it}(T) = \gamma(T)$. Suppose that v_4 and v_5 are not supports. If $\deg(v_4) = \deg(v_5) = 2$, then $T = P_8$ and $sd_{\gamma_{it}}(P_8) = 2$. Suppose that at least one of $\deg(v_4)$, $\deg(v_5)$ is at least 3. If both v_4 and v_5 are adjacent to supports, then subdividing the edges v_1v_2 and v_7v_8, we see that $\gamma(T') = \gamma(T) + 2$. Hence, $\gamma_{it}(T') > \gamma_{it}(T)$ and therefore $sd_{\gamma_{it}}(T) \leq 2$. If at least one of v_4 or v_5 is not adjacent to a support, than choose S such that $v_4 \in S$. Now, $(S \setminus \{v_2\}) \cup \{v_1\}$ is an $ITDS$ of T which implies that $\gamma_{it}(T) = \gamma(T)$. Thus as discussed earlier, $sd_{\gamma_{it}}(T) \leq 3$.

Subcase 2.4. $\mathrm{diam}(T) \geq 8$.

Suppose that v_3 is a support. In this case $|epn(v_3, S)| \leq 2$. If $|epn(v_3, S)| = 1$, then $(S \setminus \{v_3, v_2\}) \cup \{v_1, y\}$ where y is the leaf vertex adjacent to v_3 is an $ITDS$ of T. Hence, $\gamma_{it}(T) = \gamma(T)$, which is not the case. Suppose that $|epn(v_3, S)| = 2$. Suppose that $v_{k-2} \in S$. Then $|epn(v_{k-2}), S| \leq 2$. If $|epn(v_{k-2}, S)| = 1$, then $(S \setminus \{v_{k-1}, v_{k-2}\}) \cup \{v_k, z\}$, where $|epn(v_{k-2}, S)| = \{z\}$ is an $ITDS$ of T. If $|epn(v_{k-2}, S)| = 2$, then subdivide the edges v_3y and $v_{k-2}z$, where $z \in epn(v_{k-2}, S))$, we see that $\gamma_{it}(T) \geq \gamma(T') = \gamma(T) + 2 > \gamma(T) + 1 = \gamma_{it}(T)$. Therefore, $sd_{\gamma_{it}}(T) \leq 2$. If $v_{k-2} \notin S$ and $\deg(v_{k-2}) \geq 3$, then $(S \setminus \{v_{k-1}\}) \cup \{v_k\}$ is an $ITDS$ of T, which implies that $\gamma_{it}(T) = \gamma(T)$. Suppose that $\deg(v_{k-2}) = 2$. If $v_{k-3} \in S$, then $(S \setminus \{v_{k-1}\}) \cup \{v_k\}$ is an $ITDS$ of T, which implies that $\gamma_{it}(T) = \gamma(T)$. If $v_{k-3} \notin S$, then subdividing the edges v_3y and v_kv_{k-1}, we see that $\gamma_{it}(T') > \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \leq 2$. Suppose that v_3 is not a support. If $\deg(v_3) \geq 3$, then $(S \setminus \{v_2, v_3\}) \cup \{v_4, v_1\}$ or $(S \setminus \{v_2\}) \cup \{v_1\}$ is an $ITDS$ of T according as $v_3 \in S$ or $v_3 \notin S$, which implies that $\gamma_{it}(T) = \gamma(T)$. Suppose that $\deg(v_3) = 2$. If $v_4 \in S$, then $(S \setminus \{v_2\}) \cup \{v_1\}$ is an $ITDS$ of T, which implies that, $\gamma_{it}(T) = \gamma(T)$. If $v_4 \notin S$, then subdividing the edges v_1v_2 and $v_{k-1}v_k$, we see that

$\gamma_{it}(T') \geq \gamma(T') = \gamma(T) + 2 > \gamma(T) + 1 = \gamma_{it}(T)$. Therefore, $sd_{\gamma_{it}}(T) \leq 2$. Summing up the above arguments, we conclude that when $\gamma_{it}(T) = \gamma(T) + 1$, $sd_{\gamma_{it}}(T) \leq 4$. $\square$

4. Trees with $sd_{\gamma_{it}}(T) = 3$

In this section we characterize the class of all trees T with $sd_{\gamma_{it}}(T) = 3$.

Aram *et al.* [2], classify the class of trees as class 1, class 2 or class 3 depending on whether their domination subdivision number is 1,2 or 3 respectively. They have also characterized all trees with $sd_\gamma(T) = 3$. In order to characterize the said trees, they have defined a family $\mathcal{F}$ of labelled trees that are of class 3 as follows. The label of a vertex is called its *status*, denoted as *sta(v)*.

Definition: Let $\mathcal{F}$ be family of labelled trees that

(1) contains P_4 where the two leaves have status A, and the two support vertices have status B, and

(2) is closed under the two operations $\mathfrak{T}_1$ and $\mathfrak{T}_2$ which extend the tree T by attaching a tree to the vertex $y \in V(T)$, called the attacher.

Operation $\mathfrak{T}_1$. Assume $sta(y) = A$. Then add a path xwv and the edge xy. Let $sta(x) = sta(w) = B$, and $sta(v) = A$.

Operation $\mathfrak{T}_2$. Assume $sta(y) = B$. Then add a path xw and the edge xy. Let $sta(x) = B$ and $sta(w) = A$.

The two operations $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are illustrated in Figure 3.

