Communications in Combinatorics and Optimization
Vol. 11, No. 2 (2026), pp. 503-519
https://doi.org/10.22049/cco.2024.28947.1789

CCO
Commun. Comb. Optim.

Research Article

Mixed double Roman domination in graphs

H. Abdollahzadeh Ahangar[1,*], M. Chellali[2], S.M. Sheikholeslami[3], J.C. Valenzuela-Tripodoro[4]

[1]Department of Mathematics, Babol Noshirvani University of Technology, Shariati Ave., Babol, I.R. Iran, Postal Code: 47148-71167
ha.ahangar@nit.ac.ir

[2]LAMDA-RO Laboratory, Department of Mathematics, University of Blida, Blida, Algeria
m_chellali@yahoo.com

[3]Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
s.m.sheikholeslami@azaruniv.ac.ir

[4]Department of Mathematics, University of Cádiz, Spain
jcarlos.valenzuela@uca.es

Received: 22 August 2023; Accepted: 6 August 2024
Published Online: 22 August 2024

Abstract: Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. A *mixed double Roman dominating function* (MDRDF) of G is a function $f : V \cup E \to \{0, 1, 2, 3\}$ satisfying (1) for any element $x \in V \cup E$ with $f(x) = 0$, there must be either element $y \in V \cup E$, with $f(y) = 3$, which is adjacent or incident to x, or either two elements $y, z \in V \cup E$, with $f(y), f(z) = 2$ which are adjacent or incident to x; (2) for any element $x \in V \cup E$ with $f(x) = 1$, there must be either element $y \in V \cup E$, with $f(y) \geq 2$, which is adjacent or incident to x. The weight of an MDRDF f is $w(f) = f(V \cup E) = \sum_{x \in V \cup E} f(x)$ and the minimum weight among all the MDRD functions the *MDRD-number*, $\gamma^*_{dR}(G)$, of the graph G. In this paper we start the study of this variation of the classic Roman domination problem by setting some basic results, giving exact values and sharp bounds of the MDRD number and we approach the study of the complexity of the decision problem associated to the MDR domination in graphs.

Keywords: Roman domination, double Roman domination; mixed double Roman domination.

AMS Subject classification: 05C69

* *Corresponding Author*

1. Introduction

In this paper we introduce the concept of mixed double Roman domination in graphs. The emergence of Roman domination in graphs is closely related to a defensive strategy decreed by the Roman Emperor Constantine I the Great around the 3rd century. Under this defensive strategy, the Emperor determined that: i) at any city of the empire could be deployed at most two legions; ii) any legion could be moved to a neighboring city to defend it from an external attack; iii) no legion, established in an strong city, could be transfered to another weaker place if it made the stronger city defenseless. Basicly, we can have some weak cities with no legions; possibly some cities with a legion, that are able to defend itself from an external attack; and, finally, the stronger cities in which we have deployed two legions and that are able to send one of its legions to defend any weak neighbor city from an external attack.

The defensive strategy described before leaded to the mathematical concept of Roman dominating function in a graph. A *Roman dominating function (RDF)* in a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2\}$ such that any vertex v with $f(v) = 0$ must have at least a neighbor $w \in V(G)$ such that $f(w) = 2$. The minimum weight $f(V) = \sum_{v \in V(G)} f(v)$ among all these RDF in a graph G is known as the *Roman domination number of* G, denoted by $\gamma_R(G)$. This topic was initialy introduced and devoloped by Stewart [16] in 1999, and Revelle and Rosing [15] in 2000, and a few years later by Cockayne et al. [9] in 2004. So far, several papers have been published regarding Roman domination including several variations of this problem as double Roman domination [4], mixed Roman domination [1, 11], strong Roman domination [3], total Roman domination [2], etc. For more details on Roman domination and its variations we refer the reader to the book chapters [5, 7] and survey [6].

In this paper we consider simple connected graphs $G = (V, E)$ with vertex set $V = V(G)$, $n = |V|$ vertices, edge set $E = E(G)$ and $m = |E|$ edges. For every vertex $v \in V(G)$, let $N_G(v) = \{w \in V : vw \in E\}$ be the *open neighborhood* of v and let $d(v) = |N_G(v)|$ denote the degree of v. The *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The minimum and maximum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *open neighborhood* of a set S is $N_G(S) = \cup_{v \in S} N_G(v)$ and the *closed neighborhood* of a set S is $N_G[S] = N_G(S) \cup S$. For any element $x \in V \cup E$ of the graph G, let us denote by $N_m(x) = \{y \in V \cup E : y \text{ is either adjacent to or incident with } x\}$ and, besides, $N_m[x] = N_m(x) \cup \{x\}$.

A dominating set of vertices is a set S such that every vertex in $V \setminus S$ has, at least, a neighbor in S, consequently, $N[S] = V$. The domination number of a graph $\gamma(G)$ is the minimum cardinality of a dominating set in G. We can extend the concept of domination to *mixed domination* by considering both vertices and edges, so that each element of $N_m(x)$ is dominated by $x \in V \cup E$. In this case, we call $X \subseteq V \cup E$ a *mixed dominating set* if every element in $(V \cup E) \setminus X$ is dominated by, at least, an element belonging to X. The *mixed domination number* of G is defined analogously and it is denoted by $\gamma^*(G)$.

With these notations, a Roman dominating function (RDF) f in a graph is $f : V \to \{0, 1, 2\}$ with the property that any vertex assigned with 0 must be adjacent to, at least, a vertex labelled with 2. The Roman domination number of a graph G is $\gamma_R(G) = \min\{f(V) = \sum_{v\in V} f(v) :\ f \text{ is an RDF in } G\ \}$.

In 2016, Beeler et al. [4] defined the double Roman domination as a matter of stronger version of Roman domination. In this case, the aim is to double the protection of any weak place so that an external attack could be defended by two legions. A double Roman dominating function (DRDF) is a function $f : V \to \{0, 1, 2, 3\}$ such that (i) any vertex assigned with 1 is adjacent to either a vertex assigned with 2 or either a vertex assigned with 3; and (ii) any vertex with $f(v) = 0$ is adjacent to a vertex assigned with 3 or either is adjacent to two different vertices assigned with 2. The double Roman domination number of a graph G is $\gamma_{dR}(G) = \min\{f(V) = \sum_{v\in V} f(v) :\ f \text{ is a DRDF in } G\ \}$.

Another variation of the Roman domination problem was the mixed Roman domination, introduced by Ahangar et al. [1] in 2017. Mixed Roman domination is based in the same principles as the Roman domination, but in this case also the defense of "roadways" is considered, so that at most two legions could be deployed either in a city (vertex) or either in the way (edge) joining two neighboring cities. More explicitly, any vertex (resp. edge) with no legions must be adjacent to a vertex, or incident with an edge, with two legions.

