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Research Article

Algorithmic results on independent Roman {2}-domination

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Abstract: An independent Roman {2}-dominating function (IR2DF) $f : V \to \{0, 1, 2\}$ in a graph $G = (V, E)$ has the properties that $\sum_{u\in N(v)} f(u) \geq 2$ if $f(v) = 0$, and $f(u) = 0$ for $u \in N(v)$ if $f(v) \geq 1$, where $v \in V$. The weight of an IR2DF in a graph G is defined as the sum of its function value for all vertices, given by $\omega(f) = \sum_{v\in V} f(v)$. The independent Roman {2}-domination number of G, denoted $i_{\{R2\}}(G)$, is the minimum weight of an IR2DF on G. In this paper, we prove that the independent Roman {2}-domination problem (IR2D) is NP-complete, even when restricted to chordal bipartite graphs. We then give an exact formula for the IR2D in corona graphs. Finally, we present two linear-time algorithms for solving IR2D for proper interval graphs and block-cactus graphs, respectively.

Keywords: independent Roman {2}-domination, linear-time algorithm, NP-completeness, proper interval graph, block-cactus graph.

AMS Subject classification: 05C69, 05C85

1. Introduction

In this paper, all the graphs we considered are undirected graphs without loops or multiple edges. In a graph $G = (V, E)$, V is the set of vertices in G and E is the set of edges in G. The open neighborhood of a vertex v in the graph G, denoted by $N(v)$, is the set of all vertices in G that are adjacent to v. The closed neighborhood of v

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in the graph G is $N[v] = N(v) \cup \{v\}$. Let $H = (V_H, E_H)$ be a subgraph of G. The open neighborhood of a vertex v in the subgraph H is denoted by $N_H(v)$, while $N_H[v]$ denotes the closed neighborhood of a vertex v in H. Denote the set $\{1, 2, \ldots, k\}$ by $[k]$. Denote the cardinality of a set S by $|S|$. For $S \subseteq V(G)$, $G - S$ denotes the subgraph obtained from G by removing all vertices in S and all edges incident to the vertices in S, and $G[S]$ denotes the subgraph of G induced by S. For a vertex v in G, $G - v$ denotes the subgraph obtained from G by deleting v and all edges of G incident to v. A chord of a cycle C in a graph G is an edge in $E(G) \setminus E(C)$ whose endpoints both lie on C. A chordal graph is a simple graph in which every cycle of length greater than three has a chord [3].

Let D be a set of vertices in a graph $G = (V, E)$ such that $N_G(v) \cap D \neq \emptyset$ for all vertices $v \in V \setminus D$, then D is a dominating set of G [20]. The minimum domination problem of a graph G is to find the minimum cardinality of D. A set $I \subseteq V$ is an independent set of $G = (V, E)$, if $\forall u, v \in I$, $N(u) \cap \{v\} = \emptyset$. Denote the cardinality of the maximum independent set of G by $\alpha(G)$ [20]. If $I \subseteq V$ is both a dominating set and an independent set of G, then I is an independent dominating set of G. Denote the minimum cardinality of an independent dominating set of G by $i(G)$ [2].

The study of the Roman domination problem originated from the defense of the Roman Empire [25]. According to the legend, the rulers of the Romans sent troops to their cities to protect the Roman Empire from invasion. The city is considered safe if there is at least one troop in the city, or if there is no troops in the city but at least two troops in one of its adjacent cities. When an enemy is encroaching on a city without troops, the rulers could send an army from an adjacent city with two troops to the unprotected city under attack. Under the premise of securing all the cities, the number of troops devoted to the protection of the cities should be as small as possible. That is where the Roman domination problem arises.

Let $f : V(G) \to \{0, 1, 2\}$ be a function of a graph $G = (V, E)$ such that if $f(v) = 0$ for $v \in V$, then $\exists u \in N(v)$ such that $f(u) = 2$. In this case, f is a Roman dominating function of G. The weight of a function f of G is $\omega(f) = \sum_{v \in V} f(v)$. The Roman domination number, denoted $\gamma_R(G)$, is the minimum weight of a Roman dominating function on G. The Roman domination problem was initially studied by Cockayne et al. in 2004 [8]. After that, more and more researchers followed in their footsteps, and many new varieties of Roman domination have been developed in the recent decade [5].

Roman {2}-domination is one of the variants of Roman domination. Also known as Italian domination, it was initially studied by Chellali *et al.* [4]. Let $f : V(G) \to \{0, 1, 2\}$ be a function of a graph $G = (V, E)$ such that $\sum_{u \in N(v)} f(u) \geq 2$ if $f(v) = 0$ for $v \in V$. Then f is defined as the Roman {2}-dominating function of G. Similar to Roman domination, the weight of the Roman {2}-dominating function of G is the sum of its function value for all vertices. The minimum weight of the Roman {2}-dominating functions on G is called the Roman {2}-domination number, denoted as $\gamma_{\{R2\}}(G)$. Compared to Roman domination, in which a vertex assigned 0 can only be dominated by a vertex assigned 2, Roman {2}-domination allows a vertex assigned 0 to be dominated not only by a vertex assigned 2 but also by two vertices each assigned 1. This means Roman domination is a subcase of Roman {2}-domination.

Independent Roman {2}-domination, which was initially studied by Rahmouni and Chellali [6], is a variant of Roman {2}-domination with the independent property. Let $f : V(G) \to \{0, 1, 2\}$ be an independent Roman {2}-dominating function (IR2DF) on a graph $G = (V, E)$ that meets the following conditions: (i) $\sum_{u\in N(v)} f(u) \geq 2$ if $f(v) = 0$ for $v \in V$; (ii)$f(u) = 0$ for $u \in N(v)$, if $f(v) \geq 1$ for $v \in V$. There are many IR2DFs on G, and the functions with minimum weight among all IR2DFs in G are called $i_{\{R2\}}$-functions. The independent Roman {2}-domination number of G, denoted by $i_{\{R2\}}(G)$, is the minimum weight of an independent Roman 2-dominating function in G. The independent Roman {2}-domination problem is abbreviated as IR2D.

In [6], Rahmouni and Chellali showed that IR2D is NP-complete even when restricted to bipartite graphs. Note that Poureidi and Rad in [23] proved that IR2D is NP-complete for planar graphs. Furthermore, Chakradhar and Venkata in [21] demonstrated that IR2D is NP-complete even when restricted to chordal graphs, star convex bipartite graphs, comb convex bipartite graphs, tree convex bipartite graphs and dually chordal graphs. Linear-time algorithms for computing the value of $i_{\{R2\}}(T)$ in any tree are propose in [28], [7], [23] and [16], addressing the question posed by Rahmouni and Chellali in [6]. In [26], it is shown that a linear-time algorithm based on dynamic programming can be used to find $i_{\{R2\}}(G)$ for any connected block graph G.

In this paper, we present several complexity and algorithmic results for independent Roman {2}-domination. In Section 2, we demonstrate that IR2D is NP-complete even when restricted to chordal bipartite graphs. In Section 3, we propose an exact formula for computing IR2D in corona graphs. In Section 4, we provide a linear-time algorithm for solving IR2D in proper interval graphs. Finally, in Section 5, we present a linear-time algorithm for solving IR2D in block-cactus graphs.

2. Complexity result

In this section, we present the complexity result for independent Roman {2}-domination problem (IR2D) restricted to chordal bipartite graphs. First, we review the definition of chordal bipartite graphs.

Definition 1 ([11]). A bipartite graph is an undirected graph whose vertices can be partitioned into two disjoint sets, X and Y, such that every edge has one vertex in X and the other in Y. A bipartite graph is chordal bipartite if each cycle of length at least 6 has a chord.

In order to prove the complexity result for IR2D restricted to chordal bipartite graphs, we reduce the well-known NP-complete problem, independent dominating set problem (IDS)[19], to IR2D. The definitions of these two problems are as follows.

Independent Roman {2}-domination problem **(IR2D)**

Instance: A graph $G = (V, E)$, a positive integer $k \leq |V|$.
Question: Does G have an IR2DF f with weight $\omega(f) \leq k$?

Independent dominating set problem **(IDS)**

Instance: A graph $G = (V, E)$, a positive integer $k \leq |V|$.
Question: Does there exist an independent dominating set I in graph G such that $|I| \leq k$?

Theorem 1. *The independent Roman {2}-domination problem (IR2D) is NP-complete for chordal bipartite graphs.*

Proof. Clearly, IR2D belongs to NP. Let $G = (V, E)$ be a chordal bipartite graph with $V = \{v_1, v_2, \ldots, v_n\}$. Let U be the graph with n isolated vertices $\{u_1, u_2, \ldots, u_n\}$. Let G' be the graph obtained from the disjoint unions of G and U by adding edges between v_i and u_i for $i \in [n]$. This construction of G' is illustrated in Figure 1. Obviously, G' is a chordal bipartite graph. We now begin the reduction from IDS to IR2D.

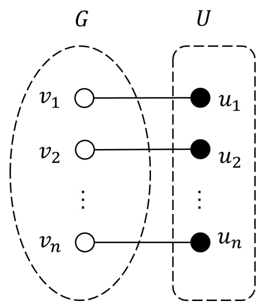

Figure 1. **The construction of the graph G'.**

Claim 1. *If there exists an independent dominating set I of G such that $|I| \leq k$, then there exists an IR2DF of G' with a weight of at most $n + k$.*

Proof of Claim 1. Suppose I is an independent dominating set of G such that $|I| \leq k$.

Let f be a function on G' defined as follows:

$$f(v) = \begin{cases} 2, & \text{if } v \in I \\ 0, & \text{if } v \in N_{G'}(u) \text{ for } u \in I \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, f is an IR2DF of G' with a weight of at most $2|I|+(n-|I|) = n+|I| \leq n+k$.♦

Claim 2. *If there exists an IR2DF of G' with a weight of at most $n+k$, then there exists an independent dominating set I of G such that $|I| \leq k$.*

Proof of Claim 2. If $k = n$, the result holds. Thus, we may assume that $k < n$. Let g be an IR2DF of G' with weight $\omega(g) \leq k+n$. Now we discuss the situations of the vertices in U under g. If $g(v_i) = 1$, then $g(u_i) = 0$ and u_i is not dominated. Therefore, the vertices in G are assigned either 0 or 2 under g.

Case 1. All the vertices in G are assigned 0 under g.
In this case, the weight of g is $2n$, which is more than the condition that $\omega(g) \leq k+n$. This leads to a contradiction.

Case 2. There exists some vertices in G assigned 2 under g.
If $g(v_t) = 0$ and $\sum_{w \in N_G(v_t)} g(w) = 0$, then we have $g(u_t) = 2$. Therefore, we can construct a new IR2DF g_1 such that $g_1(v_t) = 2$, $g_1(u_t) = 0$, and $g_1(x) = g(x)$ otherwise. Based on the IR2DF g_1, we can similarly construct another IR2DF g_2. Repeatedly, we can obtain a completely new IR2DF g_k such that if $g_k(v_i) = 0$, then $\sum_{w \in N_G(v_i)} g_k(w) \geq 2$, and $\omega(g_t) = \omega(g)$.
Now, assume that there are x vertices in U assigned a value greater than 0. Then, there are $n-x$ vertices in U assigned a value of 0. Moreover, there are $n-x$ vertices in G assigned 2 and x vertices in G assigned 0. Clearly, $\sum_{i \in [n]} g(u_i) \geq x$ and $\omega(g) = \sum_{i \in [n]} g(u_i) + 2(n-x) \geq x + 2(n-x) = 2n-x$. Since $\omega(g) \leq k+n$, it follows that $2n-x \leq k+n$. Hence, $n-x \leq k$. Let S be the set of vertices assigned 2 in G. We have $|S| \leq k$ and for all $v, u \in S$, $u \notin N(v)$. All vertices assigned 0 in G can be dominated by the vertices in S. Consequently, S is an independent dominating set of G with no more than k vertices. ♦
By Claim 1 and Claim 2, we can conclude that there exists an independent dominating set I in G such that $|I| \leq k$ if and only if there exists an IR2DF with a weight of at most $k+n$ in G'. Therefore, IR2D is an NP-complete problem for chordal bipartite graphs. □

3. IR2D on corona graphs

In this section, we propose an exact formula for IR2D in corona graphs, focusing specifically on the corona of two graphs. First, we will review the definition of corona graphs.