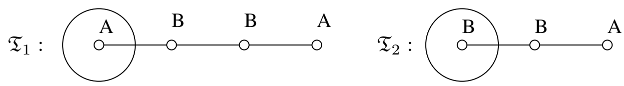

Figure 3. **The two operations.**

If $T \in \mathcal{F}$, they defined $A(T)$ and $B(T)$ to be the set of vertices of statuses A and B respectively, in T. The sets $A(T)$ and $B(T)$ depend a priori on the way the tree T is constructed from an initial P_4. We state below three results proved by them in [2].

Lemma 2. *[2] If $T \in \mathcal{F}$ and T is obtained from $T_0 = P_4$ by a sequence $\mathfrak{T}^1$, $\mathfrak{T}^2, \ldots, \mathfrak{T}^m$, then $A(T)$ is a $\gamma(T)$-set and $\gamma(T) = m + 2$.*

Lemma 3. *[2] Let $T \in \mathcal{F}$, T^* be obtained from T by subdividing one edge of T and $z \in A(T)$. Then $\gamma(T^*) = \gamma(T)$ and there is a γ-set of T^* containing z.*

Theorem 7. *[2] A tree T of order $n \geq 3$ is in class 3 if and only if $T \in \mathcal{F}$.*

In the following theorem we prove that if $T \in \mathcal{F}$, then $\gamma_{it}(T) = \gamma(T)$.

Theorem 8. *If $T \in \mathcal{F}$, then $\gamma_{it}(T) = \gamma(T)$.*

Proof. If $T = P_4$, then $\gamma_{it}(T) = \gamma(T) = 2$. Let T be obtained from P_4 by a sequence of operations $\mathfrak{T}^1, \mathfrak{T}^2, \ldots, \mathfrak{T}^m$. By Lemma 2, $A(T)$ is a γ-set of T. If the m^{th} operation is $\mathfrak{T}_1$, then a path xwv and an edge xy where $y \in V(T)$ is added to T. Further by definition, $y, v \in A(T)$. Now any maximum independent set in T contains either y or v. Thus, $A(T)$ is a γ_{it}-set of T. Hence, $\gamma_{it}(T) = \gamma(T)$.

If the m^{th} operation is $\mathfrak{T}_2$, then a path xw and an edge xy, where $y \in V(T)$ is added to T. By definition of $\mathcal{F}$, $y, x \in B(T)$ and $w \in A(T)$. Now there exists a vertex u in $A(T)$ such that $u \in N(y)$. Thus, any maximum independent set of T contains either w or u. Thus, $A(T)$ is a γ_{it}-set of T. Hence, $\gamma_{it}(T) = \gamma(T)$. □

In the following theorem we characterize trees T with $\gamma_{it}(T) = \gamma(T)$ and $sd_{\gamma_{it}}(T) = 3$.

Theorem 9. *For any tree T with $\gamma_{it}(T) = \gamma(T)$, $sd_{\gamma_{it}}(T) = 3$ if and only if $T \in \mathcal{F}$ and diam$(T) \geq 6$.*

Proof. Suppose that $sd_{\gamma_{it}}(T) = 3$. By Theorem 5, $sd_{\gamma_{it}}(T) \leq 3$. Hence, $sd_{\gamma}(T) = 3$ and by Theorem 7, $T \in \mathcal{F}$. Since T does not have a strong support, diam$(T) \geq 3$. We claim that diam$(T) \geq 6$. Suppose to the contrary that $3 \leq$ diam$(T) \leq 5$. Consider P_4. Then by definition of $\mathcal{F}$, we see that operation $\mathfrak{T}_1$ cannot be performed at any vertex of P_4. Further, operation $\mathfrak{T}_2$ can be performed only at the internal vertices of P_4. Hence, clearly T is either P_4 or T_9 or T_{10}. (Refer Figure 4).

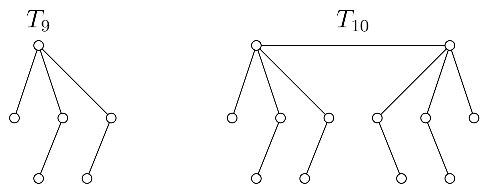

Figure 4. **Trees illustrating the proof of Theorem 9**

By Theorem 2, $sd_{\gamma_{it}}(P_4) = 2$, which is a contradiction. If $T = T_9$ or T_{10}, subdividing the two edges incident at a weak support of degree two, we see that the set of all supports of the resulting graph T' from a γ-set of T'. Further any γ-set of T' contains two weak supports at a distance 3. Hence given any γ-set S of T', there is a maximum independent set of T' which does not intersect S. Hence, $\gamma_{it}(T') > \gamma_{it}(T)$. Thus, $sd_{\gamma_{it}}(T) \leq 2$, which is a contradiction. Thus, diam$(T) \geq 6$.

Conversely suppose that $T \in \mathcal{F}$ and diam$(T) \geq 6$. We claim that $sd_{\gamma_{it}}(T) = 3$. Clearly we see that subdividing any single edge in T will not increase $\gamma_{it}(T)$. Hence,

$sd_{\gamma_{it}}(T) \geq 2$. Let T^* be obtained from T by subdividing any two edges e, f of T. Consider the length m of the sequence of operations needed to construct the tree T. Let $T \in \mathcal{F}$ be obtained from $\mathfrak{T}^1, \mathfrak{T}^2, \ldots, \mathfrak{T}^m$. Let T be obtained from T_{m-1} by the operation $\mathfrak{T}^m$.

Case 1. $e \in E(T_{m-1})$, $f \notin E(T_{m-1})$.
Subcase 1.1. T is obtained from T_{m-1} using operation $\mathfrak{T}_1$.