In this paper we deal with the new concept of mixed double Roman domination. Namely, a mixed double Roman domination function (MDRDF) is a function $f : V \cup E \to \{0, 1, 2, 3\}$ such that any element $x \in V \cup E$ could "be defended" by, at least, two legions settled in the element x itself or either deployed in neighboring elements of x. Clearly, there is an binunivocal relation between the set of MDRD functions and all the possible partitions $\{V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2, V_3 \cup E_3\}$ of the set $V \cup E$, by defining $V_j \cup E_j = \{x \in V \cup E : f(x) = j\}$.
In other words, an MDRDF is a function $f : V \cup E \to \{0, 1, 2, 3\}$, with the notation $f \equiv \{V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2, V_3 \cup E_3\}$ satisfying the following conditions

a) For every element $x \in V \cup E$ that $f(x) = 0$, the element x must have at least two neighbours in $V_2 \cup E_2$ or at least one neighbour in $V_3 \cup E_3$.

b) For every element $x \in V \cup E$ that $f(x) = 1$, the element x must have at least one neighbour in $V_2 \cup V_3 \cup E_2 \cup E_3$.

The minimum weight of an MDRDF $w(f) = f(V) = \sum_{x\in V\cup E} f(x)$ is the *mixed double Roman domination number* of the graph G and it is denoted by $\gamma^*_{dR}(G)$. An MDRDF f with minimum weight $w(f) = \gamma^*_{dR}(G)$ is called a $\gamma^*_{dR}(G)$-function.

Let $x \in V \cup E$ be an element of G, we denote by $f(x) = \sum_{y\in N_m(x)} f(y)$. For a set $S \subseteq V \cup E$ of a graph, G let us define the function f_S by assigning 3 to every element of S, 0 to every element in $N_m[S] \setminus S$, and 1 to all remaining elements in $V \cup E$. We note that f_S is a MRDF for any set $S \in V \cup E$.
We end this section by the following observation.

Observation 1. Without lost of generality, we may assume that if $f : V \cup E \to \{0,1,2,3\}$ is a $\gamma^*_{dR}(G)$-function then $f(x) \neq 1$ for all $x \in V \cup E$.

Proof. It is sufficient to observe that if $f(x) = 1$, then it must be $y \in N_m(x)$ such that $f(y) \geq 2$. Then, we may construct $g : V \cup E \to \{0,1,2,3\}$ such that $g(z) = f(z)$ for all $z \neq x, y$, $g(x) = 0$ and $g(y) = 3$ leading to a new MDRDF either with no 1 assigned to x or either with $g(V) < f(V)$, which is a contradiction. □

2. Complexity

Our aim in this section is to study the complexity of the following decision problem, to which we shall refer as MIXED DOUBLE ROMAN DOMINATION:

MIXED DOUBLE ROMAN DOMINATION (Mixed DRD)
Instance: Graph $G = (V, E)$, positive integer k.
Question: Does G have a mixed double Roman dominating function of weight at most k?

We will show that this problem is NP-complete by reducing the special case of Exact Cover by 3-sets (X3C) to which we refer as X3C3. The X3C problem considers a set of $3q$ elements, X, and a set of 3-elements clauses, C and asks wether there exists a subset C' of C that covers all elements of X exactly once. The variation X3C3 considers the input restricted to those sets of clauses C such that every element of X appears in exactly three of these clauses. The NP-completeness of X3C3 was proven in 2008 by Hickey et al. [12].

X3C3
Instance: A set of elements X with $|X| = 3q$, and a collection C of 3-element subsets of X, with $|C| = 3q$, such that each element occurs in exactly 3 members of C.
Question: Does C contain an exact cover for X, i.e. does there exist a subcollection $C' \subset C$ such that every element of X occurs in exactly one member of C'?

Theorem 2. *Problem Mixed DRD is NP-Complete for bipartite graphs.*

Proof. MRD is a member of $\mathcal{NP}$, since we can check in polynomial time that a function $f : V \cup E \to \{0,1,2,3\}$ has weight at most k and is a mixed double Roman dominating function. Now let us show how to transform any instance of X3C3 into an instance G of Mixed DRD so that one of them has a solution if and only if the other one has a solution. Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_{3q}\}$ be an arbitrary instance of X3C3.

For each $x_i \in X$, we create a vertex w_i. Let $W = \{w_1, w_2, \ldots, w_{3q}\}$. For each $C_j \in C$ we build a connected graph H_j of order 8 obtained from two disjoint stars $K_{1,3}$, one centered at a_j and the other star with center r_j and leaves b_j, c_j, d_j by adding the edges a_jb_j and a_jd_j. Let $Y = \{c_1, c_2, \ldots, c_{3q}\}$. Now to obtain a graph G, we add edges c_jw_i if $x_i \in C_j$. Clearly G is a bipartite graph. Set $k = 21q$, and let H be the subgraph of G induced by all $V(H_j)$. Observe that for every mixed double Roman dominating function f on G, each H_j has weight at least 6, and so $f(H) \geq 18q$. We also note that if $f(c_j) = m$ with $m \in \{2, 3\}$ for some vertex c_j, then $f(H_j) \geq 6 + m$. Suppose that the instance X, C of X3C3 has a solution C'. We construct a mixed double Roman dominating function f on G of weight k. We assign the value 0 to every w_i and to every edge incident with w_i. For every $C_j \in C'$, assign the value 3 to c_j, r_j and a_j, and 0 to the remaining elements of H_j. Also for every $C_j \notin C'$, assign the value 3 to edge c_jr_j and vertex a_j, and 0 to the remaining elements of H_j. Note that since C' exists, its cardinality is precisely q, and so the number of c_j's with weight 3 is q, having disjoint neighborhoods in $\{x_1, x_2, \ldots, x_{3q}\}$, Since C' is a solution for X3C3, every vertex in W is adjacent to a vertex of Y assigned a 3. Moreover, every edge incident with a vertex of Y is adjacent to an element assigned 3 under f. Hence, it is straightforward to see that f is a mixed double Roman dominating function with weight $f(V) = 18q + 3q = k$.

