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Research Article

Hyperbolic k-Mersenne and k-Mersenne-Lucas Quaternions with it's associated Spinor algebra

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Abstract: In this article, we introduce and study hyperbolic k-Mersenne and k-Mersenne-Lucas spinors. First, we give hyperbolic k-Mersenne and k-Mersenne-Lucas quaternions with some algebraic properties. Next we introduce the spinor family of k-Mersenne and k-Mersenne-Lucas numbers using the hyperbolic k-Mersenne and k-Mersenne-Lucas quaternions. Here, we start with Binet-type formulas and algebraic properties such as Catalan's identity, Cassini's identity, d'Ocagne's identity, etc. Additionally, we obtain various types of generating functions. Moreover, we give partial sum formulas in closed form.

Keywords: hyperbolic k-Mersenne quaternions, hyperbolic k-Mersenne Spinors, Binet formula, Catalan identity, generating function.

AMS Subject classification: 15A66, 11B37, 11B39, 11R52

1. Introduction

Number sequences have been studied by researchers for a long time. One of these numbers are the Mersenne numbers, which is named after Marin Mersenne, a french Minim friar who studied them in the early 17th century. Similar to the k-Fibonacci sequence as defined by Falcón et al. [5] and other known k-sequences, recently Uslu, Deniz [8] introduced and studied the k-Mersenne numbers as a generalization of Mersenne numbers, and Mourad, Ali [3] investigated some properties of k-Mersenne-Lucas numbers. For $n \in N$ and $k \in R^+$, the k-Mersenne sequence is denoted by $\{M_{k,n}\}_{n\in N}$ and the k-Mersenne-Lucas sequence is denoted by $\{m_{k,n}\}_{n\in N}$, respectively, by the following recurrences.

$$M_{k,n+2} = 3kM_{k,n+1} - 2M_{k,n}, n \geq 0 \text{ with } M_{k,0} = 0, M_{k,1} = 1, \tag{1.1}$$
$$m_{k,n+2} = 3km_{k,n+1} - 2m_{k,n}, n \geq 0 \text{ with } m_{k,0} = 2, m_{k,1} = 3k. \tag{1.2}$$

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Notice that, for $k = 1$ in the above recurrences, the standard sequences of Mersenne and Mersenne-Lucas numbers are obtained. The roots of the characteristic equation $x^2 - 3kx + 2 = 0$ are $\alpha = \frac{3k+\sqrt{9k^2-8}}{2}$ and $\beta = \frac{3k-\sqrt{9k^2-8}}{2}$, which satisfy the relations

$$\alpha + \beta = 3k, \ \alpha - \beta = \sqrt{9k^2 - 8}, \ \alpha\beta = 2. \tag{1.3}$$

The Binet formulas of the k-Mersenne and k-Mersenne-Lucas sequences are, respectively, given by

$$M_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta} \ \ \textit{and} \ \ m_{k,n} = \alpha^n + \beta^n. \tag{1.4}$$

The concept of quaternion was first introduced in 1843 by William Rowan Hamilton. A quaternion with real coefficients is of the form $q = a + be_1 + ce_2 + de_3$, where $\{1, e_1, e_2, e_3\}$ is the quaternion basis satisfying

$$e_1^2 = e_2^2 = e_3^2 = -1, e_1e_2 = -e_2e_1 = e_3, e_2e_3 = -e_3e_2 = e_1, e_3e_1 = -e_1e_3 = e_2. \tag{1.5}$$

Like the quaternions, the set of hyperbolic quaternions forms a vector space over the real numbers of dimension 4, which was described by Macfarlane [7]. Unlike the ordinary quaternion, the hyperbolic quaternions are not associative, anti-commutative also not an alternative algebra. A hyperbolic quaternion h has the form $h = h_1i_1 + h_2i_2 + h_3i_3 + h_4i_4$ where $\{i_1, i_2, i_3, i_4\}$ are hyperbolic quaternion units, which adhere to the rules

$$\begin{gathered} i_2^2 = i_3^2 = i_4^2 = i_2i_3i_4 = 1, \quad i_1 = 1, \\ i_3i_4 = -i_4i_3 = i_2, \quad i_4i_2 = -i_2i_4 = i_3, \quad i_2i_3 = -i_3i_2 = i_4. \end{gathered} \tag{1.6}$$

Consider an isotropic vector $(x, y, z) \in \mathbb{C}^3$, where $\mathbb{C}^3$ is the three-dimensional space referred to a system of orthogonal coordinates. Then the vector (x, y, z) satisfy $x^2 + y^2 + z^2 = 0$. Two numbers η_1 and η_2 can be associated with this vector as

$$x = \eta_1^2 - \eta_2^2, \quad y = i(\eta_1^2 + \eta_2^2), \quad z = -2\eta_1\eta_2.$$

By solving the above equations, we get

$$\eta_1 = \pm\sqrt{\frac{x - iy}{2}} \quad \text{and} \quad \eta_2 = \pm\sqrt{\frac{-x - iy}{2}}.$$

Thus, the spinor introduced by Cartan [1] can be defined as

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}. \tag{1.7}$$

In 1984, Vivarelli [9] defined a linear and injective correspondence between the quaternions and spinors. Let the sets of quaternions and spinors be denoted as $\mathbb{H}$ and $\mathbb{S}$, respectively. Then the correspondence is defined as follows.

Definition 1. Let $\phi : \mathbb{H} \to \mathbb{S}$ be a correspondence between a quaternion $q = a + be_1 + ce_2 + de_3 \in \mathbb{H}$ and a spinor $\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \in \mathbb{S}$. It is given by:

$$\phi(a + be_1 + ce_2 + de_3) = \begin{bmatrix} d + ia \\ b + ic \end{bmatrix} \equiv \eta. \tag{1.8}$$

Also, Vivarelli [9] has defined the correspondence between the products of two quaternions and a spinor product matrix given by

$$qp \to -i\hat{Q}P, \tag{1.9}$$

where P is the spinor corresponding to the quaternion p and $\hat{Q}$ is the complex, unitary, square matrix defined as

$$\begin{bmatrix} d + ia & b - ic \\ b + ic & -d + ia \end{bmatrix}. \tag{1.10}$$

Spinor conjugate to η is defined by Élie Cartan [1] as

$$\tilde{\eta} = iA\overline{\eta}, \tag{1.11}$$

where $\overline{\eta}$ is the complex conjugate of η and $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Finally, the mate of spinor η introduced by Castillo [2] is

$$\check{\eta} = -A\overline{\eta}. \tag{1.12}$$

Recently, Ericsir and Gungor [4] introduced the Fibonacci spinors using Fibonacci quaternions and studied their algebraic properties. Building on this, Kumari *et al.* [6], generalized the concept of Fibonacci spinors by introducing k-Fibonacci and k-Lucas spinors. Inspired by their work, this paper introduces and explores hyperbolic k-Mersenne and k-Mersenne-Lucas spinors.

2. Hyperbolic k-Mersenne and k-Mersenne-Lucas quaternions

In this section, we introduce the hyperbolic k-Mersenne and k-Mersenne-Lucas quaternions and their properties with some identities.

