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Research Article

Weak signed Roman k-domatic number of a digraph

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Abstract: Let D be a digraph with vertex set $V(D)$, and let $k \geq 1$ be an integer. A weak signed Roman k-dominating function on a digraph D is a function $f : V(D) \longrightarrow \{-1, 1, 2\}$ such that $\sum_{u \in N^-[v]} f(u) \geq k$ for every $v \in V(D)$, where $N^-[v]$ consists of v and all vertices of D from which arcs go into v. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct weak signed Roman k-dominating functions on D with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(D)$, is called a weak signed Roman k-dominating family (of functions) on D. The maximum number of functions in a weak signed Roman k-dominating family on D is the weak signed Roman k-domatic number of D, denoted by $d^k_{wsR}(D)$. In this paper we initiate the study of the weak signed Roman k-domatic number in digraphs, and we present sharp bounds for $d^k_{wsR}(D)$. In addition, we determine the weak signed Roman k-domatic number of some digraphs.

Keywords: digraphs, weak signed Roman k-dominating function, weak signed Roman k-domination number, weak signed Roman k-domatic number.

AMS Subject classification: 05C69

1. Introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [7]. Specifically, let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A graph G is *regular* or *r-regular* if $d(v) = r$ for each vertex v of G. The complement of a graph G is denoted by $\overline{G}$. We write K_n for the *complete graph* of order n, $K_{p,q}$ for the *complete bipartite graph* with partite sets X and Y, where $|X| = p$ and $|Y| = q$, and C_n for the *cycle* of length n.

Let now D be a finite and simple digraph with vertex set $V(D)$ and arc set $A(D)$. The integers $n = n(D) = |V(D)|$ and $m = m(D) = |A(D)|$ are the *order* and the *size* of the digraph D, respectively. The sets $N_D^+(v) = N^+(v) = \{x|(v,x) \in A(D)\}$ and $N_D^-(v) = N^-(v) = \{x|(x,v) \in A(D)\}$ are called *out-neighborhood* and *in-neighborhood* of the vertex v. Likewise, $N_D^+[v] = N^+[v] = N^+(v) \cup \{v\}$ and $N_D^-[v] = N^-[v] = N^-(v) \cup \{v\}$. We write $d_D^+(v) = d^+(v) = |N^+(v)|$ for the *out-degree* of a vertex v and $d_D^-(v) = d^-(v) = |N^-(v)|$ for its *in-degree*. The *minimum* and *maximum in-degree* are $\delta^- = \delta^-(D)$ and $\Delta^- = \Delta^-(D)$ and the *minimum* and *maximum out-degree* are $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$. A digraph D is *regular* or *δ-regular*, if $\delta^-(D) = \Delta^-(D) = \delta^+(D) = \Delta^+(D) = \delta$. A digraph D is *in-regular* or *δ-in-regular*, if $\delta^-(D) = \Delta^-(D) = \delta$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. For an arc $(x,y) \in A(D)$, the vertex y is an *out-neighbor* of x and x is an *in-neighbor* of y, and we also say that x *dominates* y or y is *dominated* by x. An *oriented cycle* is an orientation of a cycle. A digraph with no arcs is the *empty digraph*. The *complement* $\overline{D}$ of a digraph D is the digraph with vertex set $V(D)$ such that for any two distinct vertices u,v the arc (u,v) belongs to $\overline{D}$ if and only if (u,v) does not belong to D. A digraph D is called a *tournament* when either $(u,v) \in A(D)$ or $(v,u) \in A(D)$, but not both, for each pair of distinct vertices $u,v \in V(D)$.

In this paper we continue the study of Roman dominating functions and Roman domatic numbers in graphs and digraphs (see, for example, the survey papers [2–5]). If $k \geq 1$ is an integer, then the *signed Roman k-dominating function* (SRkDF) on a graph G is defined in [8] as a function $f : V(G) \longrightarrow \{-1,1,2\}$ such that $\sum_{u \in N[v]} f(u) \geq k$ for each $v \in V(G)$, and such that every vertex $u \in V(G)$ for which $f(u) = -1$ is adjacent to at least one vertex w for which $f(w) = 2$. The *weight* of an SRkDF f is the value $\omega(f) = \sum_{v \in V} f(v)$. The *signed Roman k-domination number* of a graph G, denoted by $\gamma_{sR}^k(G)$, equals the minimum weight of an SRkDF on G. A *$\gamma_{sR}^k(G)$-function* is a signed Roman k-dominating function of G with weight $\gamma_{sR}^k(G)$. If $k = 1$, then we write $\gamma_{sR}^1(G) = \gamma_{sR}(G)$. This case was introduced and studied in [1].

A *weak signed Roman k-dominating function* (WSRkDF) on a graph G is defined in [18] as a function $f : V(G) \longrightarrow \{-1,1,2\}$ such that $\sum_{u \in N[v]} f(u) \geq k$ for each $v \in V(G)$. The *weight* of a WSRkDF f is the value $\omega(f) = \sum_{v \in V} f(v)$. The *weak signed Roman k-domination number* of a graph G, denoted by $\gamma_{wsR}^k(G)$, equals the minimum weight of a WSRkDF on G. A *$\gamma_{wsR}^k(G)$-function* is a weak signed Roman k-dominating function of G with weight $\gamma_{wsR}^k(G)$. The special case $k = 1$ was introduced and investigated by Volkmann [16].

If $k \geq 1$ is an integer, then the *signed Roman k-dominating function* (SRkDF) on a digraph D is defined in [15] as a function $f : V(D) \longrightarrow \{-1,1,2\}$ such that $\sum_{u \in N^-[v]} f(u) \geq k$ for each $v \in V(D)$, and such that every vertex $u \in V(D)$ for which $f(u) = -1$ has an in-neighbor w for which $f(w) = 2$. The *weight* of an SRkDF f is the value $\omega(f) = \sum_{v \in V(D)} f(v)$. The *signed Roman k-domination number* of a digraph D, denoted by $\gamma_{sR}^k(D)$, equals the minimum weight of an SRkDF on D. A *$\gamma_{sR}^k(D)$-function* is a signed Roman k-dominating function of D with weight $\gamma_{sR}^k(D)$. If $k = 1$, then we write $\gamma_{sR}^1(D) = \gamma_{sR}(D)$. This case was introduced and studied in [11].