Conversely, suppose that G has a mixed double Roman dominating function with weight at most k. Among all such functions, let $g = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2, V_3 \cup E_3,)$ with a fewest elements assigned the value 1. According to our choice of g, we claim that every vertex of Y is assigned either 0 or 3. Indeed, suppose that $g(c_j) \in \{1, 2\}$ for some j. Then it is easy to see that $g(H_j) \geq 7$. In this case, let g' be the function defined on G by $g'(c_jr_j) = 3$, $g'(a_j) = 3$ and $g'(y) = 0$ for any other element y of H_j, and $g'(x) = g(x)$ for every element x of G not in H_j. Clearly g' is either a MRDF on G with weight $g'(V) < g(V)$ or having the same weight but with fewer vertices assigned the value 1 than those under g, contradicting our choice of g, which proves the claim. Now, on the basis of the previous fact and since $g(H) \geq 18q$, we may assume that if $g(c_j) = 0$, then $g(c_jr_j) = 3$. Hence regardless of the value assigned to every c_j; all edges of the form w_ic_j are assigned the value 0. Now since $g(H) \geq 18q$ and $k \leq 21q$ we need to assign a weight of at most $3q$ on the elements G in order to mixed double Roman dominate the vertices of W. Observe that since $|W| = 3q$, we have $W \cap V_0 \neq \emptyset$. Note that if $g(w_i) = 3$ for some i, then we can reassign w_i the value 2 instead of 3. If $g(w_i) = 1$ for some i, then w_i must have a neighbor in $E_2 \cup V_2$, say w_i^*. If $w_i^* \in E_2$, then every vertex of $N(w_i) \cap Y$ is assigned a 0, and so we can reassign w_i, w_i^* and one vertex of $N(w_i) \cap Y$ the values $0, 0, 3$, respectively. Hence we can assume that $w_i^* \in V_2$. But then as above, we reassign w_i and w_i^* the values 0 and 3 respectively. Thus, without loss of generality, we may suppose that $W \subseteq V_0 \cup V_2$. Let $W_0 = W \cap V_0$ and $W_2 = W \cap V_2$. Moreover, since $W_0 = W \cap V_0 \neq \emptyset$, let $Y_2 = Y \cap V_2$ and $Y_3 = Y \cap V_3$. Clearly, a vertex of W_2 has no neighbor in Y_3 for otherwise we can reassign it the value 0 instead of 2. Likewise, if some vertex $w_i \in W_2$ has a neighbor in Y_2, then we can reassign them the values 0 and 3 which makes us gain a unit in the weight of g. Therefore, we assume that no vertex of W_2 has a neighbor in $Y_2 \cup Y_3$. Let W_0' be the subset of all vertices of W_0 having a neighbor in Y_3 and let $W_0'' = W_0 - W_0'$.

Thus

$$|W_0'| + |W_0''| + |W_2| = 3q. \tag{2.1}$$

Since every vertex of W_0'' must have at least two neighbors in Y_2, the following inequality holds

$$3\,|Y_2| \geq 2\,|W_0''|\,. \tag{2.2}$$

Also, since every vertex of Y_3 has three neighbors in W, we must have

$$3\,|Y_3| \geq |W_0'|\,. \tag{2.3}$$

Using the facts that $k \leq 21q$ and $f(H_j) \geq 6+m$ for every c_j with $f(c_j) = m \in \{2,3\}$, we deduce that

$$3\,|Y_3| + 2\,|Y_2| + 2\,|W_2| \leq 3q \tag{2.4}$$

By (2.2) and (2.3), inequality (2.4) becomes

$$|W_0'| + \frac{4}{3}\,|W_0''| + 2\,|W_2| \leq 3q. \tag{2.5}$$

Now using (2.1), inequality (2.5) yields $\frac{1}{3}\,|W_0''| + |W_2| \leq 0$, implying that $|W_0''| = |W_2| = 0$. Therefore $Y_2 = \emptyset, |W_0'| = 3q$, and so $|Y_3| = q$. Consequently, one can easily show that X3C3 has a solution $C' = \{C_j : g(c_j) = 3\}$. □

Next, we show that the decision problem corresponding to the mixed double Roman domination number may be solved in linear time, under certain restrictions in the underlying graphs. Namely, we prove that it is possible to have a solution for this Roman-domination type problem in linear time as long as the corresponding graph has bounded clique-width. Moreover, it can be deduced from this fact that the decision problem can be solvable in linear time also for the class of trees.

In order to demonstrate this fact, we use some objects and results from logical structure, whose formal definition the reader can find in [10, 14]. More specifically, we call a k-expression or a k-presentation on the vertices v_i of a graph G with set of labels $\{1,\ldots,k\}$ to an expression that defines the graph structure of G by using the operations described below

- $i(x)$: To create a new vertex, x, with an i assigned as a label.

$G_1 \oplus G_2$: To create a new graph as the disjoint union of G_1 and G_2.

$\eta_{ij}(G)$: To create all edges in G that join i-vertices with j-vertices.

$\rho_{i\to j}(G)$: To change the label of all i-vertices into label j.

The minimum positive integer k which is needed to construct the graph G by means of a k-expression is called the *clique-width* of the graph G. For example, the complete bipartite graph $K_{2,3}$ with partite set of vertices $\{a_1, a_2\} \cup \{b_1, b_2, b_3\}$ can be described by the following 2-expression.

$$\eta_{12} \left(\left(\left(\left(\bullet 1(a_1) \oplus \bullet 1(a_2) \right) \oplus \bullet 2(b_1) \right) \oplus \bullet 2(b_2) \right) \oplus \bullet 2(b_3) \right)$$

Let us recall that MSOL(τ_1) represents the monadic second order logic with quantification over subsets of elements of the logic structure $G(\tau_1)$. Namely, $G(\tau_1)$ is the logic structure $< V(G) \cup E(G), R >$ where R is a binary relation such that $R(x, y)$ is satisfied if and only if x, y are either adjacent or incident elements of $V(G) \cup E(G)$.

We say that an optimization problem belongs to the class of *LinEMSOL(τ) optimization problems* if it can be described in the following way (see [14] for more details, since this is a version of the definition given by [10] restricted to finite simple graphs),

$$Opt \left\{ \sum_{1 \le i \le l} a_i |X_i| \ : < G(\tau_1), X_1, \ldots, X_l > \models \theta(X_1, \ldots, X_l) \right\}$$

where θ is an MSOL(τ_1) formula that contains free set-variables $X_1, \ldots, X_l$, integers a_i and Opt is either min or max .

In what follows, we make use of a result concerning LinEMSOL optimization problems by Courcelle *et al.* [10].

Theorem 3. (Courcelle et al. [10]) *Let $k \in \mathbb{N}$ and let $\mathcal{C}$ be a class of graphs of clique-width at most k. Then every LinEMSOL(τ_1) optimization problem on $\mathcal{C}$ can be solved in linear time if a k-presentation of the graph is part of the input.*

Theorem 3 was used by Liedloff et al. [14] to prove a result related to the complexity of the Roman domination decision problem. We prove an analogous result regarding the decision problem associated to the mixed double Roman domination problem.

Theorem 4. *The mixed double Roman domination problem belongs to the class of optimization problems LinEMSOL(τ_1).*

Proof. In order to prove the result we only have to show that the mixed double Roman domination problem can be expressed as a LinEMSOL(τ_1) optimization problem.