Definition 2. For $n \geq 0$, the nth hyperbolic k-Mersenne and k-Mersenne-Lucas quaternions $HM_{k,n}$ and $Hm_{k,n}$ are defined, respectively, as

$$HM_{k,n} = M_{k,n}i_1 + M_{k,n+1}i_2 + M_{k,n+2}i_3 + M_{k,n+3}i_4 = (M_{k,n}, M_{k,n+1}, M_{k,n+2}, M_{k,n+3}),$$
$$Hm_{k,n} = m_{k,n}i_1 + m_{k,n+1}i_2 + m_{k,n+2}i_3 + m_{k,n+3}i_4 = (m_{k,n}, m_{k,n+1}, m_{k,n+2}, m_{k,n+3}).$$

Definition 3. For $n \geq 0$, the conjugates of $HM_{k,n}$ and $Hm_{k,n}$ are defined by

1. $\overline{HM}_{k,n} = M_{k,n}i_1 - M_{k,n+1}i_2 - M_{k,n+2}i_3 - M_{k,n+3}i_4$.
2. $\overline{Hm}_{k,n} = m_{k,n}i_1 - m_{k,n+1}i_2 - m_{k,n+2}i_3 - m_{k,n+3}i_4$.

Definition 4. The norms of $HM_{k,n}$ and $Hm_{k,n}$ are defined by

1. $N_{HM_{k,n}} = M^2_{k,n} - M^2_{k,n+1} - M^2_{k,n+2} - M^2_{k,n+3}$.
2. $N_{Hm_{k,n}} = m^2_{k,n} - m^2_{k,n+1} - m^2_{k,n+2} - m^2_{k,n+3}$.

Theorem 1. *For $n \geq 0$, the following recurrence relations for $HM_{k,n}$ and $Hm_{k,n}$ are provided:*

$$HM_{k,n+2} = 3kHM_{k,n+1} - 2HM_{k,n}, \tag{2.1}$$

with $HM_{k,0} = i_2 + 3ki_3 + (9k^2 - 2)i_4$, $HM_{k,1} = i_1 + 3ki_2 + (9k^2 - 2)i_3 + (27k^3 - 12k)i_4$, *and*

$$Hm_{k,n+2} = 3kHm_{k,n+1} - 2Hm_{k,n}. \tag{2.2}$$

with $Hm_{k,0} = 2i_1 + 3ki_2 + (9k^2 - 4)i_3 + (27k^3 - 18k)i_4$, $Hm_{k,1} = 3ki_1 + (9k^2 - 4)i_2 + (27k^3 - 18k)i_3 + (81k^4 - 72k^2 + 8)i_4$.

Proof. It can be easily proved using Eqn.(1.1) and Definition 2. □

Theorem 2 (Binet's formula). *For $n \geq 0$, we have*

$$HM_{k,n} = \frac{\overline{\alpha}\alpha^n - \overline{\beta}\beta^n}{\alpha - \beta} \quad \text{and} \quad Hm_{k,n} = \overline{\alpha}\alpha^n + \overline{\beta}\beta^n,$$

where $\overline{\alpha} = i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4 = (1, \alpha, \alpha^2, \alpha^3)$, $\overline{\beta} = i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4 = (1, \beta, \beta^2, \beta^3)$.

Proof. Using Definition 2 and Eqn. (1.4), We have

$$\begin{aligned}
HM_{k,n} &= M_{k,n}i_1 + M_{k,n+1}i_2 + M_{k,n+2}i_3 + M_{k,n+3}i_4 \\
&= \frac{\alpha^n - \beta^n}{\alpha - \beta}i_1 + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}i_2 + \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}i_3 + \frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta}i_4 \\
&= \frac{\alpha^n(i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4) - \beta^n(i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4)}{\alpha - \beta} \\
&= \frac{\overline{\alpha}\alpha^n - \overline{\beta}\beta^n}{\alpha - \beta},
\end{aligned}$$

which completes the proof. The proof is similar for $Hm_{k,n}$. □

Corollary 1. *For $\overline{\alpha} = i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4$ and $\overline{\beta} = i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4$, we have*

1. $\overline{\alpha}\overline{\beta} = 15 + (5\beta - 3\alpha)i_2 + (3\alpha^2 - \beta^2)i_3 + (\alpha^3 + \beta^3 + 2\beta - 2\alpha)i_4.$
2. $\overline{\beta}\overline{\alpha} = 15 + (5\alpha - 3\beta)i_2 + (3\beta^2 - \alpha^2)i_3 + (\alpha^3 + \beta^3 + 2\alpha - 2\beta)i_4.$

Theorem 3 (Catalan's identity). *For any positive integers n, r such that $n \geq r$, we have*

$$\begin{aligned} HM_{k,n-r}HM_{k,n+r} - HM_{k,n}^2 = \frac{2^{n-r}}{9k^2-8}\Big[&15(2^{r+1} - m_{k,2r}) + (6k2^r - 5m_{k,2r+1} + 6m_{k,2r-1})i_2 \\ &+ (2^{r+1}m_{k,2} - 12m_{k,2r-2} + m_{k,2r+2})i_3 \\ &+ (2^{r+1}m_{k,3} - 8m_{k,2r-3} - m_{k,2r+3} - 2m_{k,2r+1} + 4m_{k,2r-1})i_4\Big] \end{aligned}$$

and

$$\begin{aligned} Hm_{k,n-r}Hm_{k,n+r} - Hm_{k,n}^2 = 2^{n+r}\Big[&15(m_{k,2r} - 2^{r+1}) + (5m_{k,2r+1} - 6m_{k,2r-1} - 2^r 6k)i_2 \\ &+ (12m_{k,2r-2} - m_{k,2r+2} + 2^r(8 - 18k^2))i_3 \\ &+ (m_{k,2r+3} + 8m_{k,2r-3} + 2m_{k,2r+1} - 4m_{k,2r-1} \\ &- 2^{r+1}(27k^3 - 18k))i_4\Big]. \end{aligned}$$

Proof. For the first identity, using Theorem 2, we write

$$\begin{aligned} HM_{k,n-r}HM_{k,n+r} - HM_{k,n}^2 &= \Big(\frac{\overline{\alpha}\alpha^{n-r} - \overline{\beta}\beta^{n-r}}{\alpha-\beta}\Big)\Big(\frac{\overline{\alpha}\alpha^{n+r} - \overline{\beta}\beta^{n+r}}{\alpha-\beta}\Big) - \Big(\frac{\overline{\alpha}\alpha^{n} - \overline{\beta}\beta^{n}}{\alpha-\beta}\Big)^2 \\ &= \frac{1}{(\alpha-\beta)^2}(-\overline{\alpha}\overline{\beta}\alpha^{n-r}\beta^{n+r} - \overline{\beta}\overline{\alpha}\beta^{n-r}\alpha^{n+r} + \overline{\alpha}\overline{\beta}\alpha^n\beta^n + \overline{\beta}\overline{\alpha}\beta^n\alpha^n) \\ &= \frac{1}{(\alpha-\beta)^2}(\overline{\alpha}\overline{\beta}\alpha^n\beta^n(1 - \alpha^{-r}\beta^r) + \overline{\beta}\overline{\alpha}\alpha^n\beta^n(1 - \alpha^r\beta^{-r})). \end{aligned}$$

After some mathematical calculations, we get

$$HM_{k,n-r}HM_{k,n+r} - HM_{k,n}^2 = \frac{2^{n-r}}{9k^2-8}(\overline{\alpha}\overline{\beta}(2^r - \beta^{2r}) + \overline{\beta}\overline{\alpha}(2^r - \alpha^{2r})).$$

Now using the corollary 1, we obtain

$$\begin{aligned} HM_{k,n-r}HM_{k,n+r} - HM_{k,n}^2 = \frac{2^{n-r}}{9k^2-8}\Big[&15(2^{r+1} - m_{k,2r}) + (6k2^r - 5m_{k,2r+1} + 6m_{k,2r-1})i_2 \\ &+ (2^{r+1}m_{k,2} - 12m_{k,2r-2} + m_{k,2r+2})i_3 \\ &+ (2^{r+1}m_{k,3} - 8m_{k,2r-3} - m_{k,2r+3} - 2m_{k,2r+1} + 4m_{k,2r-1})i_4\Big]. \end{aligned}$$

The second identity follows analogously. □

Here, we find the Cassini's identity which is the special case of Catalan's identity by putting $r = 1$.

