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Short Note

Irredundance chromatic number and gamma chromatic number of trees

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Abstract: A vertex subset S of a graph $G = (V, E)$ is irredundant if every vertex in S has a private neighbor with respect to S. An irredundant set S of G is maximal if, for any $v \in V - S$, the set $S \cup \{v\}$ is no longer irredundant. The lower irredundance number of G is the minimum cardinality of a maximal irredundant set of G and is denoted by $ir(G)$. A coloring $\mathcal{C}$ of G is said to be the irredundance coloring if there exists a maximal irredundant set R of G such that all the vertices of R receive different colors. The minimum number of colors required for an irredundance coloring of G is called the irredundance chromatic number of G, and is denoted by $\chi_i(G)$. A vertex subset D of a graph G is dominating if every vertex in $V - D$ is adjacent to a vertex in D. The domination number of G is the minimum cardinality of a dominating set of G and is denoted by $\gamma(G)$. A coloring $\mathcal{C}$ of G is said to be the gamma coloring if there exists a dominating set D of G such that all the vertices of D receive different colors. The minimum number of colors required for a gamma coloring of G is called the gamma chromatic number of G, and is denoted by $\chi_\gamma(G)$. In this paper, we prove that every tree T satisfies $\chi_i(T) = ir(T)$ unless T is a star. Further, we prove that $\gamma(T) \le \chi_\gamma(T) \le \gamma(T) + 1$. We characterize all trees satisfying the upper bound.

Keywords: irredundance chromatic number, gamma chromatic number, irredundance coloring, gamma coloring.

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1. Introduction

The graphs $G = (V, E)$ considered in this paper are simple graphs without isolates with vertex set $V(G)$ and edge set $E(G)$. The order of a graph G is $|V(G)|$. The

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open neighborhood of the vertex v of G is set of all vertices of G that are adjacent to v, and is denoted by $N(v)$. The *closed neighborhood* of v is denoted by $N[v]$, and is defined as $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v of G is denoted by $d(v)$, and is defined as $d(v) = |N(v)|$. A vertex of degree one is called a *pendant* vertex. A *tree* is a connected graph that has no cycles. A *star* $K_{1,n-1}$ is a tree of order n with maximum degree $n-1$. Let $S \subseteq V(G)$. The *private neighbor set* of $v \in S$ with respect to S is $pn[v, S] = N[v] - N[S - \{v\}]$. If $pn[v, S] \neq \emptyset$ for some v, then a vertex in $pn[v, S]$ is called a *private neighbor* of v. A private neighbor $u \in V - S$ of $v \in S$ with respect to S is said to be an *external private neighbor* of v. For more graph theoretical definitions, refer to [2].

Graph coloring, domination and irredundance in graphs are well studied and applicative areas in graph theory. A (proper) coloring of a graph G is the assignment of colors to the vertices of G such that no two adjacent vertices receive the same color. The minimum number of colors required for coloring G is called as the *chromatic number* of G, and is denoted by $\chi(G)$. A coloring $\mathcal{C} = (V_1, V_2, \ldots, V_k)$ partitions $V(G)$ into independent sets $V_i (1 \leq i \leq k)$ and each V_i is said to be a *color class* with respect to the coloring $\mathcal{C}$. The color of the vertex v is denoted by $col(v)$. A coloring of G with $\chi(G)$ number of colors is said to be the χ-coloring of G. Equivalently, a χ-coloring of G is an onto function $f : V(G) \to \{1, 2, \cdots, \chi(G)\}$ such that if v and u are adjacent in G, then $f(u) \neq f(v)$. A subset $S \subseteq V(G)$ is said to be *colorful* with respect to the coloring $\mathcal{C}$ if all the vertices of S receive different colors in the coloring $\mathcal{C}$. A set $D \subseteq V(G)$ is said to be the *dominating* set of G if every vertex in $V - D$ is adjacent to some vertex in D. The minimum cardinality of a dominating set of G is called the *domination number* of G, and is denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is said to be a γ-set of G. A dominating set D of G is said to be *minimal* if $D - \{v\}$ is not a dominating set of G, for every $v \in D$. A set $D \subseteq V(G)$ is said to be an *independent dominating* set if D is independent and dominating set of G. The minimum cardinality of an independent dominating set of G is called the *independence domination number* of G, and is denoted by $i(G)$. An independent dominating set of cardinality $i(G)$ is said to be an i-set of G. A set $S \subseteq V(G)$ is said to be an *irredundant* set of G if $pn[v, S] \neq \emptyset$, for every $v \in S$ or equivalently S is said to be irredundant if for each $v \in S$, either v is isolated in $G[S]$ (subgraph induced by S) or v has a private neighbor in $V - S$. A set S is said to be a *maximal irredundant set* of G if $S \cup \{v\}$ is not an irredundant set of G, for every $v \in V(G) - S$. The minimum cardinality of a maximal irredundant set of G is called the *lower irredundance number*, and is denoted by $ir(G)$. A maximal irredundant set of cardinality $ir(G)$ is called an ir-set of G. For more details on coloring, domination and irredundance in graphs, refer to [1, 4, 5].