A *weak signed Roman k-dominating function* (WSRkDF) on a digraph D is defined in [20] as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $\sum_{u \in N^-[v]} f(u) \geq k$ for each $v \in V(D)$. The *weight* of a WSRkDF f is the value $\omega(f) = \sum_{v \in V(D)} f(v)$. The *weak signed Roman k-domination number* of a digraph D, denoted by $\gamma^k_{wsR}(D)$, equals the minimum weight of a WSRkDF on D. A $\gamma^k_{wsR}(D)$*-function* is a weak signed Roman k-dominating function of D with weight $\gamma^k_{wsR}(D)$. The special case $k = 1$ was introduced and investigated by Volkmann [17].

The weak signed Roman k-domination number of a graph (digraph) exists when $\delta \geq \frac{k}{2} - 1$ ($\delta^- \geq \frac{k}{2} - 1$). Therefore we assume in this paper that $\delta \geq \frac{k}{2} - 1$ and $\delta^- \geq \frac{k}{2} - 1$. The definitions lead to $\gamma^k_{wsR}(G) \leq \gamma^k_{sR}(G)$ and $\gamma^k_{wsR}(D) \leq \gamma^k_{sR}(D)$.

A concept dual in a certain sense to the domination number is the domatic number, introduced by Cockayne and Hedetniemi [6]. They have defined the domatic number $d(G)$ of a graph G by means of sets. A partition of $V(G)$, all of whose classes are dominating sets in G, is called a *domatic partition*. The maximum number of classes of a domatic partition of G is the *domatic number* $d(G)$ of G. But Rall has defined a variant of the domatic number of G, namely the *fractional domatic number* of G, using functions on $V(G)$. (This was mentioned by Slater and Trees in [12].) Analogous to the fractional domatic number we may define the (weak) signed Roman k-domatic number.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct (weak) signed Roman k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(G)$, is called in [10, 13, 19] a *(weak) signed Roman k-dominating family* (of functions) on G. The maximum number of functions in a (weak) signed Roman k-dominating family ((W)SRkD family) on G is the *(weak) signed Roman k-domatic number* of G, denoted by ($d^k_{wsR}(G)$) $d^k_{sR}(G)$. The (weak) signed Roman k-domatic number is well-defined and $d^k_{wsR}(G) \geq d^k_{sR}(G) \geq 1$ for all graphs G with $\delta(G) \geq \frac{k}{2} - 1$, since the set consisting of any (W)SRkDF forms a (W)SRkD family on G. For more information on the Roman domatic problem, we refer the reader to the survey article [5].

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed Roman k-dominating functions on a digraph D with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(D)$, is called in [14] a *signed Roman k-dominating family* (of functions) on D. The maximum number of functions in a signed Roman k-dominating family on D is the *signed Roman k-domatic number* of D, denoted by $d^k_{sR}(D)$. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct weak signed Roman k-dominating functions on a digraph D with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(D)$, is called a *weak signed Roman k-dominating family* (of functions) on D. The maximum number of functions in a weak signed Roman k-dominating family on D is the *weak signed Roman k-domatic number* of D, denoted by $d^k_{wsR}(D)$.

The (weak) signed Roman k-domatic number is well-defined and $d^k_{wsR}(D) \geq d^k_{sR}(D) \geq 1$ for all digraphs D with $\delta^-(D) \geq \frac{k}{2} - 1$, since the set consisting of any (W)SRkDF forms a (W)SRkD family on D.

Our purpose in this paper is to initiate the study of the weak signed Roman k-domatic number in digraphs. We first derive basic properties and bounds for the weak signed

Roman k-domatic number of a digraph. In addition, we present upper bounds on the sums $\gamma^k_{wsR}(D) + d^k_{wsR}(D)$ and $d^k_{wsR}(D) + d^k_{wsR}(\overline{D})$. Furthermore, we determine the weak signed Roman k-domatic number of some classes of digraphs.

The *associated digraph* G^* of a graph G is the digraph obtained from G when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{G^*}[v] = N_G[v]$ for each vertex $v \in V(G) = V(G^*)$, the following useful observation is valid,

Observation 1. *If G^* is the associated digraph of the graph G, then $\gamma^k_{wsR}(G^*) = \gamma^k_{wsR}(G)$ and $d^k_{wsR}(G^*) = d^k_{wsR}(G)$.*

We make use of the following known results in this paper.

Theorem A. ([20]) *If $k \geq 1$ and $n \geq \frac{k}{2}$ are integers, then $\gamma^k_{wsR}(K^*_n) = k$.*

Theorem B. ([19]) *If $n \geq k \geq 1$ are integers, then $d^k_{wsR}(K_n) = n$, unless $n = k = 2$, in which case $d^2_{wsR}(K_2) = 1$.*

Theorem C. ([19]) *If $k, n \geq 1$ are integers such that $n+1 \leq k \leq 2n-1$, then $d^k_{wsR}(K_n) = n$.*

Using Observation 1 and Theorems B, C we obtain the next results immediately.

Corollary 1. If $n \geq k \geq 1$ are integers, then $d^k_{wsR}(K^*_n) = n$, unless $n = k = 2$, in which case $d^2_{wsR}(K^*_2) = 1$.

Corollary 2. If $k, n \geq 1$ are integers such that $n + 1 \leq k \leq 2n - 1$, then $d^k_{wsR}(K^*_n) = n$.

Theorem D. ([18, 19]) *If C_{3t} is a cycle of length $3t$ with an integer $t \geq 1$, then $\gamma^4_{wsR}(C_{3t}) = 4t$ and $d^4_{wsR}(C_{3t}) = 3$.*

Theorem E. ([19]) *If C_n is a cycle of length $n \geq 3$, then $\gamma^5_{wsR}(C_n) = \gamma^5_{sR}(C_n) = \lceil \frac{5n}{3} \rceil$.*

Using Observation 1 and Theorems D and E, we obtain the next corollaries.