Corollary 2 (Cassini's identity). *For $n \geq 1$, we have*

$$\begin{aligned} HM_{k,n-1}HM_{k,n+1} - HM_{k,n}^2 = \frac{2^{n-1}}{9k^2-8}[&15(8-9k^2) - (135k^3-120k)i_2 \\ &+ (81k^4-36k^2-32)i_3 - (243k^5-324k^3+96k)i_4] \end{aligned}$$

and

$$\begin{aligned} Hm_{k,n-1}Hm_{k,n+1} - Hm_{k,n}^2 = 2^{n+1}[&15(9k^2-8) + (135k^3-120k)i_2 \\ &- (81k^4-36k^2-32)i_3 + (243k^5-324k^3+96k)i_4]. \end{aligned}$$

Theorem 4 (d'Ocagne's identity). *For $n, t \in \mathbb{N}$ such that $n \geq t$, we have*

$$\begin{aligned} HM_{k,t+1}HM_{k,n} - HM_{k,t}HM_{k,n+1} = 2^t[&15M_{k,n-t} + (5M_{k,n-t+1} - 6M_{k,n-t-1})i_2 \\ &+ (12M_{k,n-t-2} - M_{k,n-t+2})i_3 \\ &+ (3kM_{k,n-t+2} - 4M_{k,n-t-1} + 8M_{k,n-t-3})i_4] \end{aligned}$$

and

$$\begin{aligned} Hm_{k,t+1}Hm_{k,n} - Hm_{k,t}Hm_{k,n+1} = 2^t(9k^2-8)[&-15M_{k,n-t} + (6M_{k,n-t-1} - 5M_{k,n-t+1})i_2 \\ &+ (M_{k,n-t+2} - 12M_{k,n-t-2})i_3 \\ &+ (4M_{k,n-t-1} - 3kM_{k,n-t+2} - 8M_{k,n-t-3})i_4]. \end{aligned}$$

Proof. The proof follows Binet fomulas with some mathematical calculations, recurrence relations Theorem 1 and using corollary 1. □

Theorem 5 (Vajda's identity). *For any natural numbers n, i, j, we have*

$$\begin{aligned} HM_{k,n+i}HM_{k,n+j} - HM_{k,n}HM_{k,n+i+j} = 2^n M_{k,i}\Big[&15M_{k,j} + (5M_{k,j+1} - 6M_{k,j-1})i_2 \\ &+ (12M_{k,j-2} - M_{k,j+2})i_3 \\ &+ (M_{k,j+3} + 8M_{k,j-3} + 2M_{k,j+1} - 4M_{k,j-1})i_4\Big] \end{aligned}$$

and

$$\begin{aligned} Hm_{k,n+i}Hm_{k,n+j} - Hm_{k,n}Hm_{k,n+i+j} = -2^n\sqrt{9k^2-8}M_{k,i}[&15M_{k,j} \\ &+ (5M_{k,j+1} - 6M_{k,j-1})i_2 + (12M_{k,j-2} - M_{k,j+2})i_3 \\ &+ (8M_{k,j-3} + M_{k,j+3} + 2M_{k,j+1} - 4M_{k,j-1})i_4]. \end{aligned}$$

Proof. Using Binet formulas and substituting the values of $\overline{\alpha}\overline{\beta}$ and $\overline{\beta}\overline{\alpha}$, we obtain the required results. □

Theorem 6. *The generating functions for the $HM_{k,n}$ and $Hm_{k,n}$ are*

$$G_{HM_{k,n}}(t) = \frac{ti_1 + i_2 + (3k-2t)i_3 + (9k^2 - 6kt - 2)i_4}{1 - 3kt + 2t^2}$$

and

$$G_{Hm_{k,n}}(t) = \frac{(2-3kt)i_1 + (3k-4t)i_2 + (9k^2 - 6kt - 4)i_3 + (27k^3 - 18k^2t - 18k + 8t)i_4}{1 - 3kt + 2t^2}.$$

Proof. Let $G_{HM_{k,n}}(t) = \sum_{n=0}^{\infty} HM_{k,n}t^n$ be the ordinary generating function for $HM_{k,n}$. Now consider the recurrence relation $HM_{k,n+2} = 3kHM_{k,n+1} - 2HM_{k,n}$. Then multiplying it by t^{n+2} and taking summation, we have

$$\begin{aligned}
& HM_{k,n+2} - 3kHM_{k,n+1} + 2HM_{k,n} = 0 \\
\implies & \sum_{n=0}^{\infty} HM_{k,n+2}t^{n+2} - 3k\sum_{n=0}^{\infty} HM_{k,n+1}t^{n+2} + 2\sum_{n=0}^{\infty} HM_{k,n}t^{n+2} = 0 \\
\implies & (G_{HM_{k,n}}(t) - HM_{k,0} - HM_{k,1}t) - 3kt(G_{HM_{k,n}}(t) - HM_{k,0}) + 2t^2 G_{HM_{k,n}}(t) = 0 \\
\implies & G_{HM_{k,n}}(t)(1 - 3kt + 2t^2) = HM_{k,0} + (HM_{k,1} - 3kHM_{k,0})t \\
\implies & G_{HM_{k,n}}(t) = \frac{HM_{k,0} + (HM_{k,1} - 3kHM_{k,0})t}{1 - 3kt + 2t^2} \\
\implies & G_{HM_{k,n}}(t) = \frac{ti_1 + i_2 + (3k-2t)i_3 + (9k^2 - 6kt - 2)i_4}{1 - 3kt + 2t^2}.
\end{aligned}$$

The second part follows analogously. □

Theorem 7 (Exponential generating function). *The exponential generating functions for the $HM_{k,n}$ and $Hm_{k,n}$ are*

$$E_{HM_{k,n}}(t) = \frac{\overline{\alpha}e^{\alpha t} - \overline{\beta}e^{\beta t}}{\alpha - \beta} \quad \textit{and} \quad E_{Hm_{k,n}}(t) = \overline{\alpha}e^{\alpha t} + \overline{\beta}e^{\beta t},$$

where $\overline{\alpha} = i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4$, $\overline{\beta} = i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4$.