In the recent years, many graph theorists have worked on problems involving the parameters of coloring, domination and irredundance in graphs, such as irredundance coloring [6] and gamma coloring [3] of graphs, to mention a few. The *Irredundance coloring* of graphs was introduced by D. A. Kalarkop and P. Kaemawichanurat [6]. It is the coloring $\mathcal{C}$ of G in which there exists an maximal irredundant set R of G such

that all the vertices of R receive different colors in the coloring $\mathcal{C}$ or equivalently $\mathcal{C}$ is an irredundance coloring of G if there exists a maximal irredundant set R such that R is $\mathcal{C}$-colorful. The minimum number of colors required for an irredundance coloring of G is called the *irredundance chromatic number* of G, and is denoted by $\chi_i(G)$. A irredundance coloring of G with $\chi_i(G)$-number of colors is said to a χ_i-coloring of G. The *Gamma coloring* of graphs was introduced by R. Gnanaprakasam and I. S. Hamid [3]. It is the coloring $\mathcal{C}$ of G in which there exists a dominating set D of G such that all the vertices of D receive different colors in the coloring $\mathcal{C}$ or equivalently $\mathcal{C}$ is a gamma coloring of G if there exists a dominating set D such that D is $\mathcal{C}$-colorful. The minimum number of colors required for a gamma coloring of G is called the *gamma chromatic number* of G, and is denoted by $\chi_\gamma(G)$. A gamma coloring of G with $\chi_\gamma(G)$-number of colors is said to a χ_γ-coloring of G.
We will need the following results.

Proposition 1. *[4] Every minimal dominating set in a graph G is a maximal irredundant set of G.*

Theorem 1. *[6] For any graph G, we have*

$$\max\{\chi(G), ir(G)\} \le \chi_i(G) \le \chi(G) + ir(G) - 1.$$

The bounds are sharp.

In this paper, we will prove some conditions on the graphs attaining the maximum upper bounds $\chi(G) + ir(G) - 1$, $\chi(G) + \gamma(G) - 1$ of $\chi_i(G)$ and $\chi_\gamma(G)$ respectively. We will prove that every tree T satisfies $\chi_i(T) = ir(T)$ unless T is a star. Further, we can prove that $\gamma(T) \le \chi_\gamma(T) \le \gamma(T) + 1$. All trees that achieves the upper bound are characterized.

2. Main Results

2.1. Irredundance chromatic number of trees

In this subsection, we will determine the irredundance chromatic number of trees.