A path xwv and an edge xy with $y \in A(T_{m-1})$ is added to T_{m-1}. Without loss of generality let $f = xw$. Since $sd_{\gamma_{it}}(T) = 3$, $\gamma(T^*) = \gamma(T)$. If there exists a $\gamma(T^*)$-set which contains x, then choose a $\gamma(T^*)$- set S such that $x, v \in S$. Now every maximum independent set of T^* contains either x or v. Thus, S is a $\gamma_{it}(T^*)$- set and $\gamma_{it}(T^*) = \gamma(T^*)$. By Lemma 3, $\gamma(T^*) = \gamma(T) = \gamma_{it}(T)$. Hence, $\gamma_{it}(T^*) = \gamma_{it}(T)$ which implies that $sd_{\gamma_{it}}(T) \geq 3$. If there does not exist a $\gamma(T^*)$- set which contains x, then every $\gamma(T^*)$- set S contains y and $|epn(y, S)| \geq 2$. Further there exist a path y, a_1, a_2, a_3 in that order such that $a_1, a_2 \notin S$, $a_3 \in S$ with $a_2 \in epn(a_3, S)$ and $|epn(a_3, S)| \geq 2$ (such a path exits as $diam(T) \geq 6$). Now, every maximum independent set of T^* contains either y or a_3. Hence, as before $sd_{\gamma_{it}}(T) \geq 3$.

Subcase 1.2. T is obtained from T_{m-1} using operation $\mathfrak{T}_2$.
A path xw and an edge xy with $y \in B(T_{m-1})$ is added to T_{m-1}. Without loss of generality let $f = xy$. If there exists a $\gamma(T^*)$-set which contains y, then choose a $\gamma(T^*)$- set S such that $y, w \in S$. Now every maximum independent set T^* contains either y or w. Thus, S is a $\gamma_{it}(T^*)$- set and $\gamma_{it}(T^*) = \gamma(T^*) = \gamma(T) = \gamma_{it}(T)$. Thus, $sd_{\gamma_{it}}(T) \geq 3$. If there does not exist a $\gamma(T^*)$- set which contains y, then every $\gamma(T^*)$-set S contains a vertex $z \in N(y) \cap V(T_{m-1})$ with $|epn(z, S)| \geq 2$. Then there exists a path y, a_1, a_2, a_3 in that order such that $a_1, a_2 \notin S$, $a_3 \in S$ with $a_2 \in epn(a_3, S)$ and $|epn(a_3, S)| \geq 2$. Now every maximum independent set of T^* contains either z or a_3. Hence, as before $sd_{\gamma_{it}}(T) \geq 3$.

Case 2. $e, f \notin E(T_{m-1})$.
Subcase 2.1. T is obtained from T_{m-1} using operation $\mathfrak{T}_1$.

As in the earlier case y, x, w, v are defined. Without loss of generality $e = yx$, $f = xw$. Since $sd_\gamma(T) = 3$, $\gamma(T^*) = \gamma(T)$. Any $\gamma(T^*)$ set will contain y' and w where y' is the subdivision vertex of xy. It is also clear that any maximum independent set of T^* will contain y'. Hence, any $\gamma(T^*)$ set say S is also a $\gamma_{it}(T^*)$- set. Thus, $\gamma_{it}(T^*) = \gamma(T^*) = \gamma(T) = \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \geq 3$.

Subcase 2.2. T is obtained from T_{m-1} using operation $\mathfrak{T}_2$.
As in the earlier case y, x, w are defined. Let $e = yx$, $f = xw, w_1, w_2$ be the subdivision vertices of e and f respectively. If there exist a $\gamma(T^*)$ which contains w_1, then choose a $\gamma(T^*)$- set S such that $w_1, w \in S$. Then every maximum independent set of T^* will contain either w_1 or w. Thus, S is a $\gamma_{it}(T^*)$- set. Therefore, $sd_{\gamma_{it}}(T) \geq 3$. Otherwise, every $\gamma(T^*)$- set S will contain y with $|epn(y, S)| \geq 2$. Further there exist a path y, a_1, a_2, a_3 as described in Subcase 2.1 of Case 2. Hence, $sd_{\gamma_{it}}(T) \geq 3$.

Case 3. $e, f \in E(T_{m-1})$.

Subcase 3.1. T is obtained from T_{m-1} using operation $\mathfrak{T}_1$.
The proof is similar to Subcase 1.2 of Case 1.

Subcase 3.2. T is obtained from T_{m-1} using operation $\mathfrak{T}_2$.
The vertices y, x, w are defined as in Subcase 1.2 of Case 1. If there exist a $\gamma(T^*)$-set containing some member $z \in N(y) \setminus \{x\}$, then choose a $\gamma(T^*)$-set S such that $z, w \in S$. Then S is a $\gamma_{it}(T^*)$- set. Therefore, $sd_{\gamma_{it}}(T) \geq 3$. Otherwise for every $\gamma_{it}(T^*)$- set S, there exist a path y, z, a_1, a_2 in that order such that $y, z, a_2 \notin S$, $x, a_1 \in S$ with $y \in epn(x, S)$, $z, a_2 \in epn(a_1, S)$. Now any maximum independent set of T^* will contain either x or a_1. Hence, S is a $\gamma_{it}(T^*)$- set. Thus, $sd_{\gamma_{it}}(T) \geq 3$.
In all the three cases we see that $sd_{\gamma_{it}}(T) \geq 3$. By Theorem 5, $sd_{\gamma_{it}}(T) \leq 3$. Hence, $sd_{\gamma_{it}}(T) = 3$. □

Next we characterize trees T with $\gamma_{it}(T) = \gamma(T) + 1$ and $sd_{\gamma_{it}}(T) = 3$. For this purpose, we first prove the following four lemmas. Consider the trees T_i, $2 \leq i \leq 5$ as in Figure 1. Let S be a γ - set of T such that S does not contain leaf vertices.