Corollary 3. If C^*_{3t} is the associated digraph of the cycle C_{3t}, then $\gamma^4_{wsR}(C^*_{3t}) = 4t$ and $d^4_{wsR}(C^*_{3t}) = 3$.

Corollary 4. If C^*_n is the associated digraph of the cycle C_n, then $\gamma^5_{wsR}(C^*_n) = \gamma^5_{sR}(C_n) = \lceil \frac{5n}{3} \rceil$.

Theorem F. ([20]) *If D is a δ-regular digraph of order n with $\delta \geq \frac{k}{2} - 1$, then*

$$\gamma^k_{sR}(D) \geq \gamma^k_{wsR}(D) \geq \frac{kn}{\delta + 1}.$$

Theorem G. ([20]) *If D is a digraph of order n with $\delta^-(D) \geq k - 1$, then $\gamma^k_{wsR}(D) \leq \gamma^k_{sR}(D) \leq n$.*

Theorem H. ([20]) *Let D be a digraph of order n with $\delta^-(D) \geq \lceil \frac{k}{2} \rceil - 1$. Then $\gamma^k_{wsR}(D) \leq 2n$, with equality if and only if k is even, $\delta^-(D) = \frac{k}{2} - 1$, and each vertex of D is of minimum in-degree or has an out-neighbor of minimum in-degree.*

2. Bounds on the weak signed Roman k-domatic number

In this section we present basic properties of $d^k_{wsR}(D)$ and sharp bounds on the weak signed Roman k-domatic number of a digraph.

Theorem 2. If D is a digraph with $\delta^-(D) \geq \frac{k}{2} - 1$, then $d^k_{wsR}(D) \leq \delta^-(D) + 1$. Moreover, if $d^k_{wsR}(D) = \delta^-(D) + 1$, then for each WSRkD family $\{f_1, f_2, \ldots, f_d\}$ on D with $d = d^k_{wsR}(D)$ and each vertex v of minimum in-degree, $\sum_{x \in N^-[v]} f_i(x) = k$ for each function f_i and $\sum_{i=1}^{d} f_i(x) = k$ for all $x \in N^-[v]$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a WSRkD family on D such that $d = d^k_{wsR}(D)$. If v is a vertex of minimum in-degree $\delta^-(D)$, then we deduce that

$$\begin{aligned} kd &\leq \sum_{i=1}^{d} \sum_{x \in N^-[v]} f_i(x) = \sum_{x \in N^-[v]} \sum_{i=1}^{d} f_i(x) \\ &\leq \sum_{x \in N^-[v]} k = k(\delta^-(D) + 1) \end{aligned}$$

and thus $d^k_{wsR}(D) \leq \delta^-(D) + 1$.
If $d^k_{wsR}(D) = \delta^-(D) + 1$, then the two inequalities occurring in the proof become equalities. Hence for the WSRkD family $\{f_1, f_2, \ldots, f_d\}$ on D and for each vertex v of minimum in-degree, $\sum_{x \in N^-[v]} f_i(x) = k$ for each function f_i and $\sum_{i=1}^{d} f_i(x) = k$ for all $x \in N^-[v]$. □

Theorem 3. If D is a digraph of order n with $\delta^-(D) \geq \frac{k}{2} - 1$, then

$$\gamma^k_{wsR}(D) \cdot d^k_{wsR}(D) \leq kn.$$

Moreover, if $\gamma^k_{wsR}(D) \cdot d^k_{wsR}(D) = kn$, then for each WSRkD family $\{f_1, f_2, \ldots, f_d\}$ on D with $d = d^k_{wsR}(D)$, each function f_i is a $\gamma^k_{wsR}(D)$-function and $\sum_{i=1}^{d} f_i(v) = k$ for all $v \in V(D)$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a WSRkD family on D such that $d = d^k_{wsR}(D)$ and let $v \in V(D)$. Then

$$\begin{aligned} d \cdot \gamma^k_{wsR}(D) &= \sum_{i=1}^{d} \gamma^k_{wsR}(D) \le \sum_{i=1}^{d} \sum_{v \in V(D)} f_i(v) \\ &= \sum_{v \in V(D)} \sum_{i=1}^{d} f_i(v) \le \sum_{v \in V(D)} k = kn. \end{aligned}$$

If $\gamma^k_{wsR}(D) \cdot d^k_{wsR}(D) = kn$, then the two inequalities occurring in the proof become equalities. Hence for the WSRkD family $\{f_1, f_2, \ldots, f_d\}$ on D and for each i, $\sum_{v \in V(D)} f_i(v) = \gamma^k_{wsR}(D)$. Thus each function f_i is a $\gamma^k_{wsR}(D)$-function, and $\sum_{i=1}^{d} f_i(v) = k$ for all $v \in V(D)$. □

Theorem A and Corollaries 1, 2 demonstrate that Theorems 2 and 3 are both sharp. For some regular digraphs we will improve the upper bound given in Theorem 2.