Let $f = (V_0 \cup E_0, \ldots, V_3 \cup E_3)$ be an MDRDF in the graph $G = (V, E)$. Clearly, the weight of f is $w(f) = |V_1 \cup E_1| + 2|V_2 \cup E_2| + 3|V_3 \cup E_3|$. Next, let us define the free set-variables $X_i : V \cup E \to \{0, 1\}$ as follows $X_i(z) = 1,$ if and only if $z \in V_i \cup E_i$ and $X_i(z) = 0$ in other case. To avoid confusion with logical notation, we denote by $|X_i| = \sum_{z \in V \cup E} X_i(z)$, although it is clear that $|X_i| = |V_i \cup E_i|$. With that structure and following these notations, we have that the mixed double Roman domination decision problem is equivalent to optimize the expression

$$\min_{X_i} \{|X_1| + 2|X_2| + 3|X_3| \ : < G(\tau_1), X_0, \ldots, X_3 > \vDash \theta(X_0, \ldots, X_3)\},$$

where θ is defined below

$$\theta(X_0, \ldots, X_3) = \forall z \Big(X_2(z) \vee X_3(z) \vee \Big(X_1(z) \wedge \exists y \left((X_2(y) \vee X_3(y)) \wedge R(z, y) \right) \Big) \vee \\ \vee \Big(X_0(z) \wedge (\exists y \left(X_3(y) \wedge R(z, y) \right) \vee \exists y, t \left(X_2(y) \wedge X_2(t) \wedge R(z, y) \wedge R(z, t) \right)) \Big) \Big)$$

It is not difficult to check that θ corresponds to the conditions required to any mixed double Roman domination function. The definition of θ consists in four main clauses, where the first and the second ones describe the cases in which z is a self-defended element of $V \cup E$. If the third clause is satisfied then z is an element assigned with a label 1 by f which is adjacent or incident with an element assigned with a label greater or equal than 2. Analogously, the last clause is satisfied when the element z is such that $f(z) = 0$ and it verifies the conditions required to an MDRDF. Therefore, we may assume that f is an MDRDF if and only if the logical expression θ is satisfied. □

The corollaries stated below is an immediate consequence of Theorem 4.

Corollary 1. *The decision problem related to the mixed double Roman domination problem can be solved in linear time on any graph G with clique-width bounded by a positive intender k, provided that either there exists a linear-time algorithm to construct a k-expression of G, or a k-expression of G is part of the input.*

Corollary 2. *The mixed double Roman domination decision problem can be solved in linear time for any tree T.*

The last corollary is derived from the fact that any bounded treewidth graph is also a bounded clique-width graph and, of course, any tree has treewidth equal to 1.
Finally, it is well known that many classes of graphs G have bounded clique-width $cw(G)$ as, for example, the cographs ($cw(G) \leq 2$) or the distance hereditary graphs ($cw(G) \leq 3$), so also for these graphs we can have a solution in linear time for the MDRD problem.

3. Bounds

In [1] the authors described the following graphs

- $G(a, b, c)$, which are the graphs obtained from a non-trivial star $K_{1,n-1}$ with center v by adding edges from its complement such that $G(a, b, c) - v = aK_1 \cup bK_2 \cup cP_3$

- for every $j \leq a$, let $G_j(a,b,c)$ be the graph obtained from $G(a,b,c)$ by subdividing (once) j pendant edges. (See Figure 1).

Let $\mathcal{H}$ be the family of graphs $\mathcal{H} = \{G(a,b,c), G_j(a,b,c) : a,b,c \geq 0, j \leq a\}$ satisfying that if $G \in \mathcal{H}$ and $(b,c) \in \{(0,0),(1,0)\}$, then either $G = G(0,1,0) = K_3$ or $a > j$.

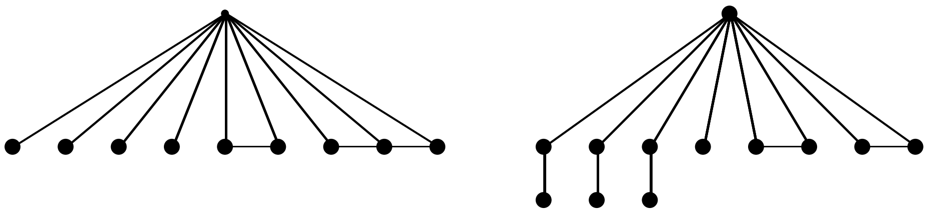

Figure 1. $G(4,1,1)$ **and** $G_3(4,2,0)$

In the same paper, the authors proved the following result, which we will use further.

Proposition 1. [1] *Let G be a connected graph of order $n \geq 2$, size m, and $\Delta(G) \geq 1$. Then $\gamma_R^*(G) \leq m + n - 2\Delta(G) + 1$, with equality if and only if $G \in \mathcal{H}$.*

Next, we give an upper bound for the mixed double Roman domination number of a graph in terms of the mixed Roman domination one.

Proposition 2. *Let G be a non trivial connected graph of order $n \geq 2$. Then*

$$\gamma_{dR}^*(G) \leq 2\gamma_R^*(G) - 1$$

and the equality holds if and only if $G \in \mathcal{H}$.

Proof. Let $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ be a $\gamma_R^*(G)$-function. Since G is a non trivial we have that $V_2 \cup E_2 \neq \emptyset$. Consider the function $g : V(G) \to \{0,1,2,3\}$ defined as follows, $g(x) = 3$ for all $x \in V_2 \cup E_2$, $g(x) = 2$ for all $x \in V_1 \cup E_1$ and $g(x) = 0$ otherwise. There are no elements labeled 1 under the function g and every element of G with a label 0 under f is adjacent to an element in $V_2 \cup E_2$, with a label 3 under the new function g. Then

$$\gamma_{dR}^*(G) \leq w(g) = 2|V_1 \cup E_1| + 3|V_2 \cup E_2| = 2w(f) - |V_2 \cup E_2| \leq 2\gamma_R^*(G) - 1$$

If the equality $\gamma_{dR}^*(G) = 2\gamma_R^*(G) - 1$ holds then $|V_2 \cup E_2| = 1$ for all mixed Roman domination function $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ in G. Let f be a $\gamma_R^*(G)$-function and let $V_2 \cup E_2 = \{x\}$. If $x \in V_2$ then $f(y) = 0$, for all $y \in N_m(x)$, because f is a MDR having minimum weight. So,

$$\begin{aligned}\gamma_R^*(G) &= w(f) = 2 + \sum_{y \notin N_m(x)} f(y) \\ &= 2 + \sum_{y \in V(G)-N_m(x)} f(y) + \sum_{y \in E(G)-N_m(x)} f(y) \\ &\geq 2 + n - 1 - d(x) + m - d(x) \\ &\geq m + n - 2\Delta(G) + 1\end{aligned}$$

and by applying Proposition 1, we have that $\gamma_R^*(G) = m + n - 2\Delta(G) + 1$ and hence, $G \in \mathcal{H}$. Assume that $x \in E_2$. Since f is a $\gamma_R^*(G)$-function then $f(y) = 0$, for all $y \in N_m(x)$. Let us denote by $\{u, v\}$ the final vertices of x. Hence, we have that

$$\begin{aligned}\gamma_R^*(G) &= w(f) = 2 + \sum_{y \notin N_m(x)} f(y) \\ &= 2 + \sum_{y \in V(G)-N_m(x)} f(y) + \sum_{y \in E(G)-N_m(x)} f(y) \\ &\geq 2 + n - 2 + m - 1 - (d(u) - 1) - (d(v) - 1) \\ &\geq m + n - 2\Delta(G) + 1\end{aligned}$$

and we can reasoning as in the previous case. This completes the proof. □

For any tree T_1, let $T_1 \circ K_1$ denote the corona of T_1 and let $\mathcal{F} = \{T_1 \circ K_1 \mid T_1 \text{ is a tree}\}$.