Lemma 4. *If $T = T_2$, then $sd_{\gamma_{it}}(T) = 3$.*

Proof. Consider the graph T_2 in Figure 1. Let $u, a_1, b_1, u_1, u_2,$ be as labelled in Figure 1. Now $S = \{u, a_1\}$ is a γ -set of T. Clearly $\gamma_{it}(T) = \gamma(T)+1$. Subdividing an edge incident at a_1 does not increase the value of $\gamma(T)$. Now subdividing any pendant edge incident at u will increase the value of $\gamma(T)$ by 1. Without loss of generality let the edge uu_1 be subdivided. Then, $(S \setminus \{u\}) \cup \{u_1, u_2\}$ is an $ITDS$ of the resulting graph, as every maximum independent set of T' contains u_1 and u_2. Therefore, $\gamma_{it}(T') = \gamma(T)+1 = \gamma_{it}(T)$, which implies that $\gamma_{it}(T') = \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \geq 2$.

The following table (Table 2) gives the respective $ITDS$ of the resulting graph and the justification. We also assume that w_i, $i = 1, 2$ are the subdivision vertices taken in the order in which the edges appear in the table, for all the tables that appear in the following Lemmas, w_i, $1 \leq i \leq k$ are defined in a similar manner.

Table 2. **Subdividing two edges**

Edges subdivided	$ITDS$	Every maximum independent set of T' contains
uu_1 and uu_2	$(S \setminus \{u\}) \cup \{u_1, u_2\}$	u_1
uu_1 and a_1b_1	$(S \setminus \{u, a_1\}) \cup \{u_1, u_2, w_1\}$	u_1 and u_2
a_1b_1 and ua_1	$(S \setminus \{a_1\}) \cup \{w_1, b_1\}$	b_1 or w_1
uu_1 and ua_1	$(S \setminus \{u\}) \cup \{u_1, u_2\}$	u_1 and u_2

Thus, in all the cases $\gamma_{it}(T') \leq \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \geq 3$. Finally, subdivide the edges uu_1, ua_1 and a_1b_1. Corresponding to any γ - set S of T', $G[V \setminus S] =$

$(k+2)K_1 \cup K_2$ and thus any maximum independent set of $G[V \setminus S]$ is of size $k+3$. Further, $\beta(T') = k+3$. Hence, there exist a maximum independent set of T' which does not intersect S. Thus, $\gamma_{it}(T') > \gamma_{it}(T)$ which implies that $sd_{\gamma_{it}}(T) = 3$. □

Lemma 5. *If $T = T_3$, then $sd_{\gamma_{it}}(T) = 3$.*

Proof. Consider the graph T_3 in Figure 1. Let $u, a_1, a_2, \ldots, a_k$, $b_1, b_2, \ldots, b_k$, u_1, u_2, be as labelled in Figure 1 and $S = \{u, a_1, a_2, \ldots, a_k\}$ is a γ - set of T. Clearly $\gamma_{it}(T) = \gamma(T) + 1$. Subdividing any edge incident at a_i, $1 \leq i \leq k$ does not increase $\gamma(T)$. Hence, $\gamma_{it}(T)$ also does not increase. Now subdividing any pendant edge incident at u will increase the value of $\gamma(T)$ by 1. Without loss of generality let the edge uu_1 be subdivided. Then, $(S \setminus \{u\}) \cup \{u_1, u_2\}$ is an $ITDS$ of the resulting graph, as every maximum independent set of T' contains u_1 and u_2. Therefore, $\gamma_{it}(T') = \gamma(T) + 2 > \gamma(T) + 1 = \gamma_{it}(T)$, which implies that $\gamma_{it}(T') = \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \geq 2$. The following table (Table 3) gives the respective $ITDS$ of the resulting graph and the justification. We define w_i, $i = 1, 2$ as earlier.

Table 3. **Subdividing two edges**

Edges subdivided	$ITDS$	Every maximum independent set of T' contains
uu_1 and uu_2	$(S \setminus \{u\}) \cup \{u_1, u_2\}$	u_1 and u_2
uu_1 and a_1b_1	$(S \setminus \{u, a_1\}) \cup \{u_1, u_2, w_1\}$	u_1 and u_2
a_1b_1 and a_2b_2	$(S \setminus \{a_1, a_2\}) \cup \{w_1, w_2, b_1\}$	b_1
ua_1 and a_1b_1	$(S \setminus \{a_1\}) \cup \{w_1, b_1\}$	b_1 or w_1

Thus in all the cases, $\gamma_{it}(T') \leq \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \geq 3$.