Lemma 1. *Let G be a graph with $\chi_i(G) = \chi(G) + ir(G) - 1$. Then the following properties hold.*

(1) *For every χ-coloring $\mathcal{C} = (V_1, V_2, \ldots, V_k)$ and any ir-set S of G, we have that $S = V_i$ for some $1 \le i \le k$.*

(2) *Every ir-set of G is an i-set. Thus $ir(G) = \gamma(G) = i(G)$.*

(3) *Every vertex v in an ir-set S of G has at least two external private neighbors (epn) with respect to S.*

Proof. **(1)** Suppose there exists a χ-coloring f in which there exists an ir-set S of G in which the vertices $u, v \in S$ receive different colors (i.e $f(u) \neq f(v)$). Let $S - \{u, v\} = \{v_i : 1 \leq i \leq ir(G) - 2\}$. Then the new coloring g of G such that $g(x) = f(x)$, for every $x \notin S - \{u, v\}$ and $g(v_i) = \chi(G) + i$, for $v_i \in S - \{u, v\}$. Then the coloring g is an irredundance coloring of G (since S is g-colorful) with $\chi(G) + ir(G) - 2$ number of colors, contradicting $\chi_i(G) = \chi(G) + ir(G) - 1$. Thus $S \subseteq V_i$ for some $1 \leq i \leq k$. Since an ir-set cannot be a proper subset of an independent set implies that $S = V_i$.

(2) By Property (1), in any χ-coloring $\mathcal{C} = (V_1, V_2, \ldots, V_k)$ of G and any ir-set S of G, we have $S = V_i$ for some $1 \leq i \leq k$. Therefore S is independent. Since S is maximal irredundant set of G, there is no vertex in $V(G) - S$ that is not adjacent to any vertex of S, otherwise $S \cup \{v\}$ would be an irredundant set contradicting the maximality of S. Thus S is independent dominating set of G. By the minimality of $i(G)$, we have that $i(G) \leq ir(G)$. Because $ir(G) \leq \gamma(G) \leq i(G)$, it follows that $ir(G) = \gamma(G) = i(G)$.

(3) Let S be an ir-set of G. By Property (2), S is a γ-set. Suppose that there exists a vertex $v \in S$ such that v has no epn with respect to S. Thus v is not adjacent to any vertex in S. Since G is without isolotes, it follows that $N(v) \neq \emptyset$. Let $u \in N(v)$. Hence $S_1 = (S - \{v\}) \cup \{u\}$ is a γ-set of G. By $ir(G) = \gamma(G)$ and Proposition 1, S_1 is an ir-set of G. But S_1 is not independent contradicting Property (2). Thus, for every $v \in S$, v has an *epn* with respect to S. Let $S = \{v_i : 1 \leq i \leq ir(G)\}$ be an ir-set of G. We suppose to the contrary that there exists $v_i \in S$ such that v_i has exactly u_i as the *epn* of v_i. Thus, $S_i = (S - \{v_i\}) \cup \{u_i\}$ is an ir-set of G. Clearly, all $v_1, ...v_{ir(G)}$ have the same color by Property (1). Since v_i is adjacent to u_i, it follows that $col(v_i) \neq col(u_i)$. Thus, all the vertices in S_i do not obtain the same color, contradicting Property (1). Thus every vertex v_i in S has at least two epn with respect to the set S. $\square$

Theorem 2. *Let T be a tree other than star. Then $\chi_i(T) = ir(T)$.*

Proof. From Theorem 1, it is clear that $ir(T) \leq \chi_i(T) \leq ir(T) + 1$ since $\chi(T) = 2$. Thus $\chi_i(T) \geq ir(T)$. Now we shall prove that $\chi_i(T) \leq ir(T)$. We shall assume that $\chi_i(T) > ir(T)$ and arrive at contradiction. Since $\chi_i(T) \leq ir(T) + 1$, it follows that $\chi_i(T) = ir(T) + 1$. Thus Properties (1), (2) and (3) of Lemma 1 are satisfied. Let $T(V_1, V_2)$ be the tree with partite sets V_1 and V_2 such that $|V_1| \leq |V_2|$. Now $\mathcal{C} = (V_1, V_2)$ is the unique χ-coloring of T. Property (1) implies that V_1 or V_2 is an ir-set of T. Since both V_1 and V_2 are maximal irredundant sets of T imply that V_1 is an ir-set of T. Because $\chi_i(T) = ir(T) + 1 = |V_1| + 1$, there exists a χ_i-coloring $\mathcal{C}'$ such that all the vertices of V_1 receive different colors while all the vertices in V_2 receive the same color.