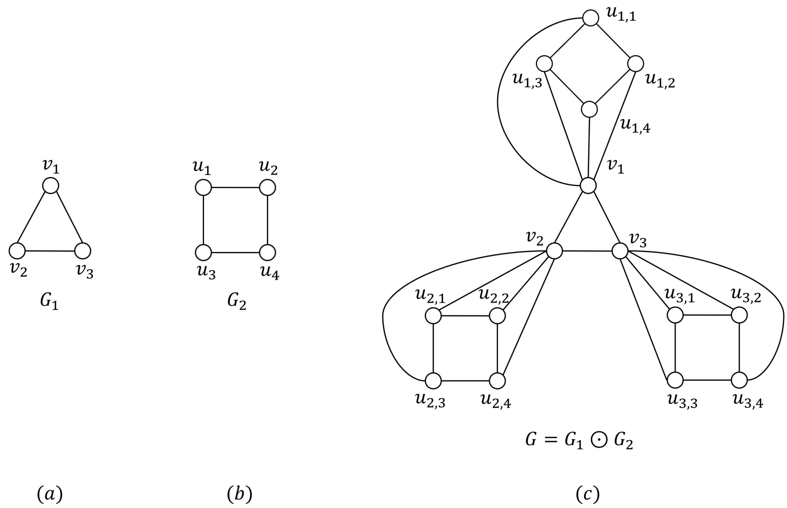

Figure 2. **the graph G_1 (a), the graph G_2 (b) and the corona graph $G = G_1 \odot G_2$ (c).**

Definition 2 ([9]). The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 (where G_k has p_k vertices and q_k edges) is defined as the graph G obtained by taking one copy of G_1 and p_1 copies of G_2, and then joining the i-th vertex of G_1 to every vertex in the i-th copy of G_2 by an edge. An example of a corona graph G is shown in Figure 2.

Let G_1 and G_2 be two graphs with p and q vertices, respectively. Let $G = G_1 \odot G_2$ be the corona graph, and V be the set of vertices in G. Denote the p copies of G_2 by $G_{2,1}, G_{2,2}, \dots, G_{2,p}$. According to Definition 2, we have $V = V(G_1) \cup V(G_{2,1}) \cup V(G_{2,2}) \cup \dots \cup V(G_{2,p})$. Denote $V(G_1) = \{v_1, v_2, \dots, v_p\}$, where v_i is adjacent to all vertices in $V(G_{2,i})$ for $i \in [p]$.

Lemma 1. *Let G_1 and G_2 be connected graphs with p and q vertices, respectively. Let $G = G_1 \odot G_2$ be a corona graph. If f is an IR2DF of G, then $f(v_i) \neq 1$ for $v_i \in V(G_1)$, where $i \in [p]$.*

Proof. Let f be an IR2DF of the corona graph $G = G_1 \odot G_2$. Assume that there is a vertex $v_i \in V(G_1)$ assigned 1 under f. By Definition 2, v_i is adjacent to all vertices in $V(G_{2,i})$. The vertices in $V(G_{2,i})$ must be assigned 0 but can not be defended, which contradicts the assumption that f is an IR2DF. Therefore, if f is an IR2DF of G, then $f(v_i) \neq 1$ for $v_i \in V(G_1)$, where $i \in [p]$. □

Proposition 1. *Let G be a connected graph with at least 2 vertices, then $i_{\{R2\}}(G) \geq 2$.*

Theorem 2. *Let G_1 and G_2 be connected graphs with $p = |V(G_1)|$ and $q = |V(G_2)|$, respectively. Let $G = G_1 \odot G_2$ be the corona graph with vertex set $V(G) = V(G_1) \cup V(G_{2,1}) \cup V(G_{2,2}) \cup \cdots \cup V(G_{2,p})$. We have*

$$i_{\{R2\}}(G) = \begin{cases} |V(G_1)| + i(G_1), & \text{if } i_{\{R2\}}(G_2) = 1 \\ 2|V(G_1)|, & \text{if } i_{\{R2\}}(G_2) = 2 \\ 2\alpha(G_1) + (|V(G_1)| - \alpha(G_1))\, i_{\{R2\}}(G_2), & \text{if } i_{\{R2\}}(G_2) > 2. \end{cases}$$

Proof. Let f be an IR2DF of the corona graph $G = G_1 \odot G_2$ with minimum weight. Let $u_{i,j}$ be the vertex in $V(G_{2,i})$ and v_i be the vertex in $V(G_1)$ for $i \in [p]$ and $j \in [q]$. By Lemma 1, we can deduce that no vertex in G_1 can be assigned the value 1. Therefore, the following discussion will not consider the case where a vertex in G_1 is assigned the value 1. We now discuss the following cases.

Case 1. $i_{\{R2\}}(G_2) = 1$.
By Proposition 1, if $i_{\{R2\}}(G_2) = 1$, then $|V(G_2)| = 1$. Let $ID \subset V(G_1)$ be an independent dominating set in G_1 with $|ID| = i(G_1)$. We can define an IR2DF f' on G in this case as follows:

$$f'(v) = \begin{cases} 2, & \text{if } v \in ID \\ 1, & \text{if } v \in N_{G_{2,i}}(u), \text{ for } i \in [p] \text{ and } u \in V(G_1) \setminus ID \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, we have $\omega(f') = 2|ID| + |V(G_1)| - |ID| = |V(G_1)| + i(G_1)$. Hence, we can deduce that $i_{\{R2\}}(G) \le |V(G_1)| + i(G_1)$.
Now we show that $i_{\{R2\}}(G) \ge |V(G_1)| + i(G_1)$. Suppose there exists a vertex in $G_{2,i}$ for $i \in [p]$ assigned 2 under f and $\sum_{w \in N_{G_1}(v_i)} f(w) \ge 1$. In this case, we can find another function f_1 such that $f_1(u_{i,1}) = 1$ and $f_1(x) = f(x)$ for $x \in V(G) \setminus \{u_{i,1}\}$. Clearly, f_1 is an IR2DF with a weight smaller than f, which contradicts the assumption that f is an $i_{\{R2\}}$-function.
If there exists a vertex $u_{i,1} \in V(G_{2,i})$ for $i \in [p]$ assigned 2 under f and $\sum_{w \in N_{G_1}(v_i)} f(w) = 0$, then the assignments of v_i and $u_{i,1}$ can be exchanged. Therefore, we consider that f is an $i_{\{R2\}}$-function of G such that $f(u) \in \{0,1\}$ for $u \in G_{2,i}$, and $f(v) \in \{0,2\}$ for $v \in G_1$. Consequently, vertices in G_1 are dominated only by vertices in G_1, and there must be at least $i(G1)$ vertices in G_1 assigned 2. Hence, we have:

$$\omega(f) \ge 2i(G_1) + |V(G_1)| - i(G_1) = |V(G_1)| + i(G_1).$$

Therefore, if $i_{\{R2\}}(G_2) = 1$, then $i_{\{R2\}}(G) = |V(G_1)| + i(G_1)$.

Case 2. $i_{\{R2\}}(G_2) = 2$.
In this case, if $f(v_i) = 0$, then the vertices in $G_{2,i}$ can be assigned 0,1, or 2. Since v_i does not dominate the vertices in $G_{2,i}$, the restriction of f to $G_{2,i}$ is an IR2DF, and thus $\sum_{v \in V(G_{2,i})} f(v) \ge i_{\{R2\}}(G_2) = 2$. If $f(v_i) = 2$, then the vertices in $G_{2,i}$ must be assigned 0. Let f_2 be an IR2DF on G that $f_2(v_i) = 0$ and $\sum_{u \in G_{2,i}} f_2(u) = i_{\{R2\}}(G_2) = 2$ for $i \in [p]$. Clearly, $\omega(f) \le \omega(f_2) = 2|V(G_1)|$.
Assume that there are x vertices in G_1 assigned 2 under f. We have:

$\omega(f) = 2x+(|V(G_1)|-x)\cdot\sum_{v\in V(G_{2,i})} f(v) \geq 2x+(|V(G_1)|-x)\cdot i_{\{R2\}}(G_2) = 2|V(G_1)|$.

Therefore, if $i_{\{R2\}}(G_2) = 2$, then $i_{\{R2\}}(G) = 2|V(G_1)|$ holds.

Case 3. $i_{\{R2\}}(G_2) > 2$.

Let f_3 be an IR2DF on G when $i_{\{R2\}}(G_2) > 2$. Assume that there are x vertices in G_1 assigned 2. We have:

$$\omega(f_3) = 2x + (|V(G_1)| - x) \cdot i_{\{R2\}}(G_2) = (2 - i_{\{R2\}}(G_2))x + i_{\{R2\}}(G_2)|V(G_1)|.$$

As we can see, $\omega(f_3)$ is monotonically decreasing function of x. As x increases, the value of $\omega(f_3)$ decreases. The value of x ranges from 0 to $\alpha(G_1)$, where $\alpha(G_1)$ is the maximum number of vertices in G_1 that can be assigned the value 2. Therefore, the minimum value of $\omega(f_3)$ is achieved when $x = \alpha(G_1)$. Thus, we have:

$$i_{\{R2\}}(G) = (2 - i_{\{R2\}}(G_2))\alpha(G_1) + i_{\{R2\}}(G_2)|V(G_1)|.$$

When $i_{\{R2\}}(G_2) > 2$, this simplifies to:

$$i_{\{R2\}}(G) = 2\alpha(G_1) + (|V(G_1)| - \alpha(G_1))i_{\{R2\}}(G_2).$$

□

4. IR2D on proper interval graphs

In this section, we present a linear-time algorithm for IR2D in proper interval graphs. If G is a disconnected proper interval graph, then clearly $i_{\{R2\}}(G)$ equals to the sum of the independent Roman {2}-domination numbers of its connected components. Therefore, in this section we only consider connected proper interval graphs.

Let $G = (V, E)$ be an interval graph with vertices $V = \{v_1, v_2, \ldots, v_n\}$. The set $F = \{I_{v_i} | v_i \in V\}$ of intervals on the real line, representing the graph G, can be abbreviated by $F = \{I_1, I_2, \ldots, I_n\}$ for $i \in [n]$. Denote the left value of the interval I_i by $l(I_i)$ and the right value of the interval I_i by $r(I_i)$ for $i \in [n]$. Let I_i and I_j be two intervals in F. If $l(I_i) < l(I_j) < r(I_j) < r(I_i)$, then I_j is properly contained in I_i [17].

Definition 3 ([10][17]). An interval graph $G = (V, E)$ is a graph with vertices $V = \{v_1, v_2, \ldots, v_n\}$ that can be represented by a set $F = \{I_{v_i} | v_i \in V\}$ of intervals on the real line, such that $v_iv_j \in E$ if and only if $I_{v_i} \cap I_{v_j} \neq \emptyset$. The set of intervals F is also called the interval model of the graph G.

Definition 4 ([18]). A proper interval graph is the intersection graph of a family of closed intervals on the real line, where no interval is properly contained within another.

A vertex v in G whose neighborhood $N_G(v)$ forms a clique is called a simplicial vertex. A perfect elimination ordering (PEO) of G is a sequence $\sigma = (v_1, v_2, \ldots, v_n)$ of all vertices in G such that each v_i is a simplicial vertex in the subgraph induced by $\{v_i, v_{i+1}, \ldots, v_n\}$. A chordal graph G has a bicompatible elimination ordering (BCO) if both σ and its reverse σ^{-1} are PEOs of G. The following theorem characterizes the classes of graphs in terms of BCO.

Theorem 3 ([15]). *A graph G has a BCO if and only if it is a proper interval graph.*

By Theorem 3, we can see that there exists a BCO for any proper interval graphs. A BCO of a proper interval graph $G = (V, E)$ can be computed sequentially in $O(|V| + |E|)$ time [22].

Lemma 2 ([22]). *Let $\sigma = (v_1, v_2, \ldots, v_n)$ be a BCO of a proper interval graph $G = (V, E)$. If $v_i v_j \in E$, then $v_k v_j \in E$ for all k, where $i \leq k \leq j - 1$.*

Denote the algorithm presented in [22] for computing a BCO of a proper interval graph by ALGORITHM-BCO. Let σ be a BCO of G obtained by ALGORITHM-BCO. In this section, we propose a linear-time algorithm for computing independent Roman $\{2\}$-domination number on proper interval graphs based on BCO.