Lemma 1. *If $T \in \mathcal{F}$, then $\gamma_{dR}^*(T) = \frac{3n(T)}{2}$.*

Proof. Let $T \in \mathcal{F}$. Then T is the corona of a tree T_1. Let $V(T_1) = \{v_1, \ldots, v_t\}$ and u_i be the leaf adjacent to v_i in T for each i. Clearly, the function f defined by $f(v_i) = 3$ and $f(u_i) = 0$ for each i, is an MDRDF of T of weight $3t$ implying that $\gamma_{dR}^*(T) \leq \frac{3n(T)}{2}$.

Next we prove the inverse inequality. We proceed by induction on t. If $t = 1$, then clearly $\gamma_{dR}^*(T) = \frac{3n(T)}{2}$. Assume that $t \geq 2$. Without loss of generality we may assume that v_1 is a leaf of T_1 and v_2 is its support vertex. Assume that f is a $\gamma_{dR}^*(T)$-function such that $f(v_1) + f(v_2)$ is as large as possible. By the choice of f, we have $f(u_1) < 3$. First let $f(u_1) = 0$. To double Roman dominate u_1 and by the choice of f, we must have $f(v_1) = 3$ and $f(v_1v_2) = 0$. If $f(v_2) \geq 2$, then clearly the function f restricted to $T - \{v_1, u_1\}$ is a MDRDF of $T - \{v_1, u_1\}$ and by the induction hypothesis we have $\gamma_{dR}^*(T) \geq 3 + \frac{3(n(T)-2)}{2} = \frac{3n(T)}{2}$. Assume that $f(v_2) = 0$. To double Roman dominate u_2 and by the choice of f, we must have $f(u_2) = 2$ and $f(u_2v_2) = 0$. It follows that to double Roman dominate the edge v_2u_2, some edge e incident to v_2 must be assigned at least 2 under f. Again the function f restricted to $T - \{v_1, u_1\}$ is a MDRDF of $T - \{v_1, u_1\}$ and as above we have $\gamma_{dR}^*(T) \geq \frac{3n(T)}{2}$.

Now let $f(u_1)=2$. To double Roman dominate the edge v_1u_1 and by the choice of f, we must have $f(v_1v_2)=2$. Since $f(u_2)+f(v_2u_2)+f(v_2)\geq 2$, the function g defined by $g(v_1)=g(v_2)=3$, $g(u_1)=g(u_2)=g(v_1u_1)=g(v_2u_2)=0$ and $g(x)=f(x)$ otherwise, is a $\gamma^*_{dR}(T)$-function which contradicts the choice of f. Hence $\gamma^*_{dR}(T)\geq \frac{3n(T)}{2}$ and thus $\gamma^*_{dR}(T)=\frac{3n(T)}{2}$. □

Theorem 5. *Let T be a tree of order $n\geq 2$. Then $\gamma^*_{dR}(T)\leq \frac{3n}{2}$, with equality if and only if $T\in\mathcal{F}$.*

Proof. We use induction on n. If $n=2$, then $T=P_2\in\mathcal{F}$ and clearly $\gamma^*_{dR}(T)=3=\frac{3n}{2}$. Assume that $n\geq 3$ and the result holds for any tree T' of order n' with $n'<n$. Let T be a tree of order n. If $\operatorname{diam}(T)=2$, then T is a star and we have $\gamma^*_{dR}(T)=3<\frac{3n}{2}$. If $\operatorname{diam}(T)=3$, then T is a double star and we have $\gamma^*_{dR}(T)=6\leq\frac{3n}{2}$ with equality if and only if $n=4$ and this if and only if $T=P_4\in\mathcal{T}$. Let $\operatorname{diam}(T)\geq 4$ and $v_1v_2\ldots v_k$ be a diametral path in T. Root T at v_k. Let $T'=T-T_{v_2}$. Obviously any $\gamma^*_{dR}(T')$-function can be extended to an MDRDF of F by assigning a 3 to v_2 and a 0 to other elements of T_{v_2} and the edge v_3v_2. It follows from the induction hypothesis that

$$\gamma^*_{dR}(T)\leq\gamma^*_{dR}(T')+3\leq\frac{3n(T')}{2}+3=\frac{3(n(T)-\deg(v_2))}{2}+3\leq\frac{3n(T)}{2}. \qquad (3.1)$$

If $\gamma^*_{dR}(T)=\frac{3n(T)}{2}$, then all inequalities occurring in the inequality chain (3.1) are equalities, and in particular we have $\deg(v_2)=2$ and $\gamma^*_{dR}(T')=\frac{3n(T')}{2}$. It follows from the induction hypothesis that $T'\in\mathcal{F}$ and so T' is the corona of a tree T_1. Let $V(T_1)=\{x_1,\ldots,x_t\}$ and u_i be the leaf adjacent to x_i in T' for each i. If $v_3=u_i$ for some i, say $i=t$, then the function g defined by $g(x_i)=3$ for $1\leq i\leq t-1$, $g(v_2)=3, g(x_tu_t)=2$ and $g(x)=0$ otherwise, is an MDRDF of T of weight less that $\frac{3n}{2}$ which is a contradiction. Hence $v_3=x_i$ for some i. Then T is the corona of $T_1+x_iv_2$ and so $T\in\mathcal{T}$. This completes the proof. □

Next, we give an upper bound for the mixed double Roman domination number of a graph in terms of the order and matching number.

Theorem 6. *Let G be a connected graph of order $n\geq 2$. Then*

$$\gamma^*_{dR}(G)\leq 2n-\alpha(G).$$

Furthermore, the bound is sharp for any tree T in $\mathcal{F}$.

Proof. Let $M=\{u_iv_i\mid 1\leq i\leq\alpha(G)\}$ be a matching of G and let $X=V(G)\backslash V(M)$ be set of M-unsaturated vertices. It is easy to see that the function f defined by $f(u_iv_i)=3$ for $1\leq i\leq\alpha(G)$, $f(x)=2$ for each $x\in X$ and $f(x)=0$ otherwise, is an MDRDF of G and this implies that $\gamma^*_{dR}(G) \leq \omega(f) = 3\alpha(G) + 2(n - 2\alpha(G)) = 2n - \alpha(G)$.