Suppose that, for any vertex $u \in V_2$, there exists a vertex $v \in V_1$ which is not adjacent to u. Hence, the vertex u can be re-colored with the color of vertex v and we get a χ_i-coloring of T with $ir(T)$ number of colors, a contradiction. Thus, there exist a vertex $w \in V_2$ such that $N(w) = V_1$ so that w receives a new color which is not assigned to any vertex in V_1 in the coloring $\mathcal{C}'$. Further, there does not exist a vertex $u \in V_2 - \{w\}$ such that u is adjacent to at least two vertices v_1 and v_2 in V_1, as otherwise there would be a cycle wv_1uv_2w contradicting T is a tree. Thus every vertex in $V_2 - \{w\}$ is a private neighbor of some vertex $v \in V_1$ with respect to V_1. In other words, every vertex $u \in V_2 - \{w\}$ is a pendant vertex. We need the following claim.

Claim 1. *The set $S = (V_1 - \{v\}) \cup \{w\}$ is an ir-set of T for any $v \in V_1$.*

Proof of the claim. By Property (3), every vertex in $V_1 - \{v\}$ has an epn in $V_2 - \{w\}$ and v is a private neighbor of w with respect to the set S. Thus S is an irredundant set of T. Now $S \cup \{v\}$ is not an irredundant set since w will not have a private neighbor. Similarly, for every $u \in V_2 - \{w\}$, the set $S \cup \{u\}$ is not an irredundant set of T since u is a pendant vertex. Thus S is maximal irredudant set and $|S| = |V_1| = ir(T)$. Thus S is an ir-set of T. This proves Claim 1.

By Claim 1, we have that S is an ir-set of T which is not independent contradicting Property (2). Thus our assumption is not correct and thus $\chi_i(T) = ir(T)$. If T is a star, then $\chi_i(T) = \chi(T) = ir(T) + 1 = 2$.

□

2.2. Gamma chromatic number of trees

In this subsection, we will determine the gamma chromatic number of trees. By similar arguments as in the proof of Theorem 1 on γ-set of a graph G, we have the observation below.

Observation 3. For any graph G, we have

$$\max\{\chi(G), \gamma(G)\} \leq \chi_\gamma(G) \leq \chi(G) + \gamma(G) - 1.$$

Further, by the similar arguments as in the proof of Lemma 1 on γ-sets of a graph G, we have the lemma below.

Lemma 2. *Let G be a graph with $\chi_\gamma(G) = \chi(G) + \gamma(G) - 1$. Then the following properties hold.*

(1) *For every χ-coloring $\mathcal{C} = (V_1, V_2, \ldots, V_k)$ and any γ-set S of G, we have that $S = V_i$ for some $1 \leq i \leq k$.*

(2) *Every γ-set of G is an i-set. Thus $\gamma(G) = i(G)$.*

(3) *Every $v \in S$ (S is a γ-set of G) has at least two external private neighbors (epn) with respect to S.*

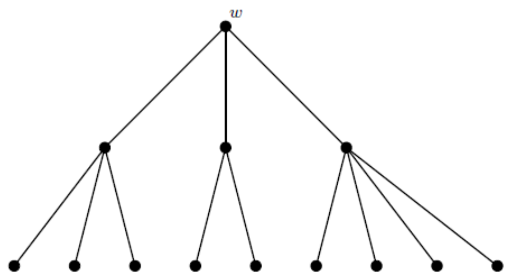

Figure 1. A tree $T \in \mathcal{F}$ when a pendant vertex of each $K_{1,4}, K_{1,3}$ and $K_{1,5}$ are identified at the vertex w.

Thus, we have the observation below.

Observation 4. Let T be a tree. Then from Observation 3 and the fact that $\chi(T) = 2$, we get

$$\gamma(T) \leq \chi_\gamma(T) \leq \gamma(T) + 1.$$

Now we will construct the graphs in family $\mathcal{F}$ of trees which will be used in our next theorem. The tree $T \in \mathcal{F}$ is obtained from stars $K_{1,n_1}, K_{1,n_2}, ..., K_{1,n_\ell}$ where $n_1, n_2, ..., n_\ell \geq 3$ by identifying a pendant vertex of all of these stars. We call this identified vertex w. An example of a tree $T \in \mathcal{F}$ is shown in Figure 1.