Theorem 4. Let D be a δ-regular digraph of order n with $\delta \ge \frac{k}{2} - 1$ such that $n = p(\delta+1) + r$ with integers $p \ge 1$ and $1 \le r \le \delta$ and $kr = t(\delta+1) + s$ with integers $t \ge 0$ and $1 \le s \le \delta$. Then $d^k_{wsR}(D) \le \delta$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a WSRkD family on D such that $d = d^k_{wsR}(D)$. It follows that

$$\sum_{i=1}^{d} \omega(f_i) = \sum_{i=1}^{d} \sum_{v \in V(D)} f_i(v) = \sum_{v \in V(D)} \sum_{i=1}^{d} f_i(v) \le \sum_{v \in V(D)} k = kn.$$

Theorem F implies

$$\begin{aligned} \omega(f_i) &\ge \gamma^k_{wsR}(D) \ge \left\lceil \frac{kn}{\delta+1} \right\rceil = \left\lceil \frac{kp(\delta+1) + kr}{\delta+1} \right\rceil \\ &= kp + \left\lceil \frac{kr}{\delta+1} \right\rceil = kp + \left\lceil \frac{t(\delta+1)+s}{\delta+1} \right\rceil = kp + t + 1. \end{aligned}$$

for each $i \in \{1, 2, \ldots, d\}$. If we suppose to the contrary that $d = \delta + 1$, then the above inequality chains lead to the contradiction

$$\begin{aligned} kn &\ge \sum_{i=1}^{d} \omega(f_i) \ge d(kp+t+1) = (\delta+1)(kp+t+1) \\ &= kp(\delta+1) + (\delta+1)(t+1) = kp(\delta+1) + t(\delta+1) + \delta + 1 \\ &= kp(\delta+1) + kr - s + \delta + 1 > kp(\delta+1) + kr = k(p(\delta+1)+r) = kn. \end{aligned}$$

Thus $d \le \delta$, and the proof is complete. □

Corollaries 1, 2 and 3 demonstrate that Theorem 4 is not valid in general.

Corollary 5. Let T be a δ-regular tournament with $\delta \geq \frac{k}{2} - 1$. If $k\delta = t(\delta + 1) + s$ with integers $t \geq 0$ and $1 \leq s \leq \delta$, then $d^k_{wsR}(T) \leq \delta$.

Proof. Since T is a δ-regular tournament, we observe that the order $n = 2\delta + 1 = (\delta + 1) + \delta$. Using Theorem 4 with $r = \delta$, we obtain $d^k_{wsR}(T) \leq \delta$. □

Theorem 5. Let D be a digraph of order $n \geq 2$ with $\delta^-(D) \geq \lceil \frac{k}{2} \rceil - 1$. Then $d^k_{wsR}(D) = n$ if and only if $G = K^*_n$, with exception of the cases $k = 2n$ or $k = n = 2$, in which cases $d^{2n}_{wsR}(K^*_n) = 1$ or $d^2_{wsR}(K^*_2) = 1$.

Proof. Let $D = K^*_n$. If $k = 2n$, then the function f with $f(x) = 2$ for each vertex $x \in V(D)$ is the unique weak signed Roman dominating function on D and so $d^{2n}_{wsR}(K^*_n) = 1$. In addition, it follows from Corollaries 1 and 2 that $d^2_{wsR}(K^*_2) = 1$ and $d^k_{wsR}(K^*_n) = n$ in the remaining cases.
Conversely, assume that $d^k_{wsR}(D) = n$. Then we deduce from Theorem 2 that $n = d^k_{wsR}(D) \leq \delta^-(D) + 1$, and so $\delta^-(D) \geq n - 1$. Thus $D = K^*_n$, and the proof is complete. □

Theorem 6. Let $k \geq 4$ be an integer, and let D be a digraph of order n with $\delta^-(D) \geq \lceil \frac{k}{2} \rceil - 1$. If $\gamma^k_{wsR}(D) \leq 2n - 1$, then $d^k_{wsR}(D) \geq 2$.

Proof. Since $\gamma^k_{wsR}(D) \leq 2n - 1$, there exists a WSRkDF f_1 with $f_1(v) \leq 1$ for at least one vertex $v \in V(D)$. Note that $f_2 : V(D) \longrightarrow \{-1, 1, 2\}$ with $f_2(x) = 2$ for each vertex $x \in V(D)$ is another WSRkDF on D. As $f_1(x) + f_2(x) \leq 4 \leq k$ for each vertex $x \in V(D)$, $\{f_1, f_2\}$ is a weak signed Roman k-dominating family on D and thus $d^k_{wsR}(D) \geq 2$. □

If D is a digraph with $\delta^-(D) = 0$, then Theorem 2 implies $d_{wsR}(D) = d^2_{wsR}(D) = 1$. Therefore Theorem 6 is not valid for $k = 1$ or $k = 2$ in general. The next example will show that Theorem 6 is also not valid for $k = 3$.

Example 1. *Let C^o_{2q+1} be an oriented cycle of odd length $2q + 1$ with an integer $q \geq 1$. Since C^o_{2q+1} is 1-regular, Theorem 4 shows with $n = 2q + 1$, $k = 3$ and $\delta = 1$ that $d^3_{wsR}(C^o_{2q+1}) = 1$.*

For $k = 2$ we will present a further example.

Example 2. *Let $Q = H \circ K_1$ be the digraph constructed from a digraph H, where for each vertex $v \in V(H)$, a new vertex v' and the arc (v, v') are added. If f is a WSR2DF on Q, then it is easy to see that $f(x) \geq 1$ for each vertex $x \in V(Q)$. Suppose that $d^2_{wsR}(Q) = 2$, and let $\{f_1, f_2\}$ be a weak signed Roman 2-dominating family on Q. Since f_1 and f_2 are* *distinct, we observe that* $f_1(w) = 2$ *or* $f_2(w) = 2$ *for at least one vertex* $w \in V(Q)$*. Hence* $f_1(w) + f_2(w) \geq 3$*, a contradiction to* $f_1(w) + f_2(w) \leq 2$*. This implies* $d^2_{wsR}(Q) = 1$*.*

Theorem 7. Let D be a digraph of order n with $\delta^-(D) \geq 1$. If the set $V_1 = \{x \,|\, d^-_D(x) = 1\}$ is independent or empty, then $d^3_{wsR}(D) \geq 2$.