To see the sharpness, Let G be the graph obtained from $k \geq 1$ paths $P_4^i = u_1^i u_2^i u_3^i u_4^i$ $(1 \leq i \leq k)$ by adding $k-1$ edges between the vertices $u_2^1, \ldots, u_2^k$ to connected the paths. It is easy to verify that $\gamma^*_{dR}(G) = 6k = 2n(G) - \alpha(G)$. □

In [13], the author proved an upper bound for the double Roman domination number of a connected graph that we use in the proof of our next result. Namely,

Proposition 3. [8] *Let G be a connected graph with order $n \geq 5$ and minimum degree at least two. If $G \notin \{C_5, C_7\}$, then $\gamma_{dR}(G) \leq \lfloor \frac{11}{10} n \rfloor$.*

Now, we prove an upper bound for the mixed double Roman domination number of a connected graph in terms of the order and the size.

Theorem 7. *Let G be a connected graph with order $n \geq 2$ and size m. Then*

$$\gamma^*_{dR}(G) \leq \left\lfloor \frac{11}{10}(n+m) \right\rfloor.$$

Proof. Let G be a connected graph with order n and size m. If $n = 2$, then clearly $\gamma^*_{dR}(G) = 3 = \lfloor \frac{33}{10} \rfloor = \lfloor \frac{11}{10}(n+m) \rfloor$. Assume that $n \geq 3$. Let $W(G)$ be a new graph whose set of vertices is $V(G) \cup E(G)$ and where two vertices $x, y \in V(W(G))$ are adjacent if and only if one of the following condition holds

- $xy \in E(G)$.
- $y \in V(G)$ is an end-vertex of $x \in E(G)$.
- $x, y \in E(G)$ are two edges sharing a common end-vertex in G.

Clearly, a function f is an MDRDF in G if and only if it is an DRDF in $W(G)$. Since G is a connected graph then $W(G)$ has minimum degree at least two. Therefore, by applying Proposition 3 we have that $\gamma^*_{dR}(G) = \gamma_{dR}(W(G)) \leq \lfloor \frac{11}{10}(n+m) \rfloor$ □

Lemma 2. *Let G be a connected non trivial graph with n vertices and m edges. Let f be a γ^*_{dR}-function in G. Then*

(i) If there is a vertex $u \in V(G)$ with $f[u] = \sum_{x \in N_m(u)} f(x) = 2$ then $f[v] \geq 4$ for every vertex $v \in N_G(u)$.

(ii) If there is an edge $e \in E(G)$ with $f[e] = \sum_{x \in N_m(e)} f(x) = 2$ then $f[e'] \geq 4$ for every edge $e' \in E(G)$ incident with e.

(iii) $\sum_{v \in V} f[v] \geq 3n$ and $\sum_{e \in E} f[e] \geq 3m$

Proof. (i) Let $u \in V(G)$ a vertex such that $f[u] = \sum_{x \in N_m(u)} f(x) = 2$. Since $2 = f[u] \geq f(u)$ then $u \notin V_3$. Clearly, if $f(u) \in \{0,1\}$ then u must be adjacent or incident to an element x_u such that $f(u) + f(x_u) \geq 3$, because f is an MDRDF, and hence $f[u]$ should be greater that or equal to 3. Therefore, $f[u] = 2$ implies that $f(u) = 2$ and $f(x) = 0$ for all $x \in N_m(u)$. So, $f(v) = 0$ for every vertex $v \in N_G(u)$ and there must exist an element $y_v \in N_m(v) - u$ with $f(y_v) \geq 2$ and we can deduce that $f[v] \geq f(u) + f(y_v) \geq 4$.
(ii) Let $e = uv \in E(G)$ an edge such that $f[e] = 2$. Reasoning as in the previous case, the only possibility is that $f(e) = 2, f(u) = f(v) = 0$ and $f(e') = 0$ for every edge e' incident with e. Since f is an MDRDF and $f(e') = 0$ then there are elements $x_{e'} \in N_m(e')$, not necessary different, such that $f(x_{e'}) \geq 2$ for all $e' \in N_m(e)$. Finally, $f[e'] \geq f(e) + f(x_{e'}) \geq 4$.
(iii) It is straightforward deduced from (i) and (ii).

□

Proposition 4. *Let G be a connected non trivial graph of order n and size m. Then*

$$\gamma^*_{dR}(G) \geq \left\lceil \frac{3(m+n)}{2\Delta + 1} \right\rceil.$$

This bound is sharp.

Proof. Let f be a γ^*_{dR}-MDRDF in G. First of all, observe that if $x \in V - (V_2 \cup E_2)$ then we have that $f[x] \geq 3$, because f is an MDRDF. Moreover, if $f[x] = 2$ for some $x \in V \cup E$ then by Proposition 2 we have that there must be element y such that $x, y \in V$ or $x, y \in E$ and $f[y] \geq 4$. Hence, we can derive that $\sum_{e \in E} f[e] \geq 3|E| = 3m$ and $\sum_{u \in V} f[u] \geq 3|V| = 2n$. Then,

$$\begin{aligned} 3(m+n) &\leq \sum_{v \in V} f[v] + \sum_{e=uv \in E} f[uv] \\ &= \sum_{v \in V} (2d(v)+1) f(v) + \sum_{e=uv \in E} (d(u)+d(v)+1) f(uv) \\ &\leq (2\Delta+1) \left(\sum_{v \in V} f(v) + \sum_{e=uv \in E} f(uv) \right) = (2\Delta+1)\, \gamma^*_R(G). \end{aligned}$$

For the sharpness, we can consider the star graph $G = K_{1,n-1}$ for which $\gamma^*_{dR}(G) = 3 = \left\lceil \frac{3(n-1+n)}{2(n-1)+1} \right\rceil$. This concludes the proof.

Proposition 5. *For a non-trivial path P_n,*

$$\gamma^*_{dR}(P_n) = \begin{cases} \lceil \frac{6n-3}{5} \rceil & if \ \ n \equiv 0, 3 \pmod 5, \\ \lceil \frac{6n-3}{5} \rceil + 1 & if \ \ n \equiv 1, 2, 4 \pmod 5. \end{cases}$$

Proof. Let $P_n = v_1v_2\dots v_n$. To prove the upper bound, we define an appropriate MRDF on P_n. Let f be the function defined as follows: $f(v_{5k-3}) = 3$ and $f(v_{5k-1}v_{5k}) = 2$ for $1 \le k \le \lfloor \frac{n}{5} \rfloor$ and $f(x) = 0$ for every $x \in V \cup E \setminus \{v_{5k-3}, v_{5k-1}v_{5k} \mid 1 \le k \le \lfloor \frac{n}{5} \rfloor\}$ except for $f(v_{n-1}) = 3$ if $n \equiv 2, 3 \pmod 5$, $f(v_n) = 2$ if $n \equiv 1 \pmod 5$, and $f(v_n) = f(v_{n-2}) = 3$ if $n \equiv 4 \pmod 5$. Since f is an MDRDF with $\omega(f) = \lceil \frac{6n-3}{5} \rceil$ if $n \equiv 0, 3 \pmod 5$ and $\omega(f) = \lceil \frac{6n-3}{5} \rceil + 1$ otherwise, the upper bound holds.