Proof. Let

$$\begin{aligned}
E_{HM_{k,n}}(t) &= \sum_{n=0}^{\infty} HM_{k,n}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \Big(\frac{\overline{\alpha}\alpha^n - \overline{\beta}\beta^n}{\alpha - \beta}\Big)\frac{t^n}{n!} \\
&= \frac{1}{\alpha - \beta}\Big(\overline{\alpha}\sum_{n=0}^{\infty}\frac{(\alpha t)^n}{n!} - \overline{\beta}\sum_{n=0}^{\infty}\frac{(\beta t)^n}{n!}\Big) \\
&= \frac{\overline{\alpha}e^{\alpha t} - \overline{\beta}e^{\beta t}}{\alpha - \beta},
\end{aligned}$$

as required and the second part of the theorem is done analogously using Binet formula for $Hm_{k,n}$. □

Theorem 8 (Finite Sum). *For any positive integer n, we have*

1. $\sum_{j=0}^{n} HM_{k,j} = \frac{1}{3(k-1)}(HM_{k,n+1} - 2HM_{k,n} - HM_{k,0} + 2HM_{k,-1})$.
2. $\sum_{j=0}^{n} HM_{k,2j} = \frac{2}{9k^2+\sqrt{9k^2-8}-10}(HM_{k,2n+2} - 4HM_{k,2n} - HM_{k,0} + 4HM_{k,-2})$.
3. $\sum_{j=1}^{n} HM_{k,2j-1} = \frac{2}{9k^2+\sqrt{9k^2-8}-10}(HM_{k,2n-2} - 4HM_{k,2n-4} - HM_{k,0} + 4HM_{k,-2})$.

Proof. 1.

$$\begin{aligned}\sum_{j=0}^{n} HM_{k,j} &= \sum_{j=0}^{n}\Big(\frac{\overline{\alpha}\alpha^j - \overline{\beta}\beta^j}{\alpha-\beta}\Big)\\ &= \frac{1}{\alpha-\beta}\Big(\overline{\alpha}\sum_{j=0}^{n}\alpha^j - \overline{\beta}\sum_{j=0}^{n}\beta^j\Big)\\ &= \frac{1}{\alpha-\beta}\Big[\overline{\alpha}\Big(\frac{\alpha^{n+1}-1}{\alpha-1}\Big) - \overline{\beta}\Big(\frac{\beta^{n+1}-1}{\beta-1}\Big)\Big]\\ &= \frac{1}{\alpha-\beta}\Big[\frac{\overline{\alpha}(\alpha^{n+1}-1)(\beta-1) - \overline{\beta}(\beta^{n+1}-1)(\alpha-1)}{(\alpha-1)(\beta-1)}\Big].\end{aligned} \tag{2.3}$$

After some mathematical calculation, we get

$$\sum_{j=0}^{n} HM_{k,j} = \frac{1}{3(k-1)}(HM_{k,n+1} - 2HM_{k,n} - HM_{k,0} + 2HM_{k,-1}).$$

For second and third identity, the argument is similar. □

Theorem 9 (Finite Sum). *For any positive integer n, we have*

1. $\sum_{j=0}^{n} Hm_{k,j} = \frac{1}{3(k-1)}(Hm_{k,n+1} - 2Hm_{k,n} - Hm_{k,0} + 2Hm_{k,-1})$.
2. $\sum_{j=0}^{n} Hm_{k,2j} = \frac{2}{9k^2+\sqrt{9k^2-8}-10}(Hm_{k,2n+2} - 4Hm_{k,2n} - Hm_{k,0} + 4Hm_{k,-2})$.
3. $\sum_{j=1}^{n} Hm_{k,2j-1} = \frac{2}{9k^2+\sqrt{9k^2-8}-10}(Hm_{k,2n-2} - 4Hm_{k,2n-4} - Hm_{k,0} + 4Hm_{k,-2})$.

Proof. The argument is very similar to Theorem 8. □

3. Hyperbolic k-Mersenne and k-Mersenne-Lucas Spinors

We start by defining the hyperbolic k-Mersenne and k-Mersenne-Lucas spinors, denoted as $\{HMS_{k,n}\}_{n\geq 0}$ and $\{HmS_{k,n}\}_{n\geq 0}$, respectively, using the spinor definition. Following this, we evaluate their conjugates, mates, Binet-type formulas, generating functions, and various algebraic identities.
We consider the correspondence defined on the sets of hyperbolic k-Mersenne and k-Mersenne-Lucas quaternions denoted as HM and Hm, respectively, to the set of spinors S. The correspondences are defined as

$$\psi : HM \to S \quad \text{and} \quad \psi : Hm \to S$$

$$\psi(M_{k,n}i_1 + M_{k,n+1}i_2 + M_{k,n+2}i_3 + M_{k,n+3}i_4) = \begin{bmatrix} M_{k,n+3} + iM_{k,n} \\ M_{k,n+1} + iM_{k,n+2} \end{bmatrix} \equiv HMS_{k,n} \tag{3.1}$$

and

$$\psi(m_{k,n}i_1 + m_{k,n+1}i_2 + m_{k,n+2}i_3 + m_{k,n+3}i_4) = \begin{bmatrix} m_{k,n+3} + im_{k,n} \\ m_{k,n+1} + im_{k,n+2} \end{bmatrix} \equiv HmS_{k,n}. \tag{3.2}$$

Note that these transformations are linear and injective but not surjective and hence not bijective.
Now, if $\overline{HM}_{k,n} = M_{k,n}i_1 - M_{k,n+1}i_2 - M_{k,n+2}i_3 - M_{k,n+3}i_4$, is the conjugate of the hyperbolic quaternion $HM_{k,n}$, then the k-hyperbolic Mersenne spinor $HMS^*_{k,n}$ corresponding to $\overline{HM}_{k,n}$ is defined by

$$HMS^*_{k,n} = \begin{bmatrix} -M_{k,n+3} + iM_{k,n} \\ -M_{k,n+1} - iM_{k,n+2} \end{bmatrix}.$$

Similarly,

$$HmS^*_{k,n} = \begin{bmatrix} -m_{k,n+3} + im_{k,n} \\ -m_{k,n+1} - im_{k,n+2} \end{bmatrix}.$$

Now, by the above defined transformation, we introduce a new family of spinors with k-Mersenne and k-Mersenne-Lucas numbers.

Definition 5. For $n \geq 0$, the hyperbolic k-Mersenne and hyperbolic k-Mersenne-Lucas spinor sequences $\{HMS_{k,n}\}_{n\geq 0}$ and $\{HmS_{k,n}\}_{n\geq 0}$ are defined respctively, as

$$HMS_{k,n+2} = 3kHMS_{k,n+1} - 2HMS_{k,n}, \tag{3.3}$$

where $HMS_{k,0} = \begin{bmatrix} 9k^2-2 \\ 1+i3k \end{bmatrix}$ and $HMS_{k,1} = \begin{bmatrix} (27k^3-12k)+i \\ 3k+i(9k^2-2) \end{bmatrix}$, and

$$HmS_{k,n+2} = 3kHmS_{k,n+1} - 2HmS_{k,n}, \tag{3.4}$$

where $HmS_{k,0} = \begin{bmatrix} (27k^3-18k)+i2 \\ 3k+i(9k^2-4) \end{bmatrix}$ and $HmS_{k,1} = \begin{bmatrix} (81k^4-72k^2+8)+i3k \\ (9k^2-4)+i(27k^3-18k) \end{bmatrix}$.