Proof. Define the functions f_1 and f_2 by $f_1(x) = 1$ if $x \in V_1$ and $f_1(x) = 2$ if $x \in V(D) \setminus V_1$ and $f_2(x) = 2$ if $x \in V_1$ and $f_2(x) = 1$ if $x \in V(D) \setminus V_1$. Since V_1 is independent, we observe that $\sum_{x \in N^-[u]} f(x) = 3$ for $u \in V_1$ and $\sum_{x \in N^-[u]} f(x) \geq 3$ for $u \in V(D) \setminus V_1$. Therefore f_1 and f_2 are weak signed Roman 3-dominating functions of D such thst $f_1(u) + f_2(u) = 3$ for each vertex $u \in V(D)$. Consequently, $\{f_1, f_2\}$ is a weak signed Roman 3-dominating family on D and thus $d^3_{wsR}(D) \geq 2$. □

Corollary 6. Let D be a digraph of order n with $\delta^-(D) \geq 1$. If $2n - 1 \geq \gamma^4_{wsR}(D) > \frac{4n}{3}$, then $d^4_{wsR}(D) = 2$.

Proof. Theorem 6 implies $d^4_{wsR}(D) \geq 2$.
Conversely, it follows from Theorem 3 that

$$d^4_{wsR}(D) \leq \frac{4n}{\gamma^4_{wsR}(D)} < \frac{4n}{\frac{4n}{3}} = 3.$$

Thus $d^4_{wsR}(D) \leq 2$, and the proof is complete. □

Corollary 3 shows that the condition $\gamma^4_{wsR}(D) > \frac{4n}{3}$ in Corollary 6 is best possible in some sense.

Example 3. *Let* C^*_n *be the associated digraph of the cycle* C_n*. Then* $d^5_{wsR}(C^*_n) = 2$ *if* $n \not\equiv 0 \pmod 3$ *and* $d^5_{wsR}(C^*_n) = 3$ *if* $n \equiv 0 \pmod 3$*.*

Proof. Let first $n = 3t + \epsilon$ with integers $t \geq 1$ and $1 \leq \epsilon \leq 2$. It follows from Theorem 3 and Corollary 4 that

$$d^5_{wsR}(C^*_n) \leq \frac{5n}{\gamma^5_{wsR}(C^*_n)} = \frac{5n}{\lceil \frac{5n}{3} \rceil} = \frac{5(3t+\epsilon)}{\left\lceil \frac{5(3t+\epsilon)}{3} \right\rceil} < 3.$$

Therefore $d^5_{wsR}(C^*_n) \leq 2$ and so Theorem 6 leads to $d^5_{wsR}(C^*_n) = 2$ in these cases.
Let now $n = 3t$ with an integer $t \geq 1$ and $C^*_{3t} = v_0 v_1 \ldots v_{3t-1} v_0$. Define the functions f_1, f_2 and f_3 by

$$f_1(v_{3i}) = 1,\ f_1(v_{3i+1}) = 2,\ f_1(v_{3i+2}) = 2,$$

$$f_2(v_{3i}) = 2,\ f_2(v_{3i+1}) = 1,\ f_2(v_{3i+2}) = 2,$$

$$f_3(v_{3i}) = 2,\ f_3(v_{3i+1}) = 2,\ f_3(v_{3i+2}) = 1$$

for $0 \le i \le t-1$. It is easy to see that f_i is a weak signed Roman 5-dominating function on C_{3t}^* of weight $5t$ for $1 \le i \le 3$, and $\{f_1, f_2, f_3\}$ is a weak signed Roman 5-dominating family of on C_{3t}^*. Therefore $d_{wsR}^5(C_{3t}^*) \ge 3$ and thus Theorem 2 implies $d_{wsR}^5(C_{3t}^*) = 3$. □

Corollary 7. Let D be a digraph of order n with $\delta^-(D) \ge 2$. If $2n-1 \ge \gamma_{wsR}^5(D) > \frac{5n}{3}$, then $d_{wsR}^5(D) = 2$.

Proof. Theorem 6 implies $d_{wsR}^5(D) \ge 2$.
Conversely, it follows from Theorem 3 that $d_{wsR}^5(D) \le \frac{5n}{\gamma_{wsR}^5(D)} < \frac{5n}{\frac{5n}{3}} = 3$. Thus $d_{wsR}^5(D) \le 2$, and the proof is complete. □

Example 3 demonstrates that the condition $\gamma_{wsR}^5(D) > \frac{5n}{3}$ in Corollary 7 is best possible in some sense.

Theorem 8. Let $k \ge 6$ be an integer, and let D be a digraph of order n with $\delta^-(D) \ge \lceil \frac{k}{2} \rceil - 1$. If $\gamma_{wsR}^k(D) \le 2n-2$, then $d_{wsR}^k(D) \ge 3$.

Proof. Since $\gamma_{wsR}^k(D) \le 2n-2$, there exists a WSRkDF f_1 with $f_1(u) = -1$ for at least one vertex $u \in V(D)$ or $f_1(v) = 1$ and $f_1(w) = 1$ for two different vertices $v, w \in V(D)$. If $f_1(u) = -1$, then $f_2(u) = 1$ and $f_2(x) = 2$ for $x \in V(D) \setminus \{u\}$ as well as $f_3(x) = 2$ for each vertex $x \in V(D)$ are further WSRkD functions on D. As $f_1(x) + f_2(x) + f_3(x) \le 6 \le k$ for each vertex $x \in V(D)$, $\{f_1, f_2, f_3\}$ is a weak signed Roman k-dominating family on D and thus $d_{wsR}^k(D) \ge 3$ in this case. If $f_1(v) = 1$ and $f_1(w) = 1$ for two different vertices $v, w \in V(D)$, then $f_2(v) = 1$ and $f_2(x) = 2$ for $x \in V(D) \setminus \{v\}$ as well as $f_3(x) = 2$ for each vertex $x \in V(D)$ are further WSRkD functions on D. As $f_1(x)+f_2(x)+f_3(x) \le 6 \le k$ for each vertex $x \in V(D)$, $\{f_1, f_2, f_3\}$ is a weak signed Roman k-dominating family on D and thus $d_{wsR}^k(D) \ge 3$ also in the second case. □