Next we prove the lower bound by induction on n. The result is trivial for $n = 2, 3, 4, 5, 6$. Let $n \ge 7$ and let the result hold for any non-trivial path $P_{n'}$ with $n' < n$. Let f be a $\gamma^*_{dR}(P_n)$-function such that (i) $f(v_n) + f(v_nv_{n-1})$ is as small as possible and (ii) subject to (i), $f(v_{n-2}v_{n-3})$ is maximized. According Proposition 1 we may assume that $f(x) \ne 1$ for each $x \in V \cup E$. By the choice of f we must have $f(v_n) \in \{0, 2\}$. We consider two cases.

Case 1. $f(v_n) = 0$.
By the choice of f, we must have $f(v_{n-1}) = 3$ and $f(v_{n-1}v_{n-2}) = f(v_{n-1}) = 0$. If $f(v_{n-2}v_{n-3}) = 3$, then the function g defined by $g(v_{n-4}) = \min\{f(v_{n-3} + f(v_{n-4}) + f(v_{n-3}v_{n-4}), 3\}$ and $g(x) = f(x)$ otherwise, is an MDRDF of P_{n-4} and by the induction hypothesis we have $\omega(f) = 6 + \omega(g) \ge 6 + \lceil \frac{6(n-4)-3}{5} \rceil \ge 1 + \lceil \frac{6n-3}{5} \rceil$. If $f(v_{n-2}v_{n-3}) = 0$, then we may assume that $f(v_{n-3}v_{n-4}) = 3$ and the function g defined by $g(v_{n-5}) = \min\{f(v_{n-5} + f(v_{n-4}) + f(v_{n-5}v_{n-4}), 3\}$ and $g(x) = f(x)$ otherwise, is an MDRDF of P_{n-5} and by the induction hypothesis we get the result. Assume that $f(v_{n-2}v_{n-3}) = 2$.Then we may assume that $f(v_{n-4}) \ge 2$ and the function g defined by $g(v_{n-4}) = 3$ and $g(x) = f(x)$ otherwise, is an MDRDF of P_{n-3} of weight $\omega(f) - 4$ and the result follows from the induction hypothesis.

Case 2. $f(v_n) = 2$.
By the choice of f, we must have $f(v_nv_{n-1}) = f(v_{n-1}) = 0$. To double Roman dominate the edge v_nv_{n-1}, we must have $f(v_{n-1}v_{n-1}) \ge 2$. If $f(v_{n-1}v_{n-1}) = 3$, then the function f restricted to $P_n - v_1$ is an MDRDF of P_{n-1} of weight $\omega(f) - 2$ and the result follows from the induction hypothesis. Assume that $f(v_{n-1}v_{n-1}) = 2$. Since f is an minimum MDRDF of P_n, we must have $f(v_{n-2}) = f(v_{n-2}v_{n-3}) = 0$. To double Roman dominate the edge v_{n-2}, we must have $f(v_{n-3}) \ge 2$. Since f is an minimum MDRDF of P_n, we have $f(v_{n-3}) = 2$. Then the function g defined by $g(v_{n-3}) = g(v_{n-1}) = 3$, $f(v_n) = f(v_{n-2}) = f(v_nv_{n-1}) = f(v_{n-1}v_{n-2}) = f(v_{n-2}v_{n-3}) = 0$ and $g(x) = f(x)$ otherwise, is a $\gamma^*_{dR}(P_n)$-function which contradicts the choice of f. This completes the proof. □

Proposition 6. *For $n \ge 3$,*

$$\gamma^*_{dR}(C_n) = \begin{cases} \lceil \frac{6n}{5} \rceil & if \;\; n \equiv 0, 2 \pmod 5, \\ \lceil \frac{6n}{5} \rceil + 1 & if \;\; n \equiv 1, 3, 4 \pmod 5. \end{cases}$$

Proof. Let $C_n = (v_1v_2\dots v_nv_1)$ be the cycle on n vertices. To prove the upper bound, we define an appropriate MRDF on P_n. Let f be the function defined as follows: $f(v_{5k-3}) = 3$ and $f(v_{5k-1}v_{5k}) = 2$ for $1 \leq k \leq \lfloor \frac{n}{5} \rfloor$ and $f(x) = 0$ for every $x \in V \cup E \setminus \{v_{5k-3}, v_{5k-1}v_{5k} \mid 1 \leq k \leq \lfloor \frac{n}{5} \rfloor\}$ except for $f(v_n) = 3$ if $n \equiv 1, 2 \pmod 5$, $f(v_{n_{n-1}}) = 3, f(v_1v_n) = 2$ if $n \equiv 3 \pmod 5$, and $f(v_n) = f(v_{n-2}) = 3$ if $n \equiv 4 \pmod 5$. Since f is an MDRDF with $\omega(f) = \lceil \frac{6n}{5} \rceil$ if $n \equiv 0, 2 \pmod 5$ and $\omega(f) = \lceil \frac{6n}{5} \rceil + 1$ otherwise, the upper bound holds.

In the sequel, we prove the lower bound. According Proposition 4, it is enough we consider $n \equiv 1, 3, 4 \pmod 5$. We proceed by induction on n. The result is trivial for $n = 3, 4, 5$. Let $n \geq 6$ and let the result hold for any non-trivial cycle $C_{n'}$ with $n' < n$. Let f be a $\gamma^*_{dR}(P_n)$-function such that (i) $f(v_1v_2)$ as large as possible (ii) subject to i, $f(v_1) + f(v_2) + f(v_2v_3) + f(v_1v_n)$ is as small as possible. According Proposition 1 we may assume that $f(x) \neq 1$ for each $x \in V \cup E$. Suppose without loss of generality that $f(v_1v_2) = \max\{f(v_iv_{i+1}) \mid 1 \leq i \leq n\}$. First let $f(v_1v_2) = 3$. By the choice of f we must have $f(v_1) = f(v_2) = f(v_2v_3) = f(v_1v_n) = 0$. Clearly the function f restricted to $C_n - \{v_1, v_2\}$ is an MDRDF of $C_n - \{v_1, v_2\}$ of weight $\omega(f) - 3$. Since $n \equiv 1, 3, 4 \pmod 5$, we have $n - 2 \equiv 1, 2$ or $4 \pmod 5$ and by Proposition 5 we have $\omega(f) \geq \lceil \frac{6(n-2)-3}{5} \rceil + 1 + 3 = \lceil \frac{6n}{5} \rceil + 1$ as desired.