From the above recurrence relation, it can be noted that the characteristic equation is the same as the k- Mersenne sequence, i.e. $x^2-3kx+2=0$, and its roots, α and β satisfy the following relations:

$$\alpha\beta = 2, \quad \alpha+\beta = 3k, \quad \alpha-\beta = \sqrt{9k^2-8}, \quad \frac{\alpha}{\beta} = \frac{\alpha^2}{2} \quad \text{and} \quad \frac{\beta}{\alpha} = \frac{\beta^2}{2}.$$

The complex conjugate of $HMS_{k,n}$ and $HmS_{k,n}$ can be written as

$$\overline{HMS}_{k,n} = \begin{bmatrix} M_{k,n+3} - iM_{k,n} \\ M_{k,n+1} - iM_{k,n+2} \end{bmatrix}, \quad \text{and} \quad \overline{HmS}_{k,n} = \begin{bmatrix} m_{k,n+3} - im_{k,n} \\ m_{k,n+1} - im_{k,n+2} \end{bmatrix}.$$

Using Eqn. (1.11), we write for the spinor conjugate of $HMS_{k,n}$ and $HmS_{k,n}$ as

$$H\tilde{M}S_{k,n} = iA\overline{HMS}_{k,n} = \begin{bmatrix} M_{k,n+2} + iM_{k,n+1} \\ -M_{k,n} - iM_{k,n+3} \end{bmatrix}, \quad \text{and} \quad H\tilde{m}S_{k,n} = \begin{bmatrix} m_{k,n+2} + im_{k,n+1} \\ -m_{k,n} - im_{k,n+3} \end{bmatrix}.$$

Also, the mate of $HMS_{k,n}$ and $HmS_{k,n}$ as defined in Eqn. (1.12) is given by

$$H\check{M}S_{k,n} = -A\overline{HMS}_{k,n} = \begin{bmatrix} -M_{k,n+1} + iM_{k,n+2} \\ M_{k,n+3} - iM_{k,n} \end{bmatrix}, \quad \text{and} \quad H\check{m}S_{k,n} = \begin{bmatrix} -m_{k,n+1} + im_{k,n+2} \\ m_{k,n+3} - im_{k,n} \end{bmatrix}.$$

Theorem 10 (Binet formula). *For $n \geq 0$, we have*

$$HMS_{k,n} = \frac{1}{\sqrt{9k^2-8}}\left(\begin{bmatrix} \alpha^3+i \\ \alpha+i\alpha^2 \end{bmatrix} \alpha^n - \begin{bmatrix} \beta^3+i \\ \beta+i\beta^2 \end{bmatrix} \beta^n \right)$$
$$\text{and} \quad HmS_{k,n} = \begin{bmatrix} \alpha^3+i \\ \alpha+i\alpha^2 \end{bmatrix} \alpha^n + \begin{bmatrix} \beta^3+i \\ \beta+i\beta^2 \end{bmatrix} \beta^n.$$

Proof. By using the theory of difference equations, we can write

$$HMS_{k,n} = C\alpha^n + D\beta^n. \tag{3.5}$$

And we have, $HMS_{k,0} = C + D = \begin{bmatrix} 9k^2-2 \\ 1+i3k \end{bmatrix}$ and $HMS_{k,1} = C\alpha + D\beta = \begin{bmatrix} (27k^3-12k)+i \\ 3k+i(9k^2-2) \end{bmatrix}$.

After some necessary calculations, we get $C = \frac{1}{\sqrt{9k^2-8}}\begin{bmatrix}\alpha^3+i\\ \alpha+i\alpha^2\end{bmatrix}$ and $D = -\frac{1}{\sqrt{9k^2-8}}\begin{bmatrix}\beta^3+i\\ \beta+i\beta^2\end{bmatrix}$. Thus, substituting the values of C and D in Eqn. (3.5), we obtain the first identity. The second identity can be proved using a similar approach. □

Theorem 11 (Catalan's identity). *For $n,\ r\in\mathbb{N}$, be such that $n\geq r$, we have*

$$
\begin{aligned}
&H\hat{M}S_{k,n-r}HMS_{k,n+r}-H\hat{M}S_{k,n}HMS_{k,n}\\
&\quad=\frac{2^{n-r}}{9k^2-8}\begin{bmatrix}15(m_{k,2r}-2^{r+1})+i(2^{r+1}m_{k,3}-8m_{k,2r-3}-m_{k,2r+3}-2m_{k,2r+1}+4m_{k,2r-1})\\(12m_{k,2r-2}-m_{k,2r+2}-2^{r+1}m_{k,2})+i(6k2^r-5m_{k,2r+1}+6m_{k,2r-1})\end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
&H\hat{m}S_{k,n-r}HmS_{k,n+r}-H\hat{m}S_{k,n}HmS_{k,n}\\
&\quad=2^{n+r}\begin{bmatrix}15(2^{r+1}-m_{k,2r})+i(m_{k,2r+3}+8m_{k,2r-3}+2m_{k,2r+1}-4m_{k,2r-1}-2^{r+1}(27k^3-18k))\\(m_{k,2r+2}-12m_{k,2r-2}-2^r(8-18k^2)+i(5m_{k,2r+1}-6m_{k,2r-1}-2^r6k)\end{bmatrix}.
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
&H\hat{M}S_{k,n-r}HMS_{k,n+r}-H\hat{M}S_{k,n}HMS_{k,n}=\frac{-1}{i}(HM_{k,n-r}HM_{k,n+r}-HM_{k,n}HM_{k,n})\\
&\quad=i\frac{2^{n-r}}{9k^2-8}\Big[15(2^{r+1}-m_{k,2r})+(6k2^r-5m_{k,2r+1}+6m_{k,2r-1})i_2\\
&\qquad+(2^{r+1}m_{k,2}-12m_{k,2r-2}+m_{k,2r+2})i_3\\
&\qquad+(2^{r+1}m_{k,3}-8m_{k,2r-3}-m_{k,2r+3}-2m_{k,2r+1}+4m_{k,2r-1})i_4\Big]\\
&\quad=i\frac{2^{n-r}}{9k^2-8}\Bigg(\begin{bmatrix}(2^{r+1}m_{k,3}-8m_{k,2r-3}-m_{k,2r+3}-2m_{k,2r+1}+4m_{k,2r-1})\\(6k2^r-5m_{k,2r+1}+6m_{k,2r-1})\end{bmatrix}\\
&\qquad+\begin{bmatrix}i(15(2^{r+1}-m_{k,2r}))\\i(2^{r+1}m_{k,2}-12m_{k,2r-2}+m_{k,2r+2})\end{bmatrix}\Bigg)\\
&\quad=\frac{2^{n-r}}{9k^2-8}\begin{bmatrix}15(m_{k,2r}-2^{r+1})+i(2^{r+1}m_{k,3}-8m_{k,2r-3}-m_{k,2r+3}-2m_{k,2r+1}+4m_{k,2r-1})\\(12m_{k,2r-2}-m_{k,2r+2}-2^{r+1}m_{k,2})+i(6k2^r-5m_{k,2r+1}+6m_{k,2r-1})\end{bmatrix}.
\end{aligned}
$$

Which completes the proof. Similarly, the second identity can be proved. □

We should note that the Catalan's identity is a generalization of the Cassini's identity. So, using Theorem 11 and putting r = 1, we have the Cassini's identity.