Example 4. *Let $p \ge 4$ be an integer, and let H_p be the graph consisting of p triangles $y_i^1 y_i^2 y_i^3 y_i^1$ for $1 \le i \le p$, a further vertex w adjacent to y_i^1 for $1 \le i \le p$ and the cycle $y_1^1 y_1^2 \dots x_1^p y_1^1$. If H_p^* is the associated digraph of H_p, then let f be a WSR6DF on H_p^*. We observe that $f(x) = 2$ for each vertex $x \in V(H_p^*) \setminus \{w\}$. Hence there exist exactly three weak signed Roman 6-dominating functions on H_p^*, namely, $f_1(w) = -1$ and $f_1(x) = 2$ for $x \ne w$, $f_2(w) = 1$ and $f_2(x) = 2$ for $x \ne w$ and $f_3(x) = 2$ for each vertex x. Thus $d_{wsR}^6(H_p^*) = 3$.*

Example 5. *Let $p \ge 5$ be an integer, and let L_p be the graph consisting of p complete graphs with vertex set $\{y_i^1, y_i^2, y_i^3, y_i^4\}$ for $1 \le i \le p$, a further vertex w adjacent to y_i^1 for $1 \le i \le p$ and the cycle $y_1^1 y_1^2 \dots x_1^p y_1^1$. If L_p^* is the associated digraph of L_p, then let f be a WSR8DF on L_p^*. We observe that $f(x) = 2$ for each vertex $x \in V(L_p^*) \setminus \{w\}$. Hence there exist exactly three weak signed Roman 8-dominating functions on L_p^*, namely, $f_1(w) = -1$*

and $f_1(x)=2$ *for* $x\neq w$*,* $f_2(w)=1$ *and* $f_2(x)=2$ *for* $x\neq w$ *and* $f_3(x)=2$ *for each vertex* x*. Thus* $d^8_{wsR}(L^*_p)=3$*.*

Examples 4 and 5 show that Theorem 8 is sharp.

3. Upper bounds on the sum $\gamma^k_{wsR}(D)+d^k_{wsR}(D)$

Theorem 9. If D is a digraph of order $n\geq 1$ and $\delta^-(D)\geq k-1$, then

$$\gamma^k_{wsR}(D)+d^k_{wsR}(D)\leq n+k.$$

Proof. If $d^k_{wsR}(D)\leq k$, then Theorem G implies $\gamma^k_{wsR}(D)+d^k_{wsR}(D)\leq n+k$ immediately. Let now $d^k_{wsR}(D)\geq k$. It follows from Theorem 3 that

$$\gamma^k_{wsR}(D)+d^k_{wsR}(D)\leq \frac{kn}{d^k_{wsR}(D)}+d^k_{wsR}(D).$$

According to Theorem 2, we have $k\leq d^k_{wsR}(D)\leq n$. Using these bounds, and the fact that the function $g(x)=x+(kn)/x$ is decreasing for $k\leq x\leq\sqrt{kn}$ and increasing for $\sqrt{kn}\leq x\leq n$, we obtain

$$\gamma^k_{wsR}(D)+d^k_{wsR}(D)\leq \frac{kn}{d^k_{wsR}(D)}+d^k_{wsR}(D)\leq \max\{n+k,k+n\}=n+k,$$

and the desired bound is proved. □

Theorem 10. Let D be a digraph of order $n\geq 2$ and $\delta^-(D)\geq\lceil\frac{k}{2}\rceil-1$. Then

$$\gamma^k_{wsR}(D)+d^k_{wsR}(D)\leq 2n+k-1,$$

with equality if and only if $k=2$ and D is the empty digraph.

Proof. If $\delta^-=\delta^-(D)\geq k-1$, then Theorem 9 implies

$$\gamma^k_{wsR}(D)+d^k_{wsR}(D)\leq n+k<2n+k-1.$$

Assume next that $\lceil\frac{k}{2}\rceil-1\leq\delta^-\leq k-2$. Then $k\geq 2$ and according to Theorem H and Theorem 2, we obtain

$$\gamma^k_{wsR}(D)+d^k_{wsR}(D)\leq 2n+\delta^-+1\leq 2n+k-1. \tag{3.1}$$

If we have equality in (3.1), then $\gamma^k_{wsR}(D)=2n$ and $d^k_{wsR}(D)=k-1$. Therefore Theorem 3 leads to $2n(k-1)=\gamma^k_{wsR}(D)\cdot d^k_{wsR}(D)\leq kn$ and so $k=2$. Thus $\delta^-=0$ and Theorem H implies that D is the empty digraph.
Clearly, if D is the empty digraph, then $\gamma^2_{wsR}(D)=2n$ and $d^2_{wsR}(D)=1$ and thus $\gamma^2_{wsR}(D)+d^2_{wsR}(D)=2n+1=2n+2-1$. □

Theorem 11. Let $k \geq 3$ be an integer, and let D be a digraph of order n with $\delta^-(D) \geq \lceil \frac{k}{2} \rceil - 1$. If $k = 2n$, then $D = K_n^*$ and $\gamma_{wsR}^k(D) + d_{wsR}^k(D) = 2n+1$. If $k \leq 2n-1$, then

$$\gamma_{wsR}^k(D) + d_{wsR}^k(D) \leq 2n + \left\lceil \frac{k}{2} \right\rceil - 1.$$

Proof. Since $n \geq \delta^-(D) + 1 \geq \lceil \frac{k}{2} \rceil \geq \frac{k}{2}$, we observe that $k \leq 2n$.
If $k = 2n$, then $\delta^-(D) + 1 = n$ and thus $D = K_n^*$. Theorem H implies $\gamma_{wsR}^k(D) = 2n$. Clearly, $d_{wsR}^k(D) = 1$ and therefore $\gamma_{wsR}^k(D) + d_{wsR}^k(D) = 2n + 1$.
Let now $k \leq 2n-1$. In this case, it is straightforward to verify that $n+k \leq 2n+\lceil \frac{k}{2} \rceil - 1$. If $\delta^- = \delta^-(D) \geq k-1$, then the last inequality and Theorem 9 lead to the desired bound.
Assume next that $\lceil \frac{k}{2} \rceil - 1 \leq \delta^- \leq k-1$. If $\gamma_{wsR}^k(D) = 2n$, then the definitions lead to $d_{wsR}^k(D) = 1$ and thus

$$\gamma_{wsR}^k(D) + d_{wsR}^k(D) = 2n + 1 \leq 2n + \left\lceil \frac{k}{2} \right\rceil - 1.$$