Definition 5. Let G be a proper interval graph with n vertices. Let $\sigma = (v_1, v_2, \ldots, v_n)$ be a BCO of G. For each i where $1 \leq i \leq n$, let G_i be the induced subgraph of G with vertex set $\{v_i, v_{i+1}, \ldots, v_n\}$. Denote $MaxN(i) = max\{j | v_j \in N_{G_i}[v_i]\}$. Denote $MaxNN(i) = max\{j | v_j \in N_{G_i}[v_{MaxN(i)}]\}$. Denote the induced subgraph of G with vertex set $\{v_i, \ldots, v_{MaxN(i)}\}$ by Mc_i.

Lemma 3. *Let G be a proper interval graph with a BCO $\sigma = (v_1, v_2, \ldots, v_n)$. For any $1 \leq i < j \leq n$, it holds that $MaxN(i) \leq MaxN(j)$.*

Proof. Assume that $MaxN(i) > MaxN(j)$ when $i < j$, then $i < j < MaxN(j) < MaxN(i)$. According to Lemma 2, the subgraph Mc_i induced by $\{v_i, \ldots, v_{MaxN(i)}\}$ is a clique. Consequently, v_j is adjacent to $v_{MaxN(i)}$. This contradicts the assumption that $MaxN(j)$ is the maximum index of vertices adjacent to v_j. Hence, it must be that $MaxN(i) \leq MaxN(j)$ for all $1 \leq i < j \leq n$.

□

Lemma 4. *Let G be a proper interval graph with a BCO $\sigma = (v_1, v_2, \ldots, v_n)$. For $1 \leq t \leq n$, let G_t be the induced subgraph of G with vertex set $\{v_t, v_{t+1}, \ldots, v_n\}$. Then, $i_{\{R2\}}(G_{i-1}) \geq i_{\{R2\}}(G_i)$ for $2 \leq i \leq n$.*

Proof. Assume that $i_{\{R2\}}(G_{i-1}) < i_{\{R2\}}(G_i)$ for $2 \leq i \leq n$. Let f be an $i_{\{R2\}}-$function on G_{i-1} with weight $i_{\{R2\}}(G_{i-1})$. Let $g : V(G_i) \to \{0, 1, 2\}$ be a function on G_i. Since $G_i = G_{i-1} - v_{i-1}$, we consider the following three cases for $f(v_{i-1})$.

Case 1. $f(v_{i-1}) = 0$.
Let $g(x) = f(x)$ for all $x \in V(G_i)$. Clearly, g is an IR2DF on G_i and $i_{\{R2\}}(G_i) \leq \omega(g) = i_{\{R2\}}(G_{i-1})$. This contradicts the assumption that $i_{\{R2\}}(G_{i-1}) < i_{\{R2\}}(G_i)$ for $2 \leq i \leq n$.

Case 2. $f(v_{i-1}) = 1$.
In this case, we should consider whether v_{i-1} dominates some vertices in $V(G_{i-1})$ with another vertex $u \in V(G_{i-1})$ assigned 1.
If there exists some vertices in $V(G_i)$ assigned 0 that are dominated by v_{i-1} and u, then let $g(x) = f(x)$ where $x \in V(G_i) \setminus \{u\}$ and $g(u) = f(u) + 1$. Intuitively, g is an IR2DF on G_i. We have $i_{\{R2\}}(G_i) \leq \omega(g) = i_{\{R2\}}(G_{i-1}) - 1 + 1 = i_{\{R2\}}(G_{i-1})$, which is a contradiction.
If there is no vertex $u \in V(G_i)$ assigned 1 such that some vertices in $V(G_i)$ assigned 0 are dominated by v_i and u, then let $g(x) = f(x)$ where $x \in V(G_i)$. Clearly, g is an IR2DF on G_i and $i_{\{R2\}}(G_i) \leq \omega(g) = i_{\{R2\}}(G_{i-1}) - 1$. This contradicts the assumption that $i_{\{R2\}}(G_i) > i_{\{R2\}}(G_{i-1})$.

Case 3. $f(v_{i-1}) = 2$.
In this case, v_{i-1} can dominate the vertices in Mc_{i-1}, and all vertices in $V(Mc_{i-1} - v_{i-1})$ are assigned 0. Since $MaxN(i-1) \leq MaxN(i)$, if $MaxN(i-1) = MaxN(i)$, then let $g(x) = f(x)$ where $x \in V(G_i) \setminus \{v_i\}$ and $g(v_i) = 2$. Clearly, g is an IR2DF on G_i. We have $i_{\{R2\}}(G_i) \leq \omega(g) = i_{\{R2\}}(G_{i-1}) - 2 + 2 = i_{\{R2\}}(G_{i-1})$, a contradiction.
Now we discuss the situations when $MaxN(i-1) < MaxN(i)$.

Subcase 3.1. $\sum_{k \in [MaxN(i-1)+1, MaxN(i)]} f(v_k) = 0$.
In this subcase, all vertices in Mc_i are assigned 0 under f. Let $g(x) = f(x)$ where $x \in V(G_i) \setminus \{v_i\}$ and $g(v_i) = 2$. Clearly, g is an IR2DF on G_i. Therefore, $i_{\{R2\}}(G_i) \leq \omega(g) = i_{\{R2\}}(G_{i-1}) - 2 + 2 = i_{\{R2\}}(G_{i-1})$, a contradiction.

Subcase 3.2. $\sum_{k \in [MaxN(i-1)+1, MaxN(i)]} f(v_k) = 1$. Let v_p be the vertex of G_{i-1} assigned 1 under f for $p \in [MaxN(i-1)+1, MaxN(i)]$. In this case, let $g(x) = f(x)$ where $x \in V(G_i) \setminus \{v_p\}$ and $g(v_p) = f(v_p) + 1$. Clearly, g is an IR2DF on G_i and $i_{\{R2\}}(G_i) \leq \omega(g) = i_{\{R2\}}(G_{i-1}) - 2 + 1 = i_{\{R2\}}(G_{i-1}) - 1$. This contradicts the assumption that $i_{\{R2\}}(G_i) > i_{\{R2\}}(G_{i-1})$.

Subcase 3.3. $\sum_{k \in [MaxN(i-1)+1, MaxN(i)]} f(v_k) = 2$.
Let v_q be the vertex of G_{i-1} assigned 2 under f for $q \in [MaxN(i-1)+1, MaxN(i)]$. In this case, let $g(x) = f(x)$ where $x \in V(G_i)$. Clearly, g is an IR2DF with a weight of $i_{\{R2\}}(G_{i-1}) - 2$. We have $i_{\{R2\}}(G_i) \leq i_{\{R2\}}(G_{i-1}) - 2$ which contradicts the assumption that $i_{\{R2\}}(G_i) > i_{\{R2\}}(G_{i-1})$.
To sum up, for any induced subgraph G_i and G_{i-1} of a proper interval graph G, it holds that $i_{\{R2\}}(G_i) \leq i_{\{R2\}}(G_{i-1})$ for $2 \leq i \leq n$. □

Corollary 1. *Let G be a proper interval graph with a BCO $\sigma = (v_1, v_2, \ldots, v_n)$. For any induced subgraphs G_t and G_s of G where G_t is defined by vertex set $\{v_t, v_{t+1}, \ldots, v_n\}$ and G_s by vertex set $\{v_s, v_{s+1}, \ldots, v_n\}$ with $1 \leq t < s \leq n$, it holds that $i_{\{R2\}}(G_t) \geq i_{\{R2\}}(G_s)$.*

Let G be a proper interval graph with a BCO $\sigma = (v_1, v_2, \ldots, v_n)$. Let G_i be the induced subgraph of G with vertex set $\{v_i, v_{i+1}, \ldots, v_n\}$ for $1 \le i \le n$. Let v_i be the root of G_i. We now consider the following three problems.

- $\gamma^0(G_i, v_i)$=min$\{\omega(f)$: f is an IR2DF of G_i and $f(v_i) = 0\}$.
- $\gamma^1(G_i, v_i)$=min$\{\omega(f)$: f is an IR2DF of G_i and $f(v_i) = 1\}$.
- $\gamma^2(G_i, v_i)$=min$\{\omega(f)$: f is an IR2DF of G_i and $f(v_i) = 2\}$.

Theorem 4. *Let G be a proper interval graph with a BCO $\sigma = (v_1, v_2, \ldots, v_n)$. Let G_i be the induced subgraph of G with vertex set $v_i, v_{i+1}, \ldots, v_n$ for $1 \le i \le n$. Let v_i be the root of G_i. We abbreviate $\gamma^k(G_i, v_i)$ as $\gamma^k(v_i)$. The following statements hold.*

(1) $\gamma^0(v_i) = 2 + \gamma(v_{MaxNN(i)+1})$.

(2) $$\gamma^1(v_i) = \begin{cases} \infty, & \text{if } MaxN(i) = MaxN(i+1) \\ 1 + \min\{\gamma^0(v_{i+1}), \gamma^1(v_{MaxN(i)+1})\}, & \text{if } MaxN(i) < MaxN(i+1). \end{cases}$$

(3) $\gamma^2(v_i) = 2 + \gamma(v_{MaxN(i)+1})$.

(4) $\gamma(v_i) = \min\{\gamma^0(v_i), \gamma^1(v_i), \gamma^2(v_i)\}$.

Proof. Let f be an IR2DF on G_i. We now consider the value of $f(v_i)$.

(1) Since Mc_i is a clique, if $f(v_i) = 0$, then v_i is dominated by a vertex $u \in V(Mc_i - v_i)$ assigned 2. Let g_1 be an $i_{\{R2\}}$-function of $G_{MaxNN(i)+1}$. Let $f(v_t) = 0$ for $t \in [i, MaxN(i) - 1] \cup [MaxN(i) + 1, MaxNN(i)]$, $f(v_{MaxN(i)}) = 2$ and $f(x) = g_1(x)$ for $x \in V(G_{MaxNN(i)+1})$. Then we have $\gamma^0(v_i) \le \omega(f) = 2 + \gamma(v_{MaxNN(i)+1})$.

Suppose v_j is the vertex assigned 2 in $Mc_i - v_i$. Then v_j dominates the vertices in $\{v_i, v_{i+1}, \ldots, v_{MaxN(j)}\}$. When v_i is assigned 0, an IR2DF g' on G_i with weight $\omega(g') = 2 + \gamma(v_{MaxN(j)+1})$ can be found. By Corollary 1, for $1 \le t < s \le n$, $\gamma(v_t) \ge \gamma(v_s)$. Since the weight of $G_{MaxN(j)+1}$ decreases with smaller values of $MaxN(j)$, we have $\gamma^0(v_i) \ge 2 + \gamma(v_{MaxNN(i)+1})$. Therefore, $\gamma^0(v_i) = 2 + \gamma(v_{MaxNN(i)+1})$.

(2) If $f(v_i) = 1$, then the vertices in G_i adjacent to v_i are assigned 0. By Lemma 3, we have $MaxN(i) \le MaxN(i + 1)$. If $MaxN(i) = MaxN(i + 1)$, then v_{i+1} can not be dominated by any vertex in Mc_i, which contradicts the assumption that f is an IR2DF. Hence, if $MaxN(i) = MaxN(i + 1)$, then $\gamma^1(v_i) = \infty$. Now we consider the situations when $MaxN(i) < MaxN(i + 1)$, and we have the following cases.

Case 1. There exists a vertex $u \in V(G_{MaxN(i)+1})$ such that all vertices in $V(Mc_i - v_i)$ are adjacent to u and $f(u) = 1$.

Clearly, in this case, the vertices in $V(Mc_i - v_i)$ are dominated by v_i and u. We deduce that u must be the vertex $v_{MaxN(i)+1}$. As we can see, v_{i+1} is not adjacent to the vertices in $G_{MaxN(i+1)+1}$. If there exists a vertex $u \in V(G_{MaxN(i)+1})$ such that all vertices in $V(Mc_i - v_i)$ are adjacent to u, then the index of u is between $MaxN(i) + 1$ and $MaxN(i + 1)$. If $MaxN(i) + 1 = MaxN(i + 1)$, the deduction holds. If $MaxN(i) + 1 < MaxN(i + 1)$, without loss of generality, we assume that u

is $v_{MaxN(i)+2}$ and $f(u) = 1$. Since $MaxN(MaxN(i) + 1) \leq MaxN(MaxN(i) + 2)$ and $v_{MaxN(i)+1}$ must be adjacent to $v_{MaxN(i)+2}$, it follows that $f(v_{MaxN(i)+1}) = 0$. This contradicts the assumption that f is an IR2DF on G_i. Therefore, $v_{MaxN(i)+1}$ must be assigned 1 in this case. Hence, we have $\gamma^1(v_i) = 1 + \gamma^1(v_{MaxN(i)+1})$.