Corollary 3 (Cassini's identity). *For $n\in\mathbb{N}$, we have*

$$
H\hat{M}S_{k,n-1}HMS_{k,n+1}-H\hat{M}S_{k,n}HMS_{k,n}=\frac{2^{n-1}}{9k^2-8}\begin{bmatrix}15(9k^2-8)+i(324k^3-243k^5-96k)\\(32+36k^2-81k^4)+i(120k-135k^3)\end{bmatrix}
$$

and

$$
H\hat{m}S_{k,n-1}HmS_{k,n+1}-H\hat{m}S_{k,n}HmS_{k,n}=2^{n+1}\begin{bmatrix}15(8-9k^2)+i(243k^5-324k^3+96k)\\(81k^4-36k^2-32)+i(135k^3-120k)\end{bmatrix}.
$$

Theorem 12 (d'Ocagne's identity). *For $n,\ t \in \mathbb{N}$ such that $n \geq t$, we have*

$$\begin{aligned}&H\hat{M}S_{k,t+1}HMS_{k,n} - H\hat{M}S_{k,t}HMS_{k,n+1}\\ &\quad = 2^t \begin{bmatrix} -15M_{k,n-t} + i(3kM_{k,n-t+2} + 8M_{k,n-t-3} - 4M_{k,n-t-1}) \\ (M_{k,n-t+2} - 12M_{k,n-t-2}) + i(5M_{k,n-t+1} - 6M_{k,n-t-1}) \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}&H\hat{m}S_{k,t+1}HmS_{k,n} - H\hat{m}S_{k,t}HmS_{k,n+1}\\ &\quad = 2^t(9k^2-8) \begin{bmatrix} 15M_{k,n-t} + i(4M_{k,n-t-1} - 3kM_{k,n-t+2} - 8M_{k,n-t-3}) \\ (12M_{k,n-t-2} - M_{k,n-t+2}) + i(6M_{k,n-t-1} - 5M_{k,n-t+1}) \end{bmatrix}.\end{aligned}$$

Proof. We have

$$\begin{aligned}&H\hat{M}S_{k,t+1}HMS_{k,n} - H\hat{M}S_{k,t}HMS_{k,n+1}\\ &\quad = \frac{-1}{i}(HM_{k,t+1}HM_{k,n} - HM_{k,t}HM_{k,n+1})\\ &\quad = \frac{-1}{i}[2^t(15M_{k,n-t} + (5M_{k,n-t+1} - 6M_{k,n-t-1})i_2\\ &\quad + (12M_{k,n-t-2} - M_{k,n-t+2})i_3\\ &\quad + (3kM_{k,n-t+2} + 8M_{k,n-t-3} - 4M_{k,n-t-1})i_4)]\\ &\quad = 2^t i \begin{bmatrix} (3kM_{k,n-t+2} + 8M_{k,n-t-3} - 4M_{k,n-t-1}) + i(15M_{k,n-t}) \\ (5M_{k,n-t+1} - 6M_{k,n-t-1}) + i(12M_{k,n-t-2} - M_{k,n-t+2}) \end{bmatrix}\\ &\quad = 2^t \begin{bmatrix} -15M_{k,n-t} + i(3kM_{k,n-t+2} + 8M_{k,n-t-3} - 4M_{k,n-t-1}) \\ (M_{k,n-t+2} - 12M_{k,n-t-2}) + i(5M_{k,n-t+1} - 6M_{k,n-t-1}) \end{bmatrix}.\end{aligned}$$

Analogously, the second identity follows. □

Theorem 13 (Vajda's identity). *For $n,\ i,\ j \in \mathbb{N}$, we have*

$$\begin{aligned}&H\hat{M}S_{k,n+i}HMS_{k,n+j} - H\hat{M}S_{k,n}HMS_{k,n+i+j}\\ &\quad = 2^n M_{k,i} \begin{bmatrix} -15M_{k,j} + i(M_{k,j+3} + 8M_{k,j-3} + 2M_{k,j+1} - 4M_{k,j-1}) \\ (M_{k,j+2} - 12M_{k,j-2}) + i(5M_{k,j+1} - 6M_{k,j-1}) \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}&H\hat{m}S_{k,n+i}HmS_{k,n+j} - H\hat{m}S_{k,n}HmS_{k,n+i+j}\\ &\quad = 2^n\sqrt{9k^2-8}M_{k,i} \begin{bmatrix} 15M_{k,j} + i(4M_{k,j-1} - 2M_{k,j+1} - M_{k,j+3} - 8M_{k,j-3}) \\ (12M_{k,j-2} - M_{k,j+2}) + i(6M_{k,j-1} - 5M_{k,j+1}) \end{bmatrix}.\end{aligned}$$

Proof. We have

$$\begin{aligned}
&H\hat{M}S_{k,n+i}HMS_{k,n+j} - H\hat{M}S_{k,n}HMS_{k,n+i+j}\\
&\qquad = \frac{-1}{i}(HM_{k,n+i}HM_{k,n+j} - HM_{k,n}HM_{k,n+i+j})\\
&\qquad = \frac{-1}{i}2^n M_{k,i}\Big[15M_{k,j} + (5M_{k,j+1} - 6M_{k,j-1})i_2\\
&\qquad\quad + (12M_{k,j-2} - M_{k,j+2})i_3\\
&\qquad\quad + (M_{k,j+3} + 8M_{k,j-3} + 2M_{k,j+1} - 4M_{k,j-1})i_4\Big]\\
&\qquad = 2^n M_{k,i} i \begin{bmatrix} (M_{k,j+3} + 8M_{k,j-3} + 2M_{k,j+1} - 4M_{k,j-1}) + i15M_{k,j} \\ (5M_{k,j+1} - 6M_{k,j-1}) + i(12M_{k,j-2} - M_{k,j+2}) \end{bmatrix}\\
&\qquad = 2^n M_{k,i} \begin{bmatrix} -15M_{k,j} + i(M_{k,j+3} + 8M_{k,j-3} + 2M_{k,j+1} - 4M_{k,j-1}) \\ (M_{k,j+2} - 12M_{k,j-2}) + i(5M_{k,j+1} - 6M_{k,j-1}) \end{bmatrix}.
\end{aligned}$$

The same approach can be followed for the second identity. □

Theorem 14. *The generating functions for $HMS_{k,n}$ and $HmS_{k,n}$ are respectively, given as*

$$G_{HMS}(x) = \frac{1}{(1-3kx+2x^2)} \begin{bmatrix} (9k^2 - 6kx - 2) + ix \\ 1 + i(3k - 2x) \end{bmatrix}$$

and

$$G_{HmS}(x) = \frac{1}{(1-3kx+2x^2)} \begin{bmatrix} (27k^3 - 18k^2x - 18k + 8x) + i(2 - 3kx) \\ (3k - 4x) + i(9k^2 - 6kx - 4) \end{bmatrix}.$$