Let now $\gamma_{wsR}^k(D) \leq 2n-1$. If $d_{wsR}^k(D) \leq \lceil \frac{k}{2} \rceil$, then the desired bound is immediate. Finally, let $d_{wsR}^k(D) \geq \lceil \frac{k}{2} \rceil + 1$. Using Theorem 2, we observe that

$$\left\lceil \frac{k}{2} \right\rceil + 1 \leq d_{wsR}^k(D) \leq \delta^- + 1 \leq k.$$

We deduce from Theorem 3 that

$$\gamma_{wsR}^k(D) + d_{wsR}^k(D) \leq \frac{kn}{d_{wsR}^k(D)} + d_{wsR}^k(D).$$

Using these bounds, we obtain analogously to the proof of Theorem 9 that

$$\gamma_{wsR}^k(D) + d_{wsR}^k(D) \leq \max\left\{ \frac{kn}{\lceil k/2 \rceil + 1} + \left\lceil \frac{k}{2} \right\rceil + 1, n+k \right\}.$$

Since $n \geq \delta^- + 1 \geq \lceil \frac{k}{2} \rceil + 1$, it is straightforward to verify that

$$\frac{kn}{\lceil k/2 \rceil + 1} + \left\lceil \frac{k}{2} \right\rceil + 1 \leq 2n + \left\lceil \frac{k}{2} \right\rceil - 1,$$

and this leads to the desired bound. □

Let k and n be integers such that $n \geq 3$ and $2n-2 \leq k \leq 2n-1$. Corollary 2 implies $d_{wsR}^k(K_n^*) = n$, and it follows from Theorem F that $\gamma_{wsR}^k(K_n^*) \geq k$. Thus

$$\gamma_{wsR}^k(K_n^*) + d_{wsR}^k(K_n^*) \geq n + k. \tag{3.2}$$

If $k = 2n-1$, then we deduce from inequality (3.2) and Theorem 11 that

$$3n-1 = n+k \leq \gamma^k_{wsR}(K^*_n) + d^k_{wsR}(K^*_n) \leq 2n + \left\lceil \frac{k}{2} \right\rceil - 1 = 3n-1$$

and therefore $\gamma^k_{wsR}(K^*_n) + d^k_{wsR}(K^*_n) = 2n + \lceil \frac{k}{2} \rceil - 1$ and $\gamma^k_{wsR}(K^*_n) = k$.
If $k = 2n-2$, then we deduce from inequality (3.2) and Theorem 11 that

$$3n-2 = n+k \leq \gamma^k_{wsR}(K^*_n) + d^k_{wsR}(K^*_n) \leq 2n + \left\lceil \frac{k}{2} \right\rceil - 1 = 3n-2$$

and therefore $\gamma^k_{wsR}(K^*_n) + d^k_{wsR}(K^*_n) = 2n + \lceil \frac{k}{2} \rceil - 1$ and $\gamma^k_{wsR}(K^*_n) = k$.
These examples demonstrate that the upper bound in Theorem 11 is sharp.

4. Nordhaus-Gaddum type results

Results of Nordhaus-Gaddum type study the extreme values of the sum or the product of a parameter on a graph or digraph and its complement. In their classical paper [9], Nordhaus and Gaddum discussed this problem for the chromatic number of graphs. We present such inequalities for the weak signed Roman k-domatic number of digraphs.

Theorem 12. If D is a digraph of order n with $\delta^-(D), \delta^-(\overline{D}) \geq \lceil \frac{k}{2} \rceil - 1$, then $d^k_{wsR}(D) + d^k_{wsR}(\overline{D}) \leq n+1$. Furthermore, if $d^k_{wsR}(D) + d^k_{wsR}(\overline{D}) = n+1$, then D is in-regular.

Proof. It follows from Theorem 2 that

$$\begin{aligned} d^k_{wsR}(D) + d^k_{wsR}(\overline{D}) &\leq (\delta^-(D)+1) + (\delta^-(\overline{D})+1) \\ &= (\delta^-(D)+1) + (n - \Delta^-(D) - 1 + 1) \leq n+1. \end{aligned}$$

If D is not in-regular, then $\Delta^-(D) - \delta^-(D) \geq 1$ and thus the inequality chain above implies the better bound $d^k_{wsR}(D) + d^k_{wsR}(\overline{D}) \leq n$. □

In the case $k = 1$ we determine all regular digraphs D with $d_{wsR}(D) + d_{wsR}(\overline{D}) = n+1$.

Theorem 13. If D is a δ-regular digraph of order n, then $d_{wsR}(D) + d_{wsR}(\overline{D}) = n+1$ if and only if $D = K^*_n$ or $\overline{D} = K^*_n$.

Proof. If $D = K^*_n$ or $\overline{D} = K^*_n$, then Corollary 1 leads to $d_{wsR}(D) + d_{wsR}(\overline{D}) = n+1$. Conversely, assume that $d_{wsR}(D) + d_{wsR}(\overline{D}) = n+1$. Since D is δ-regular, $\overline{D}$ is $(n-1-\delta)$-regular. If $\delta = n-1$ or $\delta = 0$, then $D = K^*_n$ or $\overline{D} = K^*_n$, and we obtain the desired result.