Case 2. There exists a vertex $u \in V(G_{MaxN(i)+1})$ such that all vertices in V_i are adjacent to u and $f(u) = 2$.
In this case, v_i do not need to dominate the vertices in $Mc_i - v_i$ and the vertices in $Mc_i - v_i$ must be assigned 0. Similar to the proof of (1), we have $\gamma^1(v_i) = 1 + \gamma^0(v_{i+1})$.

(3) If $f(v_i) = 2$, then v_i can dominate all the vertices assigned 0 in $Mc_i - v_i$. Similar to the proof of (1), we have $\gamma^2(v_i) = 2 + \gamma(v_{MaxN(i)+1})$. □

By Theorem 4, if we obtain $\gamma^0(G_n, v_n)$, $\gamma^1(G_n, v_n)$ and $\gamma^2(G_n, v_n)$, then we can compute $\gamma^0(G_{n-1}, v_{n-1})$, $\gamma^1(G_{n-1}, v_{n-1})$ and $\gamma^2(G_{n-1}, v_{n-1})$. Recursively, we can get the parameters $\gamma^0(G_1, v_1)$, $\gamma^1(G_1, v_1)$ and $\gamma^2(G_1, v_1)$. The minimum of these three values is the result of $i_{\{R2\}}(G)$.
Initialize $\gamma^0(G_n, v_n)$, $\gamma^1(G_n, v_n)$ and $\gamma^2(G_n, v_n)$ by ∞, 1, and 2 respectively. The issue of index out-of-bounds in Theorem 4 may occur in some cases. To avoid this error, we set a virtual vertex v_{n+1} with all its states initialized to 0 in the algorithm. Now we present a linear-time algorithm for IR2D for proper interval graphs in Algorithm 1.

Theorem 5. *The independent Roman {2}-domination problem can be solved in linear time on proper interval graphs.*

Proof. The correctness of Algorithm 1 is proved in Theorem 4. Now, we discuss the time complexity of Algorithm 1. Let $G = (V, E)$ be a proper interval graph. Firstly, obtaining a BCO using ALGORITHM-BCO can be accomplished in $O(|V| + |E|)$ time [22]. Next, calculating the values of $MaxN(i)$ and $MaxNN(i)$ for each vertex v_i in G can be completed in $O(|E|)$. Setting a virtual vertex v_{n+1} with all its parameters initialized to 0 and initializing the parameters of v_n can be done in $O(1)$ time. Finally, the for-loop for computing $\gamma^0(v_i)$, $\gamma^1(v_i)$, $\gamma^2(v_i)$ and $\gamma(v_i)$ takes $O(|V|)$ times. Therefore, the independent Roman {2}-domination problem can be solved in linear time on proper interval graphs. □

Algorithm 1 An algorithm for computing $i_{\{R2\}}(G)$ on any proper interval graph

Input: A connected proper interval graph G.
Output: The independent Roman {2}-domination number $i_{\{R2\}}(G)$.

Get a BCO $\sigma = (v_1, v_2, \ldots, v_n)$ by ALGORITHM-BCO in [22];
Calculate $MaxN(i)$ and $MaxNN(i)$ for each vertex v_i in G;
Set a virtual vertex v_{n+1} with all its parameters initialized to 0;
$\gamma^0(v_n) \leftarrow \infty$; $\gamma^1(v_n) \leftarrow 1$; $\gamma^2(v_n) \leftarrow 2$; $\gamma(v_n) \leftarrow 1$;
for i = n-1 to 1 **do**
    $\gamma^0(v_i) \leftarrow 2 + \gamma(v_{MaxNN(i)+1})$;

if MaxN(i)<MaxN(i+1) **then**
  $\gamma^1(v_i) \leftarrow 1 + \min\{\gamma^0(v_{i+1}), \gamma^1(v_{MaxN(i)+1})\}$;
else
  $\gamma^1(v_i) \leftarrow \infty$;
end if
$\gamma^2(v_i) \leftarrow 2 + \gamma(v_{MaxN(i)+1})$;
$\gamma(v_i) \leftarrow \min\{\gamma^0(v_i), \gamma^1(v_i), \gamma^2(v_i)\}$;
end for
return $\gamma(v_1)$;

Example 1. Given a proper interval graph G with 11 vertices, as shown in Figure 3. Using ALGORITHM-BCO, we can obtain a BCO $\sigma = (v_1, v_2, \ldots, v_{11})$.

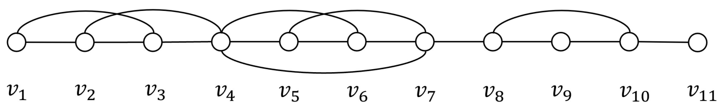

Figure 3. **An example of a proper interval graph** G.

By traversing the graph G, we can record the values of $MaxN(i)$ and $MaxNN(i)$ for $i \in [11]$. These values are summarized in Table 1.

i	1	2	3	4	5	6	7	8	9	10	11
MaxN(i)	3	4	4	7	7	7	8	10	10	11	11
MaxNN(i)	4	7	7	8	8	8	10	11	11	11	11

Table 1. **The values of MaxN(i) and MaxNN(i) of the graph G for** $i \in [11]$.

Base on BCO $\sigma = (v_1, v_2, \ldots, v_{11})$, we first set a virtual vertex v_{12} with $\gamma^0(v_{12}) = 0$, $\gamma^1(v_{12}) = 0$ and $\gamma^2(v_{12}) = 0$. Next, we initialize the parameters for v_{11} as $\gamma^0(v_{11}) = \infty$, $\gamma^1(v_{11}) = 1$, $\gamma^2(v_{11}) = 2$. We then compute $\gamma(v_i)$ for $i \in [10]$ in reverse order, following Theorem 4. The process of Algorithm 1 for G is detailed in Table 2. For this example, the independent Roman $\{2\}$-domination number of G is $i_{\{R2\}}(G) = \gamma^1(v_1) = 5$.

i	$\gamma^0(v_i)$	$\gamma^1(v_i)$	$\gamma^2(v_i)$	$\gamma(v_i)$
11	∞	1	2	1
10	2	∞	2	2
9	2	3	3	2
8	2	∞	3	2
7	3	3	4	3
6	4	4	4	4
5	4	∞	4	4
4	4	∞	4	4
3	4	5	6	4
2	4	∞	6	4
1	6	5	6	5

Table 2. **The progress of computing $i_{\{R2\}}(G)$ by Algorithm 1.**

5. IR2D on block-cactus graphs

In this section, we present a linear-time algorithm for solving IR2D in block-cactus graphs based on dynamic programming. If G is a disconnected block-cactus graph, then $i_{\{R2\}}(G)$ is simply the sum of the independent Roman $\{2\}$-domination numbers of its connected components. Therefore, we will focus on connected block-cactus graphs in this section. We begin by reviewing the definitions of blocks, block graphs and cactus graphs.

Definition 6 ([27]). In a connected graph $G = (V, E)$, a vertex $v \in V$ is called a cut vertex if removing v and all edges incident to v increases the number of connected components in the graph G.

Definition 7 ([12]). Let G be a graph such that G is connected and has no cut vertices, then the graph G is called a block. A block of a graph G is a maximal subgraph of G which is itself a block.

Definition 8 ([12]). A graph G is a block graph if every block (maximal 2-connected component) is a clique.

Definition 9 ([13]). A graph G is a cactus if each block is either an edge or a cycle.

Definition 10 ([24][14]). A graph G is a block-cactus graph if every block in G is either a clique or a cycle with at least 4 vertices.

Worth mentioned that the graph C_3 is not only a cycle but also a complete graph. In the definition of block-cactus graphs, a block consisting three vertices forms a clique, and cycles must have at least four vertices. An example of a block-cactus graph G is illustrated in Figure 4(a).

Definition 11. Let G be a block-cactus graph with p blocks $B_1, B_2, \ldots, B_p$ and q cut vertices $c_1, c_2, \ldots, c_q$. A cut-tree $T_G = (V_T, E_T)$ of G is defined as follows:

- $V_T = \{B_1, \ldots, B_p, c_1, \ldots, c_q\}$, where V_T consists of all blocks and cut vertices of G.
- $E_T = \{B_i c_j | c_j \in B_i, 1 \leq i \leq p, 1 \leq j \leq q\}$, where E_T includes an edge between each block B_i and each cut vertex c_j that is contained in B_i.

In the graph G depicted in Figure 4(a), there are six blocks, defined as follows:

- $V(B_1) = \{v_1, v_2, v_{16}, v_{17}\}$
- $V(B_2) = \{v_{16}, v_3\}$
- $V(B_3) = \{v_5, v_6, v_7, v_8, v_9, v_{10}\}$
- $V(B_4) = \{v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$
- $V(B_5) = \{v_3, v_4\}$
- $V(B_6) = \{v_4, v_5, v_{11}\}$

The cut vertices of G, illustrated as black vertices in Figure 4(a), are distributed among the blocks. are represented by black vertices. Each block contains at least one cut vertex. Blocks that include exactly one cut vertex are referred to as end block. In this case, B_1, B_3 and B_4 are end blocks.
According to Definition 11, the corresponding cut-tree T_G of G is shown in Figure 4(b).

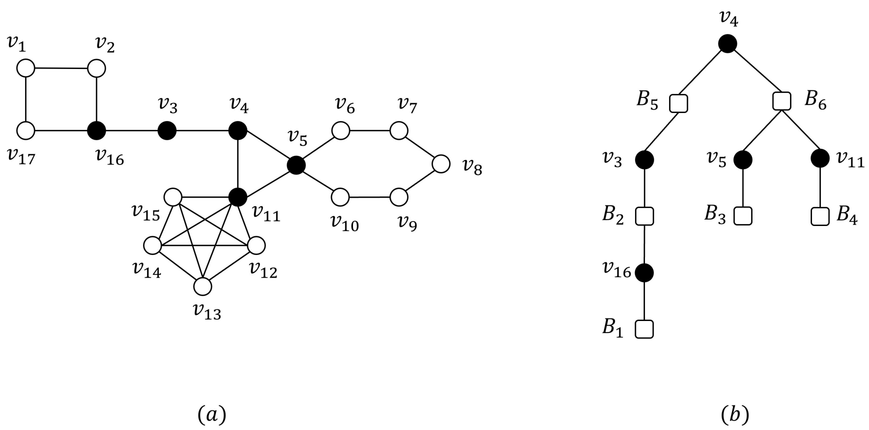

Figure 4. **An example of a block-cactus graph G (a) and its cut-tree T_G (b).**

Let $G = (V, E)$ be a graph. For a vertex $v \in V$, we define the following five problems:

- $\gamma^0(G, v)$=min$\{\omega(f)$: f is an IR2DF of G and $f(v) = 0\}$.

- $\gamma^1(G,v)$=min$\{\omega(f)$: f is an IR2DF of G and $f(v)=1\}$.
- $\gamma^2(G,v)$=min$\{\omega(f)$: f is an IR2DF of G and $f(v)=2\}$.
- $\gamma^{0^-}(G,v)$=min$\{\omega(f)$: f_{G-v} (the restriction of f on $G-v$) is an IR2DF of $G-v$ and $f(v)=0$, $\sum_{u\in N_G(v)} f(u)=1\}$.
- $\gamma^{0^+}(G,v)$=min$\{\omega(f)$: f_{G-v} (the restriction of f on $G-v$) is an IR2DF of $G-v$ and $f(u)=0$ for $u\in N_G[v]\}$.

Lemma 5. *For any graph G with a specific vertex v, we have:*

$$i_{\{R2\}}(G,v)=\min\{\gamma^0(G,v),\gamma^1(G,v),\gamma^2(G,v)\}.$$

The proof of Lemma 5 is straightforward and thus omitted. For a trivial graph G with a vertex v, no IR2DF exists if v is assigned 0. Additionally, no IR2DF f on $G-v$ satisfies $f(v)=0$ and $\sum_{u\in N_G(v)} f(u)=1$. Hence, we have the following lemma.