Proof. Let $G_{HMS}(x) = \sum_{n=0}^{\infty} HMS_{k,n}x^n$ be the ordinary generating function for $HMS_{k,n}$. Now consider the recurrence relation $HMS_{k,n+2} = 3kHMS_{k,n+1} - 2HMS_{k,n}$. Then multiplying it by x^{n+2} and taking summation, we have

$$\begin{aligned}
&HMS_{k,n+2} - 3kHMS_{k,n+1} + 2HMS_{k,n} = 0\\
&\implies \sum_{n=0}^{\infty} HMS_{k,n+2}x^{n+2} - 3k\sum_{n=0}^{\infty} HMS_{k,n+1}x^{n+2} + 2\sum_{n=0}^{\infty} HMS_{k,n}x^{n+2} = 0\\
&\implies (G_{HMS_{k,n}}(x) - HMS_{k,0} - HMS_{k,1}x) - 3kx(G_{HMS_{k,n}}(x) - HMS_{k,0})\\
&\qquad + 2x^2 G_{HMS_{k,n}}(x) = 0\\
&\implies G_{HMS_{k,n}}(x)(1 - 3kx + 2x^2) = HMS_{k,0} + (HMS_{k,1} - 3kHMS_{k,0})x\\
&\implies G_{HMS_{k,n}}(x) = \frac{HMS_{k,0} + (HMS_{k,1} - 3kHMS_{k,0})x}{1 - 3kx + 2x^2}\\
&\implies G_{HMS_{k,n}}(x) = \frac{1}{(1-3kx+2x^2)} \begin{bmatrix} (9k^2 - 6kx - 2) + ix \\ 1 + i(3k - 2x) \end{bmatrix}.
\end{aligned}$$

The second part follows analogously. □

For a sequence $\{a_n\}_{n\geq 0}$, let $G(x)=\sum_{n=0}^{\infty} a_n x^n$ be the ordinary generating function. Then, for sequences $\{a_{2n}\}_{n\geq 0}$ and $\{a_{2n+1}\}_{n\geq 0}$, the even and odd generating functions are given by, respectively

$$G_{a_{2n}}(x)=\frac{G(\sqrt{x})+G(-\sqrt{x})}{2} \quad \text{and} \quad G_{a_{2n+1}}(x)=\frac{G(\sqrt{x})-G(-\sqrt{x})}{2\sqrt{x}}. \tag{3.6}$$

Theorem 15. *The ordinary generating functions of even and odd indexed for $HMS_{k,n}$ are respectively, given by*

$$G_{HMS_{2n}}(x)=\frac{1}{1+4x^2+x(4-9k^2)}\left(\begin{bmatrix}(9k^2-2-4x)+i3kx\\(1+2x)+i3k\end{bmatrix}\right)$$

and

$$G_{HMS_{2n+1}}(x)=\frac{1}{1+4x^2+x(4-9k^2)}\left(\begin{bmatrix}3k(9k^2-4x-4)+i(1+2x)\\3k+i(9k^2-4x-2)\end{bmatrix}\right).$$

Theorem 16. *The ordinary generating functions of even and odd indexed for $HmS_{k,n}$ are respectively, given by*

$$G_{HmS_{2n}}(x)=\frac{1}{1+4x^2+x(4-9k^2)}\left(\begin{bmatrix}3k(9k^2-4x-6)+i(2+4x-9k^2x)\\3k(1-2x)+i(9k^2-8x-4)\end{bmatrix}\right)$$

and

$$G_{HmS_{2n+1}}(x)=\frac{1}{1+4x^2+x(4-9k^2)}\left(\begin{bmatrix}(8-72k^2+16x-36k^2x+81k^4)+i(3k-6kx)\\(9k^2-8x-4)+i(27k^3-18k-12kx)\end{bmatrix}\right).$$

Theorem 17. *The exponential generating functions for $HMS_{k,n}$ and $HmS_{k,n}$ are*

$$E_{HMS}(x)=\frac{1}{\sqrt{9k^2-8}}\left(\begin{bmatrix}\alpha^3+i\\\alpha+i\alpha^2\end{bmatrix}e^{\alpha x}-\begin{bmatrix}\beta^3+i\\\beta+i\beta^2\end{bmatrix}e^{\beta x}\right)$$

and

$$E_{HmS}(x)=\left(\begin{bmatrix}\alpha^3+i\\\alpha+i\alpha^2\end{bmatrix}e^{\alpha x}+\begin{bmatrix}\beta^3+i\\\beta+i\beta^2\end{bmatrix}e^{\beta x}\right).$$

Proof. Let $E_{HMS}(x)=\sum_{n=0}^{\infty} HMS_{k,n}\frac{x^n}{n!}$ be the exponential generating function for $HMS_{k,n}$, and then using Binet's formula, we write

$$\begin{aligned}E_{HMS}(x)&=\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{9k^2-8}}\left(\begin{bmatrix}\alpha^3+i\\\alpha+i\alpha^2\end{bmatrix}\alpha^n-\begin{bmatrix}\beta^3+i\\\beta+i\beta^2\end{bmatrix}\beta^n\right)\right)\frac{x^n}{n!}\\&=\frac{1}{\sqrt{9k^2-8}}\left(\begin{bmatrix}\alpha^3+i\\\alpha+i\alpha^2\end{bmatrix}\sum_{n=0}^{\infty}\frac{(\alpha x)^n}{n!}-\begin{bmatrix}\beta^3+i\\\beta+i\beta^2\end{bmatrix}\sum_{n=0}^{\infty}\frac{(\beta x)^n}{n!}\right)\\&=\frac{1}{\sqrt{9k^2-8}}\left(\begin{bmatrix}\alpha^3+i\\\alpha+i\alpha^2\end{bmatrix}e^{\alpha x}-\begin{bmatrix}\beta^3+i\\\beta+i\beta^2\end{bmatrix}e^{\beta x}\right).\end{aligned}$$

Similarly, the second identity can be proved using Binet's formula of $HmS_{k,n}$. □

Let $E(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ be the exponential generating function for a sequence $\{a_n\}_{n\geq 0}$. Then the exponential generating functions for even and odd indexed terms of the sequences $\{a_{2n}\}_{n\geq 0}$ and $\{a_{2n+1}\}_{n\geq 0}$, respectively, are

$$E_{a_{2n}}(x) = \frac{E(\sqrt{x}) + E(-\sqrt{x})}{2} \quad \text{and} \quad E_{a_{2n+1}}(x) = \frac{E(\sqrt{x}) - E(-\sqrt{x})}{2\sqrt{x}}. \tag{3.7}$$

Theorem 18. *The exponential generating functions of even and odd indexed for $HMS_{k,n}$ respectively, are*

$$E_{HMS_{2n}}(x) = \frac{1}{2\sqrt{9k^2-8}}\Big(\begin{bmatrix} \alpha^3 + i \\ \alpha + i\alpha^2 \end{bmatrix} cosh(\alpha\sqrt{x}) - \begin{bmatrix} \beta^3 + i \\ \beta + i\beta^2 \end{bmatrix} cosh(\beta\sqrt{x})\Big)$$

$$\text{and} \quad E_{HMS_{2n+1}}(x) = \frac{1}{2\sqrt{x(9k^2-8)}}\Big(\begin{bmatrix} \alpha^3 + i \\ \alpha + i\alpha^2 \end{bmatrix} sinh(\alpha\sqrt{x}) - \begin{bmatrix} \beta^3 + i \\ \beta + i\beta^2 \end{bmatrix} sinh(\beta\sqrt{x})\Big).$$