Now let $f(v_1v_2) = 2$. By the choice of f we must have $f(v_1) = f(v_2) = f(v_2v_3) = f(v_1v_n) = 0$. To double Roman dominate v_1, v_n we have $f(v_3) \geq 2$ and $f(v_n) \geq 2$. Without loss of let $f(v_3) \geq f(v_n)$. If $n \equiv 1, 4 \pmod 5$, then the function f restricted to $C_n - \{v_1, v_2, v_n\}$ is an MDRDF of the cycle $(C_n - \{v_1, v_2, v_n\}) + v_3v_{n-1}$ of weight $\omega(f) - 4$ and by the induction hypothesis we have $\omega(f) \geq \lceil \frac{6(n-3)}{5} \rceil + 1 + 4 > \lceil \frac{6n}{5} \rceil + 1$ as desired. Assume that $n \equiv 3 \pmod 5$. Then the function f restricted to $C_n - \{v_1\}$ is an MDRDF of the cycle $(C_n - \{v_1\}) + v_2v_n$ of weight $\omega(f) - 2$ and by the induction hypothesis we can see that $\omega(f) \geq \lceil \frac{6n}{5} \rceil + 1$.

Finally let $f(v_1v_2) = 0$. It follows from (i) that $f(x) = 0$ for each $x \in E(G)$. Suppose without loss of generality that $f(v_2) \geq f(v_1)$. Since $f(v_2v_3) = f(v_1v_n) = 0$, we must have $f(v_2) \geq 2$. Suppose first that $f(v_2) = 3$. It follows from (ii) and $f(v_1v_n) = 0$ that $f(v_1) = 0$ and $f(v_n) = 3$ (to double dominate the edge v_1v_n). Then the function f restricted to $C_n - \{v_1, v_n\}$ is an MDRDF of the cycle $(C_n - \{v_1, v_n\}) + v_2v_{n-1}$ of weight $\omega(f) - 3$ and using the induction hypothesis we can see that $\omega(f) \geq \lceil \frac{6n}{5} \rceil + 1$ as desired. Assume that $f(v_2) = 2$. By our earlier assumption and to double Roman dominate the edge v_1v_2 we must have $f(v_1) = 2$. Since $f(v_2v_3) = f(v_nv_1) = 0$, we must have $f(v_3), f(v_n) \geq 2$. Let $f(v_3) \geq f(v_n)$ (the case $f(v_3) \leq f(v_n)$ is similar). Then the function f restricted to $C_n - \{v_1, v_2, v_n\}$ is an MDRDF of the cycle $(C_n - \{v_1, v_2, v_n\}) + v_3v_{n-1}$ of weight $\omega(f) - 6$ and using the induction hypothesis we can see that $\omega(f) > \lceil \frac{6n}{5} \rceil + 1$. This completes the proof. □

Acknowledgements. This research is funded by Babol Noshirvani University of Technology, research grant No. P/M/1094.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

References

[1] H. Abdollahzadeh Ahangar, T.W. Haynes, and J.C. Valenzuela-Tripodoro, *Mixed Roman domination in graphs*, Bull. Malays. Math. Sci. Soc. **40** (2017), no. 4, 1443–1454.
https://doi.org/10.1007/s40840-015-0141-1.

[2] H. Abdollahzadeh Ahangar, M.A. Henning, V. Samodivkin, and I.G. Yero, *Total Roman domination in graphs*, Appl. Anal. Discrete Math. **10** (2016), no. 2, 501–517.

[3] M.P. Alvarez-Ruiz, T. Mediavilla-Gradolph, S.M. Sheikholeslami, J.C. Valenzuela-Tripodoro, and I.G. Yero, *On the strong Roman domination number of graphs*, Discrete Appl. Math. **231** (2017), 44–59.
https://doi.org/10.1016/j.dam.2016.12.013.

[4] R.A. Beeler, T.W. Haynes, and S.T. Hedetniemi, *Double Roman domination*, Discrete Appl. Math. **211** (2016), 23–29.
https://doi.org/10.1016/j.dam.2016.03.017.

[5] M. Chellali, N. Jafari Rad, S.M. Sheikholeslami, and L. Volkmann, *Roman domination in graphs*, Topics in Domination in Graphs (T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, eds.), Springer International Publishing, Cham, 2020, pp. 365–409.

[6] ______, *Varieties of Roman domination II*, AKCE Int. J. Graphs Comb. **17** (2020), no. 3, 966–984.
https://doi.org/10.1016/j.akcej.2019.12.001.

[7] ______, *Varieties of roman domination*, Structures of Domination in Graphs (T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, eds.), Springer International Publishing, Cham, 2021, pp. 273–307.

[8] X. Chen, *A note on the double Roman domination number of graphs*, Czechoslovak Math. J. **70** (2020), no. 1, 205–212.
https://doi.org/10.21136/CMJ.2019.0212-18.

[9] E.J. Cockayne, P.A. Dreyer Jr, S.M. Hedetniemi, and S.T. Hedetniemi, *Roman domination in graphs*, Discrete Math. **278** (2004), no. 1-3, 11–22.
https://doi.org/10.1016/j.disc.2003.06.004.

[10] B. Courcelle, J.A. Makowsky, and U. Rotics, *Linear time solvable optimization problems on graphs of bounded clique-width*, Theory Comput. Syst. **33** (2000), no. 2, 125–150.
https://doi.org/10.1007/s002249910009.

[11] N. Dehgardi, *Mixed Roman domination and 2-independence in trees*, Commun. Comb. Optim. **3** (2018), no. 1, 79–91.
https://doi.org/10.22049/cco.2018.25964.1062.

[12] G. Hickey, F. Dehne, A. Rau-Chaplin, and C. Blouin, *SPR distance computation for unrooted trees*, Evol. Bioinform. **4** (2008), 17–27.
https://doi.org/10.4137/EBO.S419.

[13] R. Khoeilar, H. Karami, M. Chellali, and S.M. Sheikholeslami, *An improved upper bound on the double Roman domination number of graphs with minimum degree at least two*, Discrete Appl. Math. **270** (2019), 159–167.
https://doi.org/10.1016/j.dam.2019.06.018.

[14] M. Liedloff, T. Kloks, J. Liu, and S.L. Peng, *Efficient algorithms for Roman domination on some classes of graphs*, Discrete Appl. Math. **156** (2008), no. 18, 3400–3415.
https://doi.org/10.1016/j.dam.2008.01.011.

[15] C.S. ReVelle and K.E. Rosing, *Defendens imperium romanum: a classical problem in military strategy*, Am. Math. Mon. **107** (2000), no. 7, 585–594.
https://doi.org/10.1080/00029890.2000.12005243.

[16] I. Stewart, *Defend the Roman empire!*, Sci. Am. **281** (1999), no. 6, 136–138.