Theorem 5. *Let T be a tree. Then $\chi_\gamma(T) = \gamma(T) + 1$ if and only if $T \in \mathcal{F} \cup \{K_{1,n-1}\}$.*

Proof. Let $\chi_\gamma(T) = \gamma(T) + 1$. Then Properties (1), (2) and (3) of Lemma 2 are satisfied. If $T \cong K_{1,n-1}$, then $\chi_\gamma(T) = \gamma(T) + 1 = 2$. So let $T \ncong K_{1,n-1}$. Let $T(V_1, V_2)$ be the tree with partite sets V_1 and V_2 such that $|V_1| \leq |V_2|$. Now $\mathcal{C} = (V_1, V_2)$ is the unique χ-coloring of T and by Property (1), it is clear that V_1 or V_2 is a γ-set of T. Since both V_1 and V_2 are dominating sets of T imply that V_1 is a γ-set of T. By the definition of χ_γ-coloring, if D is a dominating set of T which is colorful in the coloring $\mathcal{C}'$, then $|\mathcal{C}'| = \gamma(T) + 1$; otherwise we get a χ_γ-coloring of T with $\gamma(T)$ number of colors, a contradiction. Let $D = V_1$ and all the vertices of V_1 receive different colors in the coloring $\mathcal{C}'$. Suppose every vertex $u \in V_2$ is non-adjacent to some vertex in $v \in V_1$, then the vertex u can be colored with the color of vertex v and we get a χ_γ-coloring of T with $\gamma(T)$ number of colors, a contradiction. Thus there exist a vertex $w \in V_2$ such that $N(w) = V_1$ so that w receives a new color (color not used in V_1) in the coloring $\mathcal{C}'$. Now there cannot be a vertex $u \in V_2$ (other than w) such that u is adjacent to at least two vertices v_1 and v_2 in V_1; otherwise there would be a cycle wv_1uv_2w contradicting that T is a tree. Thus every vertex in $V_2 - \{w\}$ is a private neighbor (with respect to V_1) of some vertex $v \in V_1$. In other words, every vertex $u \in V_2 - \{w\}$ is a pendent vertex and by Property (3), every vertex in V_1 has at least two external private neighbors in V_2 with respect to the set V_1. Thus $T \in \mathcal{F}$.

Conversely, let $T \in \mathcal{F}$. Any dominating set other than V_1 is of cardinality at least $\gamma(T)+1$. Thus any gamma coloring of T in which $D \neq V_1$ is colorful, requires at least $\gamma(T)+1$ number of colors. Also any χ_γ-coloring of T in which V_1 is colorful, requires at least $\gamma(T)+1$ number of colors since the root vertex w of T has to be given a new color. Thus $\chi_\gamma(T) \geq \gamma(T)+1$ and from Observation 4, $\chi_\gamma(T) \leq \gamma(T)+1$. $\square$

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References

[1] L.W. Beineke and R.J. Wilson, *Topics in Chromatic Graph Theory*, Cambridge University Press, 2015.

[2] G. Chartrand and L. Lesniak, *Graphs and Digraphs, sixth edition*, Chapman & Hall London, 2016.

[3] R. Gnanaprakasam and I.S. Hamid, *Gamma coloring of mycielskian graphs*, Indian J. Sci. Technol. **15** (2022), no. 20, 976–982 https://doi.org/10.17485/IJST/v15i20.2324.

[4] T.W. Haynes, S. Hedetniemi, and P. Slater, *Fundamentals of Domination in Graphs*, CRC press, 2013.

[5] T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, *Domination in Graphs: Volume 2: Advanced Topics (1st ed.)*, Chapman & Hall/CRC Pure and Applied Mathematics, 1998.

[6] D.A. Kalarkop and P. Kaemawichanurat, *On irredundance coloring and irredundance compelling coloring of graphs*, arXiv preprint arXiv:2311.06161 (2023).