Next assume that $1 \le \delta \le n-2$ and $1 \le n-1-\delta \le n-2$. We assume, without loss of generality, that $\delta \le (n-1)/2$. If $n \not\equiv 0\,(\mathrm{mod}(n-\delta))$, then it follows from Theorems 2 and 4 that

$$n+1 = d_{wsR}(D) + d_{wsR}(\overline{D}) \le (\delta+1) + (n-1-\delta) = n,$$

a contradiction. Therefore assume that $n \equiv 0\,(\mathrm{mod}(n-\delta))$. Then $n = q(n-\delta)$ with an integer $q \ge 2$. Since $\delta \le (n-1)/2$, we obtain the contradiction

$$n = q(n-\delta) \ge q\left(n - \frac{n-1}{2}\right) = \frac{q(n+1)}{2} \ge n+1.$$

This completes the proof. □

In the case $k = 2$ we determine almost all regular digraphs D with $d^2_{wsR}(D) + d^2_{wsR}(\overline{D}) = n+1$.

Theorem 14. Let D be a δ-regular digraph of order $n \ge 3$, and assume that neither D nor $\overline{D}$ is 2-regular of order 6 or 5-regular of order 15. Then $d^2_{wsR}(D) + d^2_{wsR}(\overline{D}) = n+1$ if and only if $D = K^*_n$ or $\overline{D} = K^*_n$.

Proof. If $D = K^*_n$ or $\overline{D} = K^*_n$, then Corollary 1 leads to $d^2_{wsR}(D)+d^2_{wsR}(\overline{D}) = n+1$. Conversely, assume that $d^2_{wsR}(D) + d^2_{wsR}(\overline{D}) = n+1$. Since D is δ-regular, $\overline{D}$ is $\overline{\delta}$-regular such that $\delta+\overline{\delta}+1 = n$. If $\delta = n-1$ or $\delta = 0$, then $D = K^*_n$ or $\overline{D} = K^*_n$, and we obtain the desired result. Next assume that $1 \le \delta, \overline{\delta} \le n-2$ and that, without loss of generality, $\overline{\delta} \le \delta$.
Let $2\overline{\delta} = t(\delta+1)+s$ with integers $t \ge 0$ and $0 \le s \le \delta$. If $s \ne 0$, then Theorems 2 and 4 imply

$$d^2_{wsR}(D) + d^2_{wsR}(\overline{D}) \le \delta + \overline{\delta} + 1 = n.$$

If $s = 0$, then the condition $1 \le \overline{\delta} \le \delta$ and the identity $2\overline{\delta} = t(\delta+1)$ show that $t = 1$ and so

$$2\overline{\delta} = \delta + 1. \tag{4.1}$$

Let now

$$n = p(\overline{\delta}+1) + r \tag{4.2}$$

with integers $p \ge 1$ and $0 \le r \le \overline{\delta}$ and when $r \ne 0$

$$2r = a(\overline{\delta}+1) + b$$

with integers $a \ge 0$ and $0 \le b \le \overline{\delta}$. If $b, r \ne 0$, then we deduce from Theorems 2 and 4 that $d^2_{wsR}(D) + d^2_{wsR}(\overline{D}) \le \delta + 1 + \overline{\delta} = n$. Now let $r \ne 0$ and $b = 0$. Then

$$2r = a(\overline{\delta}+1) = \overline{\delta} + 1. \tag{4.3}$$

Using (4.1), (4.2) and (4.3), we obtain

$$6r - 3 = \delta + \overline{\delta} + 1 = n = p(\overline{\delta} + 1) + r = 2pr + r$$

and thus $p = 1$ or $p = 2$. If $p = 1$, then $r = 1$ and so $\overline{\delta} = 1$, $\delta = 1$ and $n = 3$. Therefore D and $\overline{D}$ are oriented cycles of length 3. In this case it is easy to see that $d^2_{wsR}(D) + d^2_{wsR}(\overline{D}) = 2 = n - 1$. If $p = 2$, then $r = 3$, $\overline{\delta} = 5$, $\delta = 9$ and $n = 15$. However, by the hypothesis, this is not allowed.
Finally, let $r = 0$. Then it follows from (4.1) and (4.2) that $3\overline{\delta} = \delta+\overline{\delta}+1 = n = p(\overline{\delta}+1)$ and thus $p = 2$ and hence $\overline{\delta} = 2$, $\delta = 3$ and $n = 6$. However, this not allowed. □

Using Theorems 2 and 4, one can prove the next result analogue to Theorem 3.4 in [14].

Theorem 15. Let $k \geq 3$ be an integer, and let D be a δ-regular digraph such that $\delta, \delta^-(\overline{D}) \geq \frac{k}{2} - 1$. Then there is only a finite number of digraphs D such that $d^k_{wsR}(D) + d^k_{wsR}(\overline{D}) = n(D) + 1$.

Conjecture 1. *Let $k \geq 3$ be an integer. If D is a δ-regular digraph of order n such that $\delta, \delta^-(\overline{D}) \geq \frac{k}{2} - 1$, then $d^k_{wsR}(D) + d^k_{wsR}(\overline{D}) \leq n$.*

For tournaments T of odd order with $\delta^-(T), \delta^-(\overline{T}) \geq k$, we improve Theorem 12.

Theorem 16. If T is a tournament of odd order $n \geq 3$ with $\delta^-(T), \delta^-(\overline{T}) \geq k$, then $d^k_{wsR}(T) + d^k_{wsR}(\overline{T}) \leq n - 1$.

Proof. If T is not regular, then $\delta^-(T) \leq (n-3)/2$ and $\delta^-(\overline{T}) \leq (n-3)/2$. Hence Theorem 2 implies that

$$d^k_{wsR}(T) + d^k_{wsR}(\overline{T}) \leq (\delta^-(T) + 1) + (\delta^-(\overline{T}) + 1) \leq \frac{n-3}{2} + 1 + \frac{n-3}{2} + 1 = n - 1.$$

Let now T be a δ-regular tournament. Then $\overline{T}$ is also a δ-regular tournament of order $n = 2\delta + 1$ such that $k\delta = (k-1)(\delta + 1) + (\delta - k + 1)$. Using Corollary 5 with $1 \leq s = \delta - k + 1 \leq \delta$, we deduce that

$$d^k_{wsR}(T) + d^k_{wsR}(\overline{T}) \leq \delta + \delta = 2\delta = n - 1,$$

and the proof is complete. □

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