Lemma 6. *Let G be a trivial graph with a vertex v, then $\gamma^0(G,v)=\infty$, $\gamma^1(G,v)=1$, $\gamma^2(G,v)=2$, $\gamma^{0^-}(G,v)=\infty$ and $\gamma^{0^+}(G,v)=0$.*

Let G be a block-cactus graph, and let T_G be a cut-tree of G with the root vertex u. Note that u is a specific vertex of G. Now, consider an induced subgraph of G. Let G_1, G_2, $\ldots$, G_k be k block-cactus graphs with specific vertices $v_1,v_2,\ldots,v_k$, respectively. Let H be a composition graph of G_1, G_2, $\ldots$, G_k formed by adding edges to make $\{v_1,v_2,\ldots,v_k\}$ a clique or a cycle. Let B be a block constructed from $\{v_1,v_2,\ldots,v_k\}$ in H. Denote $B^-=B-v_1$.
Note that a connected block-cactus graph can be obtained from trivial graphs by repeatedly applying such graph composition. Therefore, we only need to focus on the calculation for each composition graph. In the following subsections, we will introduce the idea of our algorithm for the cases when B is a clique and when B is a cycle, respectively.

5.1. IR2D on clique blocks

In this subsection, we introduce the idea of calculating the independent Roman {2}-domination number for a composition graph H in which the block B is a clique. By Definition 8, if all blocks of a graph are cliques, then the graph is a block graph. Clearly, block graphs are a subclass of block-cactus graphs. In 2019, Wei and Lu [26] presented a linear-time algorithm to find the independent Roman {2}-domination number for any connected block graph G based on dynamic programming. In this subsection, we propose a solution to calculate the independent Roman {2}-domination number for clique blocks using dynamic programming, which differs from [26].
Let G_1, G_2, $\ldots$, G_k be k block-cactus graphs with specific vertices $v_1,v_2,\ldots,v_k$ respectively. Let H be the composition graph of G_1, G_2, $\ldots$, G_k, obtained from their

disjoint unions by adding edges to make $\{v_1, v_2, \ldots, v_k\}$ a clique, as shown in Figure 5. Let v_1 be the specific vertex of H. Let B be the block constructed by $\{v_1, v_2, \ldots, v_k\}$. Clearly, B is a clique and we denote $B^- = B - v_1$.

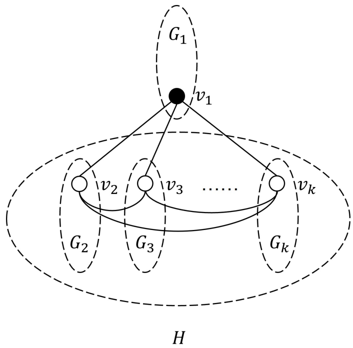

Figure 5. **The construction of a composition graph H when B is a clique block.**

To determine the value of $\gamma^i(H, v_1)$ for $i \in \{0, 1, 2, 0^-, 0^+\}$, we find the minimum value among the possible combinations of parameters for the graphs $G_1, G_2, \ldots, G_k$ in the following theorem.

Theorem 6. *Let G_1, G_2, ..., G_k be k block-cactus graphs with specific vertices v_1, v_2, ..., v_k respectively. Let H be a graph with specific vertex v_1. H is constructed from the disjoint unions of G_1, G_2, ..., G_k by adding edges to make $\{v_1, v_2, \ldots, v_k\}$ a clique. Let B be the block constructed by $\{v_1, v_2, \ldots, v_k\}$. Denote $B^- = B - v_1$. Let*

- $C_0 = \sum_{t \in [k] \setminus \{1\}} \gamma^0(G_t, v_t)$.
- $C_1 = \min_{s \in [k] \setminus \{1\}} \{\gamma^1(G_s, v_s) + \sum_{t \in [k] \setminus \{1, s\}} \min\{\gamma^0(G_t, v_t), \gamma^{0^-}(G_t, v_t)\}\}$.
- $C_2 = \min_{s \in [k] \setminus \{1\}} \{\gamma^2(G_s, v_s) + \sum_{t \in [k] \setminus \{1, s\}} \min\{\gamma^0(G_t, v_t), \gamma^{0^-}(G_t, v_t), \gamma^{0^+}(G_t, v_t)\}\}$.

Then the following statements hold.

(1) $\gamma^0(H, v_1) = \min \begin{cases} \gamma^0(G_1, v_1) + \min\{C_0, C_1, C_2\}, \\ \gamma^{0^-}(G_1, v_1) + \min\{C_1, C_2\}, \\ \gamma^{0^+}(G_1, v_1) + C_2. \end{cases}$

(2) $\gamma^1(H, v_1) = \gamma^1(G_1, v_1) + \sum_{t \in [k] \setminus \{1\}} \min\{\gamma^0(G_t, v_t), \gamma^{0^-}(G_t, v_t)\}$.

(3) $\gamma^2(H, v_1) = \gamma^2(G_1, v_1) + \sum_{t \in [k] \setminus \{1\}} \min\{\gamma^0(G_t, v_t), \gamma^{0^-}(G_t, v_t), \gamma^{0^+}(G_t, v_t)\}$.

(4) $\gamma^{0^-}(H, v_1) = \min\{\gamma^{0^-}(G_1, v_1) + C_0, \gamma^{0^+}(G_1, v_1) + C_1\}$.

(5) $\gamma^{0^+}(H, v_1) = \gamma^{0^+}(G_1, v_1) + C_0$.

Proof. (1) Let f be an IR2DF of H. By the definition of $\gamma^0(H, v_1)$, we have $f(v_1) = 0$ and $\sum_{u \in N_H(v_1)} f(u) \geq 2$. Then we discuss the following three cases.

Case 1. $\sum_{u \in N_{G_1}(v_1)} f(u) \geq 2$.
In this case, v_1 is dominated by the vertices in G_1 and can not dominate the vertices in B^-. Thus, we have $\gamma^0(H, v_1) = \gamma^0(G_1, v_1) + \min\{C_0, C_1, C_2\}$.

Case 2. $\sum_{u \in N_{G_1}(v_1)} f(u) = 1$.
In this case, to satisfy the condition that $\sum_{u \in N_H(v_1)} f(u) \geq 2$, it requires a vertex in B^- to be assigned a value of 1 or 2. Therefore, we have $\gamma^0(H, v_1) = \gamma^{0^-}(G_1, v_1) + \min\{C_1, C_2\}$.

Case 3. $\sum_{u \in N_{G_1}(v_1)} f(u) = 0$.
In this case, to satisfy the condition that $\sum_{u \in N_H(v_1)} f(u) \geq 2$, it requires a vertex in B^- to be assigned a value of 2. Thus, $\gamma^0(H, v_1) = \gamma^{0^+}(G_1, v_1) + C_2$ holds.

(2) Let f be an IR2DF of H. By the definition of $\gamma^1(H, v_1)$, we have $f(v_1) = 1$ and $\sum_{u \in N_H(v_1)} f(u) = 0$. Consequently, all vertices in B^- are assigned 0, and the vertex $v_t \in B^-$ can be adjacent to a vertex in G_t that is assigned 1 or 2 for $t \in [k] \setminus \{1\}$. Therefore, we have $\gamma^1(H, v_1) = \gamma^1(G_1, v_1) + \sum_{t \in [k] \setminus \{1\}} \min\{\gamma^0(G_t, v_t), \gamma^{0^-}(G_t, v_t)\}$.

(3) Let f be an IR2DF of H. By the definition of $\gamma^2(H, v_1)$, we have $f(v_1) = 2$ and $\sum_{u \in N_H(v_1)} f(u) = 0$. In this case, the vertex $v_t \in B^-$ does not need to be dominated by the graph G_t for $t \in [k] \setminus \{1\}$. Therefore, we have $\gamma^2(H, v_1) = \gamma^2(G_1, v_1) + \sum_{t \in [k] \setminus \{1\}} \min\{\gamma^0(G_t, v_t), \gamma^{0^-}(G_t, v_t), \gamma^{0^+}(G_t, v_t)\}$.

(4) Let f_{H-v_1} be an IR2DF of $H - v_1$, which is the restriction of f to $H - v_1$. Since $\gamma^{0^-}(H, v_1)$ is the minimum weight of f when $f(v_1) = 0$ and $\sum_{u \in N_H(v_1)} f(u) = 1$, we have the following two cases:

Case 1. $\sum_{u \in N_{G_1}(v_1)} f(u) = 1$.
In this case, we have $\sum_{u \in B^-} f(u) = 0$. According to the definition of $\gamma^{0^-}(H, v_1)$, the vertex $v_t \in B^-$ must be dominated by the vertices in G_t for $t \in [k] \setminus \{1\}$. Hence, we have $\gamma^{0^-}(H, v_1) = \gamma^{0^-}(G_1, v_1) + C_0$.

Case 2. $\sum_{u \in N_{G_1}(v_1)} f(u) = 0$.
In this case, it is necessary for a vertex in B^- to be assigned a value of 1 to ensure that $\sum_{u \in N_H(v_1)} f(u) = 1$. Therefore, we have $\gamma^{0^-}(H, v_1) = \gamma^{0^+}(G_1, v_1) + C_1$.

(5) Let f_{H-v_1} be an IR2DF of $H - v_1$, which is the restriction of f on $H - v_1$. Since $\gamma^{0^+}(H, v_1)$ is the minimum weight of f when $\sum_{u \in N_H[v_1]} f(u) = 0$, it requires that $\sum_{u \in N_{G_1}(v_1)} f(u) = 0$ and $\sum_{u \in B^-} f(u) = 0$. Consequently, each vertex $v_t \in B^-$ must be dominated by the vertices in G_t for $t \in [k] \setminus \{1\}$. Therefore, we have $\gamma^{0^+}(H, v_1) = \gamma^{0^+}(G_1, v_1) + C_0$. □

Now we present an algorithm, called STATES-FOR-CLIQUE-BLOCK, to compute these five parameters $\gamma^i(H, v_1)$ for $i \in \{0, 1, 2, 0^-, 0^+\}$ on a composition graph H with the specific vertex v_1, where B is a clique block, described in Algorithm 2.

Algorithm 2 STATES-FOR-CLIQUE-BLOCK

Input: A composition graph H with a specific vertex v_1 where B is a clique block.
Output: The five parameters $\gamma^i(H, v_1)$ for $i \in \{0, 1, 2, 0^-, 0^+\}$.

1: $C_0 \leftarrow \sum_{t\in[k]\setminus\{1\}} \gamma^0(G_t, v_t)$;
2: $C_1 \leftarrow \min_{s\in[k]\setminus\{1\}} \{\gamma^1(G_s, v_s) + \sum_{t\in[k]\setminus\{1,s\}} \min\{\gamma^0(G_t, v_t), \gamma^{0^-}(G_t, v_t)\}\}$;
3: $C_2 \leftarrow \min_{s\in[k]\setminus\{1\}} \{\gamma^2(G_s, v_s) + \sum_{t\in[k]\setminus\{1,s\}} \min\{\gamma^0(G_t, v_t), \gamma^{0^-}(G_t, v_t), \gamma^{0^+}(G_t, v_t)\}\}$;
4: Compute $\gamma^0(H, v_1)$ by using Theorem 6 (1);
5: Compute $\gamma^1(H, v_1)$ by using Theorem 6 (2);
6: Compute $\gamma^2(H, v_1)$ by using Theorem 6 (3);
7: Compute $\gamma^{0^-}(H, v_1)$ by using Theorem 6 (4);
8: Compute $\gamma^{0^+}(H, v_1)$ by using Theorem 6 (5);
9: **return** $\gamma^0(H, v_1)$,$\gamma^1(H, v_1)$,$\gamma^2(H, v_1)$,$\gamma^{0^-}(H, v_1)$,$\gamma^{0^+}(H, v_1)$;

5.2. IR2D on cycle blocks

Now we focus on the case where B is a cycle block in H. According to Definition 9, if all blocks in a graph are either cycles or edges, then the graph is a cactus graph. Clearly, cactus graphs are a subclass of block-cactus graphs.
Let G_1, G_2, ..., G_k be k graphs with specific vertices v_1, v_2, ..., v_k, respectively. Let H be the composition graph of G_1, G_2, ..., G_k, obtained by taking the disjoint unions of these graphs and adding edges to form a cycle with the vertices $\{v_1, v_2, \ldots, v_k\}$. The specific vertex of H is v_1. Let B be the block constructed by $\{v_1, v_2, \ldots, v_k\}$. Denote $B^- = B - v_1$. Clearly, B is a cycle block. The construction of such a composition graph H with a cycle block is illustrated in Figure 6.