Finally, subdivide the edges uu_1, ua_1 and a_1b_1. Corresponding to any γ-set S of T', $G[V \setminus S] = (k+2)K_1 \cup K_2$ and thus any maximum independent set of $G[V \setminus S]$ is of size $k+3$. Further, $\beta(T') = k+3$. Hence, there exist a maximum independent set of T' which does not intersect S. Thus, $\gamma_{it}(T') > \gamma_{it}(T)$ which implies that $sd_{\gamma_{it}}(T) = 3$. □

Lemma 6. *If $T = T_5$, then $sd_{\gamma_{it}}(T) = 3$.*

Proof. Consider the graph T_5 in Figure 1. Let $x, x_1, x_2, a_1, a_2, \ldots, a_k$, $b_1, b_2, \ldots, b_k$, $y_1, y_2, \ldots, y_k$, $z_1, z_2, \ldots, z_k$, $c_1, c_1, \ldots, c_k$, be as labelled in Figure 2 and $S = \{x, a_1, a_2, \ldots, a_k, c_1, c_1, \ldots, c_k\}$ is a γ-set of T. Clearly $\gamma_{it}(T) = \gamma(T) + 1$. Subdividing any edge incident at a_i does not increase the value of $\gamma(T)$. Now subdividing any pendant edge incident at u will increase the value of $\gamma(T)$ by 1. Without loss of generality let the edge xx_1 be subdivided. Then, $S \cup \{x_1\}$ is an $ITDS$ of the resulting graph, as every maximum independent set of T' contains x_1 and x. Therefore,

$\gamma_{it}(T') = \gamma(T)+1 = \gamma_{it}(T)$, which implies that $\gamma_{it}(T') = \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \geq 2$.

The following table (Table 4) gives the respective $ITDS$ of the resulting graph and the justification. We define w_i, $i = 1, 2$ as earlier.

Table 4. **Subdividing two edges**

Edges subdivided	$ITDS$	Every maximum independent set of T' contains
xx_1 and xx_2	$(S \setminus \{x\}) \cup \{x_1, w_2\}$	x_1
xx_1 and xa_1	$(S \setminus \{a_1\}) \cup \{w_1, b_1\}$	b_1
xx_1 and z_1x_2	$(S \setminus \{x\}) \cup \{x_1, x_2\}$	x_1
a_1b_1 and a_2b_2	$(S \setminus \{a_1, a_2\}) \cup \{w_1, w_2, x_1\}$	x_1
y_1c_1 and y_2c_2	$(S \setminus \{y_1, y_2, x\}) \cup \{w_1, w_2, x_1, x_2\}$	x_1
xx_2 and y_1c_1	$(S \setminus \{y_1\}) \cup \{w_2, c_1\}$	c_1
a_1b_1 and y_1c_1	$(S \setminus \{a_1, y_1\}) \cup \{w_1, w_2, x_1\}$	x_1
xx_1 and y_1c_1	$(S \setminus \{y_1, x\}) \cup \{w_2, x_1, x_2\}$	x_1

Thus, in all the cases $\gamma_{it}(T') \leq \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \geq 3$. Finally, subdivide the edges y_1c_1, y_1z_1 and y_2z_2, then we see that $\gamma_{it}(T') = \gamma(T') = \gamma(T) + 2 > \gamma(T) + 1 = \gamma_{it}(T)$. Therefore, $\gamma_{it}(T') > \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \leq 4$ which implies that $sd_{\gamma_{it}}(T) = 4$. □

Theorem 10. *Let T be a tree with $\gamma_{it}(T) = \gamma(T) + 1$. Then, $sd_{\gamma_{it}}(T) = 3$ if and only if T is T_2 or T_3 or T_4 or T_5*

Proof. The sufficiency follows from Lemmas 1, 4, 5 and 6. To prove the necessity, suppose that $sd_{\gamma_{it}}(T) = 3$. As in the proof of Theorem 6, we see that if T has either a strong support adjacent to at least three leaf vertices or two strong supports adjacent to exactly two leaf vertices respectively, then $sd_{\gamma_{it}}(T) \leq 2$. Suppose T contains exactly one strong support adjacent to exactly two leaf vertices. Then as in the proof of Theorem 6, if $\text{diam}(T) \geq 5$, then $sd_{\gamma_{it}}(T) \leq 2$. If $\text{diam}(T) = 3$, then $T = T_2$. If $\text{diam}(T) = 4$, then T is $T_1(a)$ or $T_2(b)$, $\gamma_{it}(T) = \gamma(T)$ which is not the case. Thus, $T = T_3$.
Suppose T contains no strong supports. Let $P = (v_1, v_2, \ldots, v_k)$, where $k = \text{diam}(T)+1$ be a diametral path in T. If $\text{diam}(T) = 3$, then $T = P_4$. If $\text{diam}(T) = 4$, then T is either a healthy spider or a wounded spider. For all the said graphs, $\gamma_{it}(T) = \gamma(T)$ which is not the case. Suppose that $\text{diam}(T) = 5$. If $\deg(v_3) = \deg(v_4) = 2$, then $T = P_6$, and by Theorem 2, $sd_{\gamma_{it}}(T) = 4$. If $\deg(v_3) \geq 3$ and $\deg(v_4) = 2$ and v_3 is not a support, then $T = T_6$. By Lemma 7, $sd_{\gamma_{it}}(T) = 4$. In all the other cases, as discussed in Theorem 6, we see that $\gamma_{it}(T) = \gamma(T)$ which is not the case.

Suppose that $\text{diam}(T) = 6$. Let S be a γ-set of T such that S does not contain leaf vertices. If $\deg(v_3) = \deg(v_4) = \deg(v_5) = 2$, then $T = P_7$. Suppose that v_3

is a support. Let x be its leaf neighbour. If $\deg(v_4) = \deg(v_5) = 2$, then $T = T_7$ and by Lemma 8, $sd_{\gamma_{it}}(T) = 4$. Suppose that $\deg(v_4) \geq 4$ and $\deg(v_5) = 2$. If v_4 is adjacent to a support, then $(S \setminus \{v_2, v_3\}) \cup \{v_1, x\}$ is an $ITDS$ of T which implies that $\gamma_{it}(T) = \gamma(T)$. If v_4 is not adjacent to a support and each member of $N(v_4) \setminus \{v_3\}$ is of degree 2, then $T = T_5$. If v_4 is not adjacent to a support and some member of $N(v_4) \setminus \{v_3, v_5\}$ is of degree at least 3, then $\gamma_{it}(T) = \gamma(T)$. Then as discussed in Subcase 2.2 of Case 2 of Theorem 6, we see that if v_4 is not a support and adjacent to a support, then $T = T_4$ and in all other cases $\gamma_{it}(T) = \gamma(T)$.