Theorem 19. *The exponential generating functions of even and odd indexed for $HmS_{k,n}$ respectively, are*

$$E_{HmS_{2n}}(x) = \frac{1}{2}\Big(\begin{bmatrix} \alpha^3 + i \\ \alpha + i\alpha^2 \end{bmatrix} cosh(\alpha\sqrt{x}) + \begin{bmatrix} \beta^3 + i \\ \beta + i\beta^2 \end{bmatrix} cosh(\beta\sqrt{x})\Big)$$

$$\text{and} \quad E_{HmS_{2n+1}}(x) = \frac{1}{2}\Big(\begin{bmatrix} \alpha^3 + i \\ \alpha + i\alpha^2 \end{bmatrix} sinh(\alpha\sqrt{x}) + \begin{bmatrix} \beta^3 + i \\ \beta + i\beta^2 \end{bmatrix} sinh(\beta\sqrt{x})\Big).$$

Theorem 20 (Finite sum formulas). *For the hyperbolic k-Mersenne spinor sequence, we have*

1. $\sum_{j=1}^{n} HMS_{k,j} = \frac{1}{3(k-1)} \begin{bmatrix} (M_{k,n+4} - 2M_{k,n+3} - 27k^3 + 18k^2 + 12k - 4) + i(M_{k,n+1} - 2M_{k,n} - 1) \\ (M_{k,n+2} - 2M_{k,n+1} + 2 - 3k) + i(M_{k,n+3} - 2M_{k,n+2} + 2 + 6k - 9k^2) \end{bmatrix}.$
2. $\sum_{j=1}^{n} HMS_{k,2j}$
$= \frac{1}{9(k^2-1)} \begin{bmatrix} ((9k^2-8)M_{k,2n+3} - 4M_{k,2n+1} - 81k^4 + 90k^2 - 12) + i(M_{k,2n+2} - 4M_{k,2n} - 3k) \\ (M_{k,2n+3} - 4M_{k,2n+1} + 6 - 9k^2) + i((9k^2-8)M_{k,2n+2} - 4M_{k,2n} + 24k - 27k^3) \end{bmatrix}.$
3. $\sum_{j=1}^{n} HMS_{k,2j-1}$
$= \frac{1}{9(k^2-1)} \begin{bmatrix} ((9k^2-8)M_{k,2n+2} - 4M_{k,2n} - 24k - 27k^3) + i(M_{k,2n+3} + (5-9k^2)M_{k,2n+1} - 3) \\ (M_{k,2n+2} - 4M_{k,2n} - 3k) + i(M_{k,2n+3} - 4M_{k,2n+1} + 6 - 9k^2) \end{bmatrix}.$

Proof. 1. Using mathematical induction on n, let $P(n) = \sum_{j=1}^{n} HMS_{k,j}$, then for the initial value $n = 1$, $P(1)$ is clearly true: $HMS_{k,1} = \begin{bmatrix} (27k^3 - 12k) + i \\ 3k + i(9k^2 - 2) \end{bmatrix}$ from Eqn. (3.3). Assume that the statement holds for some arbitrary positive integer t, i.e. ,

$$\sum_{j=1}^{t} HMS_{k,j} = \frac{1}{3(k-1)} \begin{bmatrix} (M_{k,t+4} - 2M_{k,t+3} - 27k^3 + 18k^2 + 12k - 4) + i(M_{k,t+1} - 2M_{k,t} - 1) \\ (M_{k,t+2} - 2M_{k,t+1} + 2 - 3k) + i(M_{k,t+3} - 2M_{k,t+2} + 2 + 6k - 9k^2) \end{bmatrix}.$$

Then, for $n = t + 1$, we have

$$\sum_{j=1}^{t+1} HMS_{k,j} = \frac{1}{3(k-1)} \begin{bmatrix} (M_{k,t+5} - 2M_{k,t+4} - 27k^3 + 18k^2 + 12k - 4) + i(M_{k,t+2} - 2M_{k,t+1} - 1) \\ (M_{k,t+3} - 2M_{k,t+2} + 2 - 3k) + i(M_{k,t+4} - 2M_{k,t+3} + 2 + 6k - 9k^2) \end{bmatrix}.$$

Starting with the left-hand side of the statement for $t+1$, using the inductive hypothesis $\sum_{j=1}^{t} HMS_{k,j}$ and Eqn. (3.1), we can write

$$\begin{aligned} \sum_{j=1}^{t+1} HMS_{k,j} &= \sum_{j=1}^{t} HMS_{k,j} + HMS_{k,t+1} \\ &= \frac{1}{3(k-1)} \begin{bmatrix} (M_{k,t+4} - 2M_{k,t+3} - 27k^3 + 18k^2 + 12k - 4) + i(M_{k,t+1} - 2M_{k,t} - 1) \\ (M_{k,t+2} - 2M_{k,t+1} + 2 - 3k) + i(M_{k,t+3} - 2M_{k,t+2} + 2 + 6k - 9k^2) \end{bmatrix} \\ &\quad + \begin{bmatrix} M_{k,t+4} + iM_{k,t+1} \\ M_{k,t+2} + iM_{k,t+3} \end{bmatrix}. \end{aligned}$$

By simplifying the above equation, we obtain the required result. Therefore, by mathematical induction, the statement is true. The second and third identities can be derived in a similar manner. □

Theorem 21 (Finite sum formulas). *For the hyperbolic k-Mersenne-Lucas spinor sequence, we have*

1. $\sum_{j=1}^{n} HmS_{k,j} = \frac{1}{3(k-1)} \begin{bmatrix} (m_{k,n+4} - 2m_{k,n+3} - 81k^4 + 54k^3 + 72k^2 - 36k - 8) + i(m_{k,n+1} - 2m_{k,n} - 3k + 4) \\ (m_{k,n+2} - 2m_{k,n+1} + 9k^2 + 6k + 4) + i(m_{k,n+3} - 2m_{k,n+2} - 27k^3 + 18k^2 + 18k - 8) \end{bmatrix}.$
2. $\sum_{j=1}^{n} HmS_{k,2j} = \frac{1}{9(k^2-1)} \begin{bmatrix} ((9k^2 - 8)m_{k,2n+3} - 4m_{k,2n+1} - 243k^5 + 378k^3 - 132k) + i(m_{k,2n+2} - 4m_{k,2n} - 9k^2 + 12) \\ (m_{k,2n+3} - 4m_{k,2n+1} - 27k^3 + 30k) + i((9k^2 - 8)m_{k,2n+2} - 4m_{k,2n} - 81k^4 + 99k^2 - 18) \end{bmatrix}.$
3. $\sum_{j=1}^{n} HmS_{k,2j-1} = \frac{1}{9(k^2-1)} \begin{bmatrix} ((9k^2 - 8)m_{k,2n+2} - 4m_{k,2n} - 81k^4 + 108k^2 - 24) + i(m_{k,2n+3} + (5 - 9k^2)m_{k,2n+1} + 3k) \\ (m_{k,2n+2} - 4m_{k,2n} - 9k^2 + 12) + i(m_{k,2n+3} - 4m_{k,2n+1} - 27k^3 + 30k) \end{bmatrix}.$

Proof. The results are obtained using the mathematical induction on n, which is very similar to Theorm 20. □

4. Conclusion

In this article, we present the hyperbolic k-Mersenne and k-Mersenne-Lucas quaternions and investigate their algebraic properties. Additionally, we introduce the hyperbolic k-Mersenne and k-Mersenne-Lucas spinors by using the relationship between spinors and hyperbolic quaternions. We also provide Binet-type formulas, several identities, generating functions, etc.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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