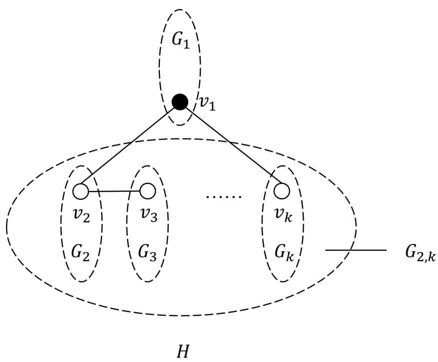

Figure 6. **The construction of a composition graph H when B is a cycle block.**

Denote $G_{t,k} = H[V(G_t) \cup V(G_{t+1}) \cup \cdots \cup V(G_k)]$ for $2 \le t \le k-1$. For each subgraph $G_{t,k}$, we consider the following problems:

- $P_{ij}(G_{t,k}, v_t, v_k)$=min$\{\omega(f)$: f is an IR2DF on $G_{t,k}$ and $f(v_t) = i$, $f(v_k) = j\}$, where $i \in \{0, 1, 2\}$ and $j \in \{0, 1, 2\}$.

- $P_{i0^+}(G_{t,k}, v_t, v_k)$=min$\{\omega(f)$: $f_{G_{t,k}-v_k}$ (the restriction of f on $G_{t,k} - v_k$) is an IR2DF on $G_{t,k} - v_k$ and $f(v_t) = i$, $f(u) = 0$ for $u \in N_{G_{t,k}}[v_k]\}$, where $i \in \{0, 1, 2\}$.

- $P_{i0^-}(G_{t,k}, v_t, v_k)$=min$\{\omega(f)$: $f_{G_{t,k}-v_k}$ (the restriction of f on $G_{t,k} - v_k$) is an IR2DF on $G_{t,k} - v_k$ and $f(v_t) = i$, $f(v_k) = 0$, $\sum_{u\in N_{G_{t,k}}(v_k)} f(u) = 1\}$, where $i \in \{0, 1, 2\}$.

- $P_{0^-j}(G_{t,k}, v_t, v_k)$=min$\{\omega(f)$: $f_{G_{t,k}-v_t}$ (the restriction of f on $G_{t,k} - v_t$) is an IR2DF on $G_{t,k} - v_t$ and $f(v_t) = 0$, $f(v_k) = j$, $\sum_{u\in N_{G_{t,k}}(v_t)} f(u) = 1\}$, where $j \in \{0, 1, 2\}$.

- $P_{0^+j}(G_{t,k}, v_t, v_k)$=min$\{\omega(f)$: $f_{G_{t,k}-v_t}$ (the restriction of f on $G_{t,k} - v_t$) is an IR2DF on $G_{t,k} - v_t$ and $f(v_k) = j$, $f(u) = 0$ for $u \in N_{G_{t,k}}[v_t]\}$, where $j \in \{0, 1, 2\}$.

- $P_{0^-0^+}(G_{t,k}, v_t, v_k)$=min$\{\omega(f)$: $f_{G_{t,k}-\{v_t,v_k\}}$ (the restriction of f on $G_{t,k} - \{v_t, v_k\}$) is an IR2DF on $G_{t,k} - \{v_t, v_k\}$ and $f(v_t) = 0$, $f(v_k) = 0$, $\sum_{v\in N_{G_{t,k}}(v_k)} f(v) = 0$, $\sum_{u\in N_{G_{t,k}}(v_t)} f(u) = 1\}$.

- $P_{0^-0^-}(G_{t,k}, v_t, v_k)$=min$\{\omega(f)$: $f_{G_{t,k}-\{v_t,v_k\}}$ (the restriction of f on $G_{t,k} - \{v_t, v_k\}$) is an IR2DF on $G_{t,k} - \{v_t, v_k\}$and $f(v_t) = 0$, $f(v_k) = 0$, $\sum_{v\in N_{G_{t,k}}(v_t)} f(v) = 1$, $\sum_{u\in N_{G_{t,k}}(v_k)} f(u) = 1$ $\}$.

- $P_{0^+0^+}(G_{t,k}, v_t, v_k)$=min$\{\omega(f)$: $f_{G_{t,k}-\{v_t,v_k\}}$ (the restriction of f on $G_{t,k} - \{v_t, v_k\}$) is an IR2DF on $G_{t,k} - \{v_t, v_k\}$ and $f(v_t) = 0$, $f(v_k) = 0$, $\sum_{v\in N_{G_{t,k}}(v_k)} f(v) = 0$, $\sum_{u\in N_{G_{t,k}}(v_t)} f(u) = 0\}$.

- $P_{0^+0^-}(G_{t,k}, v_t, v_k)$=min$\{\omega(f)$: $f_{G_{t,k}-\{v_t,v_k\}}$ (the restriction of f on $G_{t,k} - \{v_t, v_k\}$) is an IR2DF on $G_{t,k} - \{v_t, v_k\}$ and $f(v_t) = 0$, $f(v_k) = 0$, $\sum_{v\in N_{G_{t,k}}(v_k)} f(v) = 1$, $\sum_{u\in N_{G_{t,k}}(v_t)} f(u) = 0\}$.

By Definition 10, a block that is a cycle must contain at least 4 vertices. As we can see, when B is a cycle block in the composition graph H, the graph H can be divided into two parts: G_1 and $G_{2,k}$. To determine the parameters $P_{i,j}(G_{2,k}, v_2, v_k)$ for $i \in \{0, 1, 2, 0^-, 0^+\}$ and $j \in \{0, 1, 2, 0^-, 0^+\}$, we use the following theorem:

Theorem 7. *Let $G_1, G_2, \ldots, G_k$ be k graphs (with $k \geq 4$) each with specific vertices $v_1, v_2, \ldots, v_k$, respectively. Let H be the composition graph of $G_1, G_2, \ldots, G_k$ with the specific vertex v_1, obtained from the disjoint unions $G_1, G_2, \ldots, G_k$ by adding edges to make $\{v_1, v_2, \ldots, v_k\}$ a cycle. Denote $G_{t,k}$ as the graph obtained by adding edges to the disjoint unions of $G_t, G_{t+1}, \ldots, G_k$ to form a path with the vertices $v_t, v_{t+1}, \ldots, v_k$, where $2 \leq t \leq k - 2$. Then the following statements hold.*

(1) $P_{0i}(G_{t,k}, v_t, v_k) = \min \begin{cases} \gamma^0(G_t, v_t) + \min\{P_{0i}(G_{t+1,k}, v_{t+1}, v_k), P_{1i}(G_{t+1,k}, v_{t+1}, v_k), \\ P_{2i}(G_{t+1,k}, v_{t+1}, v_k)\}, \\ \gamma^{0^-}(G_t, v_t) + \min\{P_{1i}(G_{t+1,k}, v_{t+1}, v_k), P_{2i}(G_{t+1,k}, v_{t+1}, v_k)\}, \\ \gamma^{0^+}(G_t, v_t) + P_{2i}(G_{t+1,k}, v_{t+1}, v_k), \end{cases}$
for $i \in \{0^+, 0^-, 0, 1, 2\}$.

(2) $P_{0^-i}(G_{t,k}, v_t, v_k) = \min\{\gamma^{0^-}(G_t, v_t) + P_{0i}(G_{t+1,k}, v_{t+1}, v_k), \gamma^{0^+}(G_t, v_t) + P_{1i}(G_{t+1,k}, v_{t+1}, v_k)\}$, *for* $i \in \{0^+, 0^-, 0, 1, 2\}$.

(3) $P_{0^+i}(G_{t,k}, v_t, v_k) = \gamma^{0^+}(G_t, v_t) + P_{0i}(G_{t+1,k}, v_{t+1}, v_k)$, *for* $i \in \{0^+, 0^-, 0, 1, 2\}$.

(4) $P_{1i}(G_{t,k}, v_t, v_k) = \gamma^1(G_t, v_t) + \min\{P_{0i}(G_{t+1,k}, v_{t+1}, v_k), P_{0^-i}(G_{t+1,k}, v_{t+1}, v_k)\}$, *for* $i \in \{0^+, 0^-, 0, 1, 2\}$.

(5) $P_{2i}(G_{t,k}, v_t, v_k) = \gamma^2(G_t, v_t) + \min\{P_{0i}(G_{t+1,k}, v_{t+1}, v_k), P_{0^-i}(G_{t+1,k}, v_{t+1}, v_k), P_{0^+i}(G_{t+1,k}, v_{t+1}, v_k)\}$, *for* $i \in \{0^+, 0^-, 0, 1, 2\}$.

Proof. (1) Let $f : V \to \{0, 1, 2\}$ be a function on $G_{t,k}$. Let $f_{G_{t,k}-v_k}$ (the restriction of f on $G_{t,k} - v_k$) be an IR2DF on $G_{t,k} - v_k$. By the definition of $P_{0i}(G_{t,k}, v_t, v_k)$ for $i \in \{0^+, 0^-, 0, 1, 2\}$, $f(v_t) = 0$ and v_t is dominated by its neighbors in $G_{t,k}$. Then we consider the following three cases.

Case 1. $\sum_{u \in N_{G_t}(v_t)} f(u) \geq 2$.
In this case, the weight of v_{t+1} can be 0, 1 or 2. Hence, we have $P_{0i}(G_{t,k}, v_t, v_k) = \gamma^0(G_t, v_t) + \min\{P_{0i}(G_{t+1,k}, v_{t+1}, v_k), P_{1i}(G_{t+1,k}, v_{t+1}, v_k), P_{2i}(G_{t+1,k}, v_{t+1}, v_k)\}$, for $i \in \{0^+, 0^-, 0, 1, 2\}$.

Case 2. $\sum_{u \in N_{G_t}(v_t)} f(u) = 1$.
In this condition, the minimum weight of v_t in G_t is $\gamma^{0^-}(G_t, v_t)$. We have $f(v_{t+1}) \geq 1$. Thus, $P_{0i}(G_{t,k}, v_t, v_k) = \gamma^{0^-}(G_t, v_t) + \min\{P_{1i}(G_{t+1,k}, v_{t+1}, v_k), P_{2i}(G_{t+1,k}, v_{t+1}, v_k)\}$,
for $i \in \{0^+, 0^-, 0, 1, 2\}$.

Case 3. $\sum_{u \in N_{G_t}(v_t)} f(u) = 0$.
In this case, v_t is not dominated by the neighbors in G_t. Therefore, v_{t+1} must be assigned a value of 2. Thus, $P_{0i}(G_{t,k}, v_t, v_k) = \gamma^{0^+}(G_t, v_t) + P_{2i}(G_{t+1,k}, v_{t+1}, v_k)$, for $i \in \{0^+, 0^-, 0, 1, 2\}$.

(2) Let $f : V \to \{0, 1, 2\}$ be a function on $G_{t,k}$. Let $f_{G_{t,k}-\{v_t,v_k\}}$ (the restriction of f on $G_{t,k} - \{v_t, v_k\}$) be an IR2DF on $G_{t,k} - \{v_t, v_k\}$. By the definition of $P_{0^-i}(G_{t,k}, v_t, v_k)$ for $i \in \{0^+, 0^-, 0, 1, 2\}$, $f(v_t) = 0$ and $\sum_{u \in N_{G_{t,k}}(v_t)} f(u) = 1$. We consider the following two cases.

Case 1. $\sum_{u \in N_{G_t}(v_t)} f(u) = 1$.
Clearly, v_{t+1} must be assigned 0 in this case and v_{t+1} is dominated by the vertices in $V(G_{t+1}) \cup \{v_{t+2}\}$. Therefore, we have $P_{0^-i}(G_{t,k}, v_t, v_k) = \gamma^{0^-}(G_t, v_t) + P_{0i}(G_{t+1,k}, v_{t+1}, v_k)$, for $i \in \{0^+, 0^-, 0, 1, 2\}$.

Case 2. $\sum_{u \in N_{G_t}(v_t)} f(u) = 0$.
In this case, we have $f(v_{t+1}) = 1$. Therefore, $P_{0^-i}(G_{t,k}, v_t, v_k) = \gamma^{0^+}(G_t, v_t) + P_{1i}(G_{t+1,k}, v_{t+1}, v_k)\}$, for $i \in \{0^+, 0^-, 0, 1, 2\}$.

(3) Let $f : V \to \{0,1,2\}$ be a function on $G_{t,k}$. Let $f_{G_{t,k}-\{v_t,v_k\}}$ (the restriction of f on $G_{t,k} - \{v_t, v_k\}$) be an IR2DF on $G_{t,k} - \{v_t, v_k\}$. For $P_{0^+i}(G_{t,k}, v_t, v_k)$ where $i \in \{0^+, 0^-, 0, 1, 2\}$, $f(v_t) = 0$ and $\sum_{u\in N_{G_{t,k}}(v_t)} f(u) = 0$. It requires that $\sum_{u\in N_{G_t}(v_t)} f(u) = 0$ and $f(v_{t+1}) = 0$. Therefore, we have $P_{0^+i}(G_{t,k}, v_t, v_k) = \gamma^{0^+}(G_t, v_t) + P_{0i}(G_{t+1,k}, v_{t+1}, v_k)$, for $i \in \{0^+, 0^-, 0, 1, 2\}$.