Suppose that $\mathrm{diam}(T) = 7$. If v_3 and v_6 are supports and $\deg(v_4) = \deg(v_5) = 2$, then $T = T_8$ and by Lemma 9, $sd_{\gamma_{it}}(T) = 4$. In all the other cases, as discussed in Theorem 6, we see that either $\gamma_{it}(T) = \gamma(T)$ or $sd_{\gamma_{it}}(T) \leq 2$, which is a contradiction. If $\mathrm{diam}(T) \geq 8$, as in the proof of Subcase 2.4 of Theorem 6, we see that either $\gamma_{it}(T) = \gamma(T)$ or $sd_{\gamma_{it}}(T) \leq 2$, which is a contradiction.
Hence, we conclude that if $sd_{\gamma_{it}}(T) = 3$, then $T = T_i$, $2 \leq i \leq 5$. $\square$

5. Trees with $sd_{\gamma_{it}}(T) = 4$

In view of Theorem 5, we observe that if $\gamma_{it}(T) = \gamma(T)$, then $sd_{\gamma_{it}}(T) \leq 3$. Hence, for trees T with $sd_{\gamma_{it}}(T) = 4$, $\gamma_{it}(T) = \gamma(T) + 1$. In this section we characterize the class of all trees T with $sd_{\gamma_{it}}(T) = 4$. For this purpose, we first prove the following three lemmas. Consider the trees T_i, $6 \leq i \leq 8$ as in Figure 5.

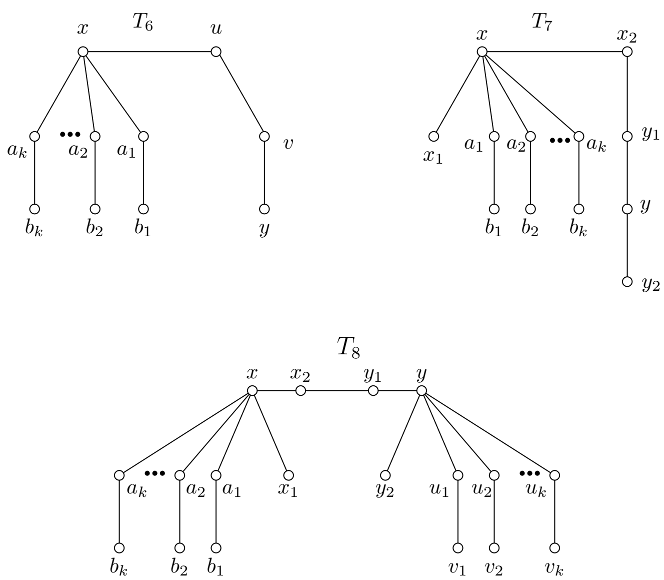

Figure 5. Examples of trees with $sd_{\gamma_{it}}(T) = 4$

Lemma 7. *If $T = T_6$, then $sd_{\gamma_{it}}(T) = 4$.*

Proof. Consider the graph T_6 in Figure 3. Let $x, a_1, a_2, \ldots, a_k$, $b_1, b_2, \ldots, b_k$, u, v, y be as labelled in Figure 5 and $S = \{x, a_1, a_2, \ldots, a_k, v\}$ is a γ-set of T. Further, $\deg(x) \geq 3$. Now subdividing any edge incident with a_i will not increase the value of $\gamma(T)$. Further subdividing any edge in the (x, y)-path will increase the value of $\gamma(T)$ by 1. Now $(S \setminus \{v\}) \cup \{u, y\}$, where w is a subdivision vertex of uv is an $ITDS$ of the resulting graph, as every maximum independent set of T' contains u or v. Therefore, $\gamma_{it}(T') = \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \geq 2$.
The following tables (Table 5 and Table 6) give the respective $ITDS$ of the resulting graph and the justification. We define w_i, $i = 1, 2$ as earlier.

Table 5. **Subdividing two edges**

Edges subdivided	$ITDS$	Every maximum independent set of T' contains
xa_1 and a_1b_1	$(S \setminus \{a_1\}) \cup \{w_1, b_1\}$	w_1 or b_1
uv and vy	$(S \setminus \{v\}) \cup \{w_1, y\}$	w_1 or y
xu and vy	$(S \setminus \{v\}) \cup \{u, y\}$	u and y
a_1b_1 and vy	$(S \setminus \{a_1, v\}) \cup \{w_1, u, y\}$	u or y
a_1b_1 and a_2b_2	$(S \setminus \{a_1, a_2\}) \cup \{w_1, w_2, y\}$	y

Thus in all the cases $\gamma_{it}(T') \leq \gamma_{it}(T)$. Hence, Table 5 implies that $sd_{\gamma_{it}}(T) \geq 3$ and Table 6 implies that $sd_{\gamma_{it}}(T) \geq 4$.