(4) Let $f : V \to \{0,1,2\}$ be a function on $G_{t,k}$. Let $f_{G_{t,k}-v_k}$ (the restriction of f on $G_{t,k} - v_k$) be an IR2DF on $G_{t,k} - v_k$. For $P_{1i}(G_{t,k}, v_t, v_k)$ where $i \in \{0^+, 0^-, 0, 1, 2\}$, $f(v_t) = 1$ and v_{t+1} must be assigned 0. If $\sum_{u\in N_{G_{t+1,k}}(v_{t+1})} f(u) = 1$, v_t can dominate v_{t+1}. Thus, we have $P_{1i}(G_{t,k}, v_t, v_k) = \gamma^1(G_t, v_t) + \min\{P_{0i}(G_{t+1,k}, v_{t+1}, v_k), P_{0^-i}(G_{t+1,k}, v_{t+1}, v_k)\}$, for $i \in \{0^+, 0^-, 0, 1, 2\}$.

(5) Let $f : V \to \{0,1,2\}$ be a function on $G_{t,k}$. Let $f_{G_{t,k}-v_k}$ (the restriction of f on $G_{t,k} - v_k$) be an IR2DF on $G_{t,k} - v_k$. For $P_{2i}(G_{t,k}, v_t, v_k)$ where $i \in \{0^+, 0^-, 0, 1, 2\}$, $f(v_t) = 2$ and v_{t+1} must be assigned 0. v_t can dominate v_{t+1} regardless of whether v_{t+1} is dominated by its neighbors in $G_{t+1,k}$. Therefore, we have $P_{2i}(G_{t,k}, v_t, v_k) = \gamma^2(G_t, v_t) + \min\{P_{0i}(G_{t+1,k}, v_{t+1}, v_k), P_{0^-i}(G_{t+1,k}, v_{t+1}, v_k), P_{0^+i}(G_{t+1,k}, v_{t+1}, v_k)\}$, for $i \in \{0^+, 0^-, 0, 1, 2\}$. □

Now, we initialize the parameters $P_{i,j}(G_{k-1,k}, v_{k-1}, v_k)$ for $i \in \{0, 1, 2, 0^-, 0^+\}$ and $j \in \{0, 1, 2, 0^-, 0^+\}$ in the following proposition.

Proposition 2. *Let G_{k-1} and G_k be graphs with specific vertices v_{k-1} and v_k, respectively. Let $G_{k-1,k}$ be the graph obtained from G_{k-1} and G_k by adding the edge $v_{k-1}v_k$. Then the following statements hold.*

- $P_{0i}(G_{k-1,k}, v_{k-1}, v_k) = \gamma^0(G_{k-1}, v_{k-1}) + \gamma^i(G_k, v_k)$, *for* $i \in \{0^-, 0^+, 0\}$.
- $P_{0^-i}(G_{k-1,k}, v_{k-1}, v_k) = \gamma^{0^-}(G_{k-1}, v_{k-1}) + \gamma^i(G_k, v_k)$, *for* $i \in \{0^-, 0^+, 0\}$.
- $P_{0^+i}(G_{k-1,k}, v_{k-1}, v_k) = \gamma^{0^+}(G_{k-1}, v_{k-1}) + \gamma^i(G_k, v_k)$, *for* $i \in \{0^-, 0^+, 0\}$.
- $P_{10}(G_{k-1,k}, v_{k-1}, v_k) = \gamma^1(G_{k-1}, v_{k-1}) + \min\{\gamma^0(G_k, v_k), \gamma^{0^-}(G_k, v_k)\}$.
- $P_{10^-}(G_{k-1,k}, v_{k-1}, v_k) = \gamma^1(G_{k-1}, v_{k-1}) + \gamma^{0^+}(G_k, v_k)$.
- $P_{01}(G_{k-1,k}, v_{k-1}, v_k) = \min\{\gamma^0(G_{k-1}, v_{k-1}), \gamma^{0^-}(G_{k-1}, v_{k-1})\} + \gamma^1(G_k, v_k)$.
- $P_{0^-1}(G_{k-1,k}, v_{k-1}, v_k) = \gamma^{0^+}(G_{k-1}, v_{k-1}) + \gamma^1(G_k, v_k)$.
- $P_{20}(G_{k-1,k}, v_{k-1}, v_k) = \gamma^2(G_{k-1}, v_{k-1}) + \min\{\gamma^0(G_k, v_k), \gamma^{0^-}(G_k, v_k), \gamma^{0^+}(G_k, v_k)\}$.
- $P_{02}(G_{k-1,k}, v_{k-1}, v_k) = \min\{\gamma^0(G_{k-1}, v_{k-1}), \gamma^{0^-}(G_{k-1}, v_{k-1}), \gamma^{0^+}(G_{k-1}, v_{k-1})\} + \gamma^2(G_k, v_k)$.
- $P_{10^+}(G_{k-1,k}, v_{k-1}, v_k) = P_{0^+1}(G_{k-1,k}, v_{k-1}, v_k) = P_{20^+}(G_{k-1,k}, v_{k-1}, v_k) = P_{20^-}(G_{k-1,k}, v_{k-1}, v_k) = P_{0^-2}(G_{k-1,k}, v_{k-1}, v_k) = P_{0^+2}(G_{k-1,k}, v_{k-1}, v_k) = \infty$
- $P_{11}(G_{k-1,k}, v_{k-1}, v_k) = P_{12}(G_{k-1,k}, v_{k-1}, v_k) = P_{21}(G_{k-1,k}, v_{k-1}, v_k) = P_{22}(G_{k-1,k}, v_{k-1}, v_k) = \infty$.

The proof of the above proposition is trivial, so it is omitted. Here we simply illustrate the parameters equivalent to infinity. Take $P_{10^+}(G_{k-1,k}, v_{k-1}, v_k)$ as an example. If $f(v_{k-1}) = 1$, then $\sum_{u \in N_{G_{k-1,k}}(v_k)} f(u) \geq 1$. This does not satisfy the condition that $f(u) = 0$ for $u \in N_{G_{t,k}}[v_k]$. Therefore, $P_{10^+}(G_{k-1,k}, v_{k-1}, v_k) = \infty$. The reasons why $P_{0^+1}(G_{k-1,k}, v_{k-1}, v_k)$, $P_{20^+}(G_{k-1,k}, v_{k-1}, v_k)$, $P_{20^-}(G_{k-1,k}, v_{k-1}, v_k)$, $P_{0^-2}(G_{k-1,k}, v_{k-1}, v_k)$ and $P_{0^+2}(G_{k-1,k}, v_{k-1}, v_k)$ are equal to ∞ are similar. In addition, if two adjacent vertices are assigned values greater than 0, this contradicts the independence property of an IR2DF. Hence we have $P_{11}(G_{k-1,k}, v_{k-1}, v_k) = P_{12}(G_{k-1,k}, v_{k-1}, v_k) = P_{21}(G_{k-1,k}, v_{k-1}, v_k) = P_{22}(G_{k-1,k}, v_{k-1}, v_k) = \infty$.

Now we present an algorithm called STATES-FOR-PATH to compute the parameters $P_{ij}(G_{2,k}, v_2, v_k)$ for $i \in \{0, 1, 2, 0^-, 0^+\}$ and $j \in \{0, 1, 2, 0^-, 0^+\}$ of $G_{2,k}$ in Algorithm 3.

Algorithm 3 STATES-FOR-PATH

Input: A subgraph $G_{2,k}$.

Output: The parameters γ^{ij} for $i \in \{0, 1, 2, 0^-, 0^+\}$ and $j \in \{0, 1, 2, 0^-, 0^+\}$ of $G_{2,k}$.

Initialize $P_{0i}(G_{k-1,k}, v_{k-1}, v_k)$, $P_{0^-i}(G_{k-1,k}, v_{k-1}, v_k)$, $P_{0^+i}(G_{k-1,k}, v_{k-1}, v_k)$, $P_{1i}(G_{k-1,k}, v_{k-1}, v_k)$, $P_{2i}(G_{k-1,k}, v_{k-1}, v_k)$ for $i \in \{0^+, 0^-, 0, 1, 2\}$ by using Proposition 2;
for j = k-2 to 2 **do**
    Compute $P_{0i}(G_{j,k}, v_j, v_k)$ for $i \in \{0^+, 0^-, 0, 1, 2\}$ by using Theorem 7 (1);
    Compute $P_{0^-i}(G_{j,k}, v_j, v_k)$ for $i \in \{0^+, 0^-, 0, 1, 2\}$ by using Theorem 7 (2);
    Compute $P_{0^+i}(G_{j,k}, v_j, v_k)$ for $i \in \{0^+, 0^-, 0, 1, 2\}$ by using Theorem 7 (3);
    Compute $P_{1i}(G_{j,k}, v_j, v_k)$ for $i \in \{0^+, 0^-, 0, 1, 2\}$ by using Theorem 7 (4);
    Compute $P_{2i}(G_{j,k}, v_j, v_k)$ for $i \in \{0^+, 0^-, 0, 1, 2\}$ by using Theorem 7 (5);
end for
return $P_{0i}(G_{2,k}, v_2, v_k)$,$P_{0^-i}(G_{2,k}, v_2, v_k)$,$P_{0^+i}(G_{2,k}, v_2, v_k)$, $P_{1i}(G_{2,k}, v_2, v_k)$,
P_{2i} $(G_{2,k},$ $v_2, v_k)$ for $i \in \{0^+, 0^-, 0, 1, 2\}$;

After obtaining the parameters $P_{ij}(G_{2,k}, v_2, v_k)$ for $i \in \{0, 1, 2, 0^-, 0^+\}$ and $j \in \{0, 1, 2, 0^-, 0^+\}$ of $G_{2,k}$, we can determine the parameters $\gamma^i(H, v_1)$ for $i \in \{0, 1, 2, 0^-, 0^+\}$ of v_1 in the composition graph H using the following theorem.

Theorem 8. *Let $G_1, G_2, \ldots, G_k$ be k graphs with the specific vertices $v_1, v_2, \ldots, v_k$, respectively. Let $G_{2,k}$ be a graph obtained from the disjoint unions of G_2, G_3, $\ldots$, G_k by adding edges to make $\{v_2, v_3, \ldots, v_k\}$ a path. Let H be a composition graph of $G_1, G_2, \ldots, G_k$ with the specific vertex v_1, obtained from the disjoint unions of G_1 and $G_{2,k}$ by adding edges v_1v_2 and v_1v_k. Then the following statements hold.*

(1) $$\gamma^0(H, v_1) = \min \begin{cases} \gamma^0(G_1, v_1) + \min\{P_{ij}(G_{2,k}, v_2, v_k)\}, \text{ for } i \in \{0,1,2\} \text{ and } j \in \{0,1,2\}, \\ \gamma^{0^-}(G_1, v_1) + \min\{P_{0i}(G_{2,k}, v_2, v_k), P_{ij}(G_{2,k}, v_2, v_k)\}, \text{ for } i \in \{1,2\} \\ \text{and } j \in \{0,1,2\}, \\ \gamma^{0^+}(G_1, v_1) + \min\{P_{02}(G_{2,k}, v_2, v_k), P_{1i}(G_{2,k}, v_2, v_k), P_{2j}(G_{2,k}, v_2, v_k)\}, \\ \text{for } i \in \{1,2\} \text{ and } j \in \{0,1,2\}. \end{cases}$$

(2) $\gamma^1(H, v_1) = \gamma^1(G_1, v_1) + \min\{P_{ij}(G_{2,k}, v_2, v_k)\}$, *for* $i \in \{0^-, 0\}$ *and* $j \in \{0^-, 0\}$.

(3) $\gamma^2(H, v_1) = \gamma^2(G_1, v_1) + \min\{P_{ij}(G_{2,k}, v_2, v_k)\},$ *for* $i \in \{0^-, 0^+, 0\}$ *and* $j \in \{0^-, 0^+, 0\}$.