Table 6. **Subdividing three edges**

Edges subdivided	$ITDS$	Every maximum independent set of T' contains
xa_1, a_1b_1 and vy	$(S \setminus \{a_1, v\}) \cup \{w_3, b_1, x\}$	x
xa_2, a_2b_2 and xu	$(S \setminus \{a_2\}) \cup \{w_1, x\}$	x
xu, uv and vy	$(S \setminus \{v\}) \cup \{w_3, u\}$	u or w_3
xa_1, uv and vy	$(S \setminus \{v\}) \cup \{u, w_3\}$	u
xa_1, xa_2 and a_1b_1	$(S \setminus \{a_1, v\}) \cup \{w_1, x, y\}$	y
a_1b_1, a_2b_2 and vy	$(S \setminus \{a_1, v, a_2\}) \cup \{w_1, w_2, u, y\}$	y or u

Finally, if we subdivide the edges incident with a_1 and two edges incident with v, then we see that $\gamma(T') = \gamma(T) + 2 > \gamma(T) + 1 = \gamma_{it}(T)$. Therefore, $\gamma_{it}(T') > \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \leq 4$ which implies that $sd_{\gamma_{it}}(T) = 4$. □

Lemma 8. *If $T = T_7$, then $sd_{\gamma_{it}}(T) = 4$.*

Proof. Consider the graph T_7 in Figure 5. Let $x, a_1, a_2, \ldots, a_k$, $b_1, b_2, \ldots, b_k$,

x_1, x_2, y, y_1, y_2 be as labelled in Figure 5 and $S = \{x, a_1, a_2, \ldots, a_k, y\}$ is a γ-set of T. Further, $\deg(x) \geq 3$. Now subdividing any edge incident at a_i will not increase the value of $\gamma(T)$. Further subdividing any edge in the (x, y_2)-path will increase $\gamma(T)$ by 1. Now $(S \setminus \{y\}) \cup \{y_1, y_2\}$, where w is a subdivision vertex of yy_2 is an $ITDS$ of the resulting graph, as every maximum independent set of T' contains y_1 or y_2. Therefore, $\gamma_{it}(T') = \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \geq 2$.

The following tables (Table 7 and Table 8) give the respective $ITDS$ of the resulting graph and the justification. We define w_i, $i = 1, 2$ as earlier. Thus in all the cases

Table 7. **Subdividing two edges**

Edges subdivided	$ITDS$	Every maximum independent set of T' contains
y_1y and yy_2	$(S \setminus \{y\}) \cup \{w_1, y_2\}$	y_2
xa_1 and yy_2	$(S \setminus \{x, y\}) \cup \{x_1, x_2, w_2\}$	x_1 and x_2
xa_1 and a_1b_1	$(S \setminus \{a_1\}) \cup \{w_1, b_1\}$	b_1
a_1b_1 and xx_2	$(S \setminus \{x, a_1\}) \cup \{x_1, w_1, w_2\}$	x_1
a_1b_1 and yy_1	$(S \setminus \{a_1, y\}) \cup \{b_1, y_1, y_2\}$	b_1
a_1b_1 and a_2b_2	$(S \setminus \{a_1, a_2\}) \cup \{w_1, w_2, b_1\}$	b_1

$\gamma_{it}(T') \leq \gamma_{it}(T)$. Hence, Table 7 implies that $sd_{\gamma_{it}}(T) \geq 3$ and Table 8 implies that $sd_{\gamma_{it}}(T) \geq 4$.

Table 8. **Subdividing three edges**

Edges subdivided	$ITDS$	Every maximum independent set of T' contains
y_1y, x_2y_1 and yy_2	$(S \setminus \{y\}) \cup \{w_3, y_1\}$	y_1
xa_1, a_1b_1 and xa_2	$(S \setminus \{a_1\}) \cup \{b_1, w_1\}$	b_1 or w_1
xa_1, a_1b_1 and yy_1	$(S \setminus \{y, a_1\}) \cup \{w_1, y_2, y_1\}$	y_1 or y_2
xx_1y, a_1b_1 and xa_1	$(S \setminus \{a_1\}) \cup \{w_3, w_1\}$	x and x_1
y_1y, a_1b_1 and yy_2	$(S \setminus \{a_1, y\}) \cup \{w_1, w_2, y_2\}$	y_2
y_1y, xx_1 and yy_2	$(S \setminus \{x\}) \cup \{x_1, y_1\}$	x_1

Finally, if we subdivide the edges xx_1, xx_2 , yy_1 and yy_2, then we see that $\gamma(T') = \gamma(T) + 2 > \gamma(T) + 1 = \gamma_{it}(T)$. Therefore, $\gamma_{it}(T') > \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \leq 4$ which implies that $sd_{\gamma_{it}}(T) = 4$. □

Lemma 9. *If $T = T_8$, then $sd_{\gamma_{it}}(T) = 4$.*

Proof. Consider the graph T_8 in Figure 5. Let $x, y, a_1, a_2, \ldots, a_k$, $b_1, b_2, \ldots, b_k$,

$u_1, u_2, \ldots, u_k$, $v_1, v_2, \ldots, v_k$, x_1, x_2, y_1, y_2 be as labelled in Figure 5 and $S = \{x, y, a_1, a_2, \ldots, a_k, u_1, u_2, \ldots, u_k\}$ is a γ-set of T. Further, $\deg(x), \deg(y) \geq 4$. Now subdividing any edge incident at a_i or u_i will not increase the value of $\gamma(T)$. Further subdividing any edge not incident with a_i or u_i will increase $\gamma(T)$ by 1. Then $(S \setminus \{x\}) \cup \{x_1, x_2\}$ is an $ITDS$ of the resulting graph, as every maximum independent set of T' contains x_1 and x_2. Therefore, $\gamma_{it}(T') = \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \geq 2$.

The following tables (Table 9 and Table 10) give the respective $ITDS$ of the resulting graph and the justification. We define w_i, $i = 1, 2, 3$ as earlier.

Table 9. **Subdividing two edges**

Edges subdivided	$ITDS$	Every maximum independent set of T' contains
xa_1 and a_1b	$(S \setminus \{a_1\}) \cup \{b_1, w_1\}$	b_1 or w_1
xx_1 and xa_1	$(S \setminus \{x\}) \cup \{x_1, x_2\}$	x_1 or x_2
xx_1 and xx_2	$(S \setminus \{x\}) \cup \{w_1, x_2\}$	w_1 or x_2
xx_2 and x_2y_1	$S \cup \{w_2\}$	w_2 or w_1
xx_1 and yy_1	$(S \setminus \{x, y\}) \cup \{w_2, x_1, x_2\}$	x_1 or x_2
a_1b_1 and xx_2	$(S \setminus \{a_1\}) \cup \{w_1, w_2\}$	w_1 or w_2
a_1b_1 and u_1v_1	$(S \setminus \{a_1, u_1\}) \cup \{w_1, w_2, x_1\}$	x_1

Thus in all the cases, $\gamma_{it}(T') \leq \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \geq 3$.

Table 10. **Subdividing three edges**

Edges subdivided	$ITDS$	Every maximum independent set of T' contains
xx_2, x_2y_1 and yy_1	$S \cup \{w_2\}$	w_2
xx_1, yy_1 and yy_2	$(S \setminus \{x, y\}) \cup \{x_1, x_2, w_3\}$	x_1 or x_2
xx_1, xa_1 and a_1b_1	$(S \setminus \{a_1\}) \cup \{w_1, w_3\}$	w_1 or x
xa_1, a_1b_1 and a_2b_2	$(S \setminus \{a_1\}) \cup \{w_2, b_2\}$	b_2
xa_1, a_1b_1 and yy_2	$(S \setminus \{a_1\}) \cup \{w_3, b_1\}$	w_3 or b_1
xx_1, xx_2 and x_2y_1	$(S \setminus \{x\}) \cup \{x_1, x_2\}$	x_2 and x_1
xx_1, xx_2 and xa_1	$(S \setminus \{x\}) \cup \{x_1, x_2\}$	x_1 or x_2

Thus in all the cases, $\gamma_{it}(T') \leq \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \geq 4$. Finally, subdivide the edges xx_1, xx_2 , yy_1 and yy_2, then we see that $\gamma(T') = \gamma(T) + 2 > \gamma(T) + 1 = \gamma_{it}(T)$. Therefore, $\gamma_{it}(T') > \gamma_{it}(T)$. Hence, $sd_{\gamma_{it}}(T) \leq 4$ which implies that $sd_{\gamma_{it}}(T) = 4$. □

Theorem 11. *For any tree T, $sd_{\gamma_{it}}(T) = 4$ if and only if T is one of the graphs: P_6 or T_i, $6 \leq i \leq 8$.*

Proof. The sufficiency holds by Lemmas 7, 8 and 9. To prove the necessity, suppose that $sd_{\gamma_{it}}(T) = 4$. Suppose T has a strong support. Then, as in the proof of Theorem 6, either $sd_{\gamma_{it}}(T) \leq 2$ or $\gamma_{it}(T) = \gamma(T)$. Hence, T does not have strong supports.

Let $P = (v_1, v_2, \ldots, v_k)$, where $k = \text{diam}(T)+1$ be a diametral path in T. Suppose that $\text{diam}(T) = 5$. If $\deg(v_3) = \deg(v_4) = 2$, then $T = P_6$, and if $\deg(v_3) \geq 3$ and $\deg(v_4) = 2$ and v_3 is not a support, then $T = T_6$. In all the other cases, as discussed in Theorem 6, we see that $\gamma_{it}(T) = \gamma(T)$ which is not the case. Suppose that $\text{diam}(T) = 6$. If v_3 is a support and $\deg(v_3) \geq 3$ and $\deg(v_4) = \deg(v_5) = 2$, then $T = T_7$. If v_4 is not a support, $\deg(v_4) \geq 3$ and $\deg(v_3) = \deg(v_5) = 2$, then $T = T_4$ and by Lemma 1, $sd_{\gamma_{it}}(T) = 3$, which is a contradiction. In all the other cases, as discussed in Theorem 6, we see that $\gamma_{it}(T) = \gamma(T)$.

Suppose that $\text{diam}(T) = 7$. If v_3 and v_6 are supports and $\deg(v_4) = \deg(v_5) = 2$, then $T = T_8$. In all the other cases, as discussed in Theorem 6, we see that $\gamma_{it}(T) = \gamma(T)$. If $\text{diam}(T) \geq 8$, as in the proof of Subcase 2.4 of Theorem 6, we see that either $\gamma_{it}(T) = \gamma(T)$ or $sd_{\gamma_{it}}(T) \leq 2$, which is a contradiction. Hence, $T = T_i$, $6 \leq i \leq 8$. □

6. Conclusion

We have proved that for any tree T, $sd_{\gamma_{it}}(T) \leq 4$. We have characterized trees T with $sd_{\gamma_{it}}(T) = 3$ and $sd_{\gamma_{it}}(T) = 4$ respectively. Characterising the class of trees T for which $sd_{\gamma_{it}}(T) = 1$ and 2 respectively are still open. One can venture into these problems which are quite challenging.

Acknowledgements: The authors are grateful to the referees whose valuable suggestions have led to the improvement in the presentation of this paper.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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