(4) $\gamma^{0^-}(H, v_1) = \min \begin{cases} \gamma^{0^-}(G_1, v_1) + P_{00}(G_{2,k}, v_2, v_k), \\ \gamma^{0^+}(G_1, v_1) + \min\{P_{01}(G_{2,k}, v_2, v_k), P_{10}(G_{2,k}, v_2, v_k)\}. \end{cases}$

(5) $\gamma^{0^+}(H, v_1) = \gamma^{0^+}(G_1, v_1) + P_{00}(G_{2,k}, v_2, v_k)$.

Proof. (1) Let f be an IR2DF on H. If $f(v_1) = 0$ and $\sum_{u\in N_H(v_1)} \geq 2$, we have the following three cases.

Case 1. $\sum_{u\in N_{G_1}(v_1)} \geq 2$.

In this case, v_1 is dominated by the neighbors in G_1, and it requires that $f(v_2) + f(v_k) \geq 0$. Therefore, we have $\gamma^0(H, v_1) = \gamma^0(G_1, v_1) + \min\{P_{ij}(G_{2,k}, v_2, v_k)\},$ for $i \in \{0, 1, 2\}$ and $j \in \{0, 1, 2\}$.

Case 2. $\sum_{u\in N_{G_1}(v_1)} = 1$.

In this case, $f(v_2) + f(v_k) \geq 1$, which implies that v_2 and v_k cannot both be assigned a value of 0 simultaneously. Therefore, we have $\gamma^0(H, v_1) = \gamma^{0^-}(G_1, v_1) + \min\{P_{0i}(G_{2,k}, v_2, v_k), P_{ij}(G_{2,k}, v_2, v_k)\},$ for $i \in \{1, 2\}$ and $j \in \{0, 1, 2\}$.

Case 3. $\sum_{u\in N_{G_1}(v_1)} = 0$.

In this case, $f(v_2) + f(v_k) \geq 2$. Therefore, we have $\gamma^0(H, v_1) = \gamma^{0^+}(G_1, v_1) + \min\{P_{02}(G_{2,k}, v_2, v_k), P_{1i}(G_{2,k}, v_2, v_k), P_{2j}(G_{2,k}, v_2, v_k)\},$ for $i \in \{1, 2\}$ and $j \in \{0, 1, 2\}$.

(2) Let f be an IR2DF in H. By the independence property of an IR2DF, if $f(v_1) = 1$, then v_2 and v_k must both be assigned 0. So we have $\gamma^1(H, v_1) = \gamma^1(G_1, v_1) + \min\{P_{ij}(G_{2,k}, v_2, v_k)\},$ for $i \in \{0^-, 0\}$ and $j \in \{0^-, 0\}$.

(3) Similar to the proof of the formula (2), we have $\gamma^2(H, v_1) = \gamma^2(G_1, v_1) + \min\{P_{ij}(G_{2,k}, v_2, v_k)\},$ for $i \in \{0^-, 0^+, 0\}$ and $j \in \{0^-, 0^+, 0\}$.

(4) Let f_{H-v_1} (the restriction of f on $H - v_1$) be an IR2DF on $H - v_1$. If $f(v_1) = 0$ and $\sum_{u\in N_H(v_1)} f(u) = 1$, we consider the following two cases.

Case 1. $\sum_{u\in N_{G_1}(v_1)} = 1$.

In this case, with $f(v_2) + f(v_k) = 0$, it follows that $\gamma^{0^-}(H, v_1) = \gamma^{0^-}(G_1, v_1) + P_{00}(G_{2,k}, v_2, v_k)$.

Case 2. $\sum_{u\in N_{G_1}(v_1)} = 0$.

In this case, with $f(v_2) + f(v_k) = 1$, it follows that $\gamma^{0^-}(H, v_1) = \gamma^{0^+}(G_1, v_1) + \min\{P_{01}(G_{2,k}, v_2, v_k), P_{10}(G_{2,k}, v_2, v_k)\}$.

(5) Let f_{H-v_1} (the restriction of f on $H - v_1$) be an IR2DF on $H - v_1$. If $f(v_1) = 0$ and $\sum_{u\in N_H(v_1)} f(u) = 0$, then $f(v_2) + f(v_k) = 0$. Since v_1 can not dominate v_2 and v_k, it follows that $\gamma^{0^+}(H, v_1) = \gamma^{0^+}(G_1, v_1) + P_{00}(G_{2,k}, v_2, v_k)$.

□

Now we present an algorithm, called STATES-FOR-CYCLE-BLOCK, to compute the parameters $\gamma^i(H, v_i)$ for $i \in \{0, 1, 2, 0^-, 0^+\}$ on a composition graph H with the specific vertex v_1, where B is a cycle block, described in Algorithm 4.

Algorithm 4 STATES-FOR-CYCLE-BLOCK

Input: A composition graph H with the specific vertex v_1 where B is a cycle block.
Output: The parameters γ^i for $i \in \{0, 1, 2, 0^-, 0^+\}$ of v_1 in H.

```
Run STATES-FOR-PATH;
Compute γ^0(H, v_1) by using Theorem 8 (1);
Compute γ^1(H, v_1) by using Theorem 8 (2);
Compute γ^2(H, v_1) by using Theorem 8 (3);
Compute γ^{0^-}(H, v_1) by using Theorem 8 (4);
Compute γ^{0^+}(H, v_1) by using Theorem 8 (5);
return γ^0(H, v_1), γ^1(H, v_1), γ^2(H, v_1), γ^{0^-}(H, v_1), γ^{0^+}(H, v_1);
```

By Theorems 6 and 8, we can now propose a dynamic programming based algorithm to solve IR2D in block-cactus graphs, as shown in Algorithm 5.

Algorithm 5 An algorithm of computing $i_{\{R2\}}(G)$ for any block-cactus graph

Input: A connected block-cactus graph G.
Output: The independent Roman $\{2\}$-domination number $i_{\{R2\}}(G)$.

```
Build the cut-tree T_G which is root at v by depth-first search;
Obtain and reverse the level-order traversal order of T_G, denoted as σ = (B_1, B_2, ..., B_p);
S ← ∅;
for i = 1 to n do
    γ^0(v_i) ← ∞; γ^1(v_i) ← 1; γ^2(v_i) ← 2; γ^{0^-}(v_i) ← ∞; γ^{0^+}(v_i) ← 0;
end for
for i = 1 to p do
    S ← S ∪ V(B_i);
    Let V(B_i) = {v_1, v_2, ..., v_k};
    Let G_i be the component with the specific vertex v_i in G[S] − E(B_i) such that V(B_i) ∩ V(G_i) = {v_i} for 1 ≤ i ≤ k;
    if B_i is a clique block then
        Let H be the graph with the specific vertex v_1 obtained from G_1, G_2, ..., G_k by adding edges to make {v_1, v_2, ..., v_k} a clique in H;
        Run STATES-FOR-CLIQUE-BLOCK(H, v_1);
    end if
    if B_i is a cycle block then
        Let H be the graph with the specific vertex v_1 obtained from G_1, G_2, ..., G_k by adding edges to make {v_1, v_2, ..., v_k} a cycle in H;
        Run STATES-FOR-CYCLE-BLOCK(H, v_1);
    end if
end for
i_{R2}(G) ← min{γ^0(G, v), γ^1(G, v), γ^2(G, v)};
return i_{R2}(G);
```

Firstly, given a connected block-cactus graph $G = (V, E)$, we build a cut-tree T_G of G using a depth-first search. Let an arbitrary cut vertex v of G be the root of T_G. We can obtain a block order $\sigma = (B_1, B_2, \ldots, B_p)$ for T_G starting at v by performing a level-order traversal and then reversing it. Let S be a set of vertices in G that have already been visited. Initialize $S = \emptyset$ and the parameters $\gamma^t(v)$ for $t \in \{0, 1, 2, 0^-, 0^+\}$ for all vertices $v \in V$ as described in Lemma 6.
Then, iterate through all the blocks in the order of σ. For each block B_i in G, add the vertices $V(B_i)$ to S. Let $V(B_i) = \{v_1, v_2, \ldots, v_k\}$. Let G_i be the connected component with the specific vertex v_i in $G[S]-E(B_i)$ such that $V(B_i) \cap V(G_i) = \{v_i\}$ for $1 \leq i \leq k$. Next, we verify the type of B_i. We can record the type of each block during the depth-first search. The time complexity for verifying the type of B_i is $O(1)$. If B_i is a clique, then let H be a composition graph of $G_1, G_2, \ldots, G_k$ obtained from the disjoint unions of $G_1, G_2, \ldots, G_k$ by adding edges $E(B_i)$ to make $\{v_1, v_2, \ldots, v_k\}$ a clique. Execute STATES-FOR-CLIQUE-BLOCK and obtain the parameters of γ^k for $k \in \{0, 1, 2, 0^-, 0^+\}$ of v_1 in H. If B_i is a cycle, then let H be a composition graph of $G_1, G_2, \ldots, G_k$ obtained from the disjoint unions of $G_1, G_2, \ldots, G_k$ by adding edges $E(B_i)$ to make $\{v_1, v_2, \ldots, v_k\}$ a cycle. Execute STATES-FOR-CYCLE-BLOCK and get the parameters of $\gamma^t(H, v_1)$ for $t \in \{0, 1, 2, 0^-, 0^+\}$. Finally, we can determine the parameters $\gamma^t(G, v)$ for $t \in \{0, 1, 2, 0^-, 0^+\}$. The result is the minimum weight among $\gamma^0(G, v), \gamma^1(G, v)$ and $\gamma^2(G, v)$.
The correctness of calculating the parameters of a specific vertex in a clique block is proved in Theorem 6. Besides, the correctness of calculating the parameters of a specific vertex in a cycle block is proved in Theorem 7 and Theorem 8. Now, we analyze the time complexity of Algorithm 5.
In Algorithm 5, given an arbitrary block-cactus graph $G = (V, E)$, the first step involves building the cut-tree T_G of the graph G. This can be accomplished in $O(|V| + |E|)$ time, as shown in [1]. Next, it takes $O(|V| + |E|)$ time to obtain and reverse the level-order traversal order of T_G. Initializing $S = \emptyset$ and the parameters $\gamma^t(v)$ for $t \in \{0, 1, 2, 0^-, 0^+\}$ for all vertices $v \in V$ by Lemma 6 takes $O(|V|)$ time. The execution time of the i-th iteration of the for-loop depends on the time required to compute the parameters of $\gamma^t(H, v_1)$ for $t \in \{0, 1, 2, 0^-, 0^+\}$. If the block B_i is a clique block, then execute STATES-FOR-CLIQUE-BLOCK(H,v_1). If the block B_i is a cycle block, then execute STATES-FOR-CYCLE-BLOCK(H,v_1). Now, we discuss the running time of the calculations in STATES-FOR-CLIQUE-BLOCK and STATES-FOR-CYCLE-BLOCK.
In Algorithm 2, the computation involves calculating the values of C_0, C_1 and C_2. Recall the formula of C_0, C_1 and C_2 from Theorem 6. Clearly, it takes $O(|V(B_i)|)$ time to compute C_0. For each vertices in B_i, it takes $O(|V(B_i)|)$ time to compute $\min\{\gamma^0(G_t, v_t), \gamma^{0^-}(G_t, v_t)\}$ and $\min\{\gamma^0(G_t, v_t), \gamma^{0^-}(G_t, v_t), \gamma^{0^+}(G_t, v_t)\}$ for $t \in [k] \setminus \{1\}$.
Denote $M_1 = \sum_{t \in [k] \setminus \{1\}} \min\{\gamma^0(G_t, v_t), \gamma^{0^-}(G_t, v_t)\}$ and
$M_2 = \sum_{t \in [k] \setminus \{1\}} \min\{\gamma^0(G_t, v_t), \gamma^{0^-}(G_t, v_t), \gamma^{0^+}(G_t, v_t)\}$. Before computing C_1 and C_2, we can compute M_1 and M_2 to reduce the number of repeated calculations. Then

we have:
$C_1 = \min_{s\in[k]\setminus\{1\}} \{\gamma^1(G_s, v_s) + M_1 - \min\{\gamma^0(G_s, v_s), \gamma^{0^-}(G_s, v_s)\}$,
$C_2 = \min_{s\in[k]\setminus\{1\}} \{\gamma^2(G_s, v_s) + M_2 - \min\{\gamma^0(G_s, v_s), \gamma^{0^-}(G_s, v_s), \gamma^{0^+}(G_s, v_s)\}$,
for $V(B_i) = \{v_1, v_2, \ldots, v_k\}$. Clearly, it takes $O(|V_{B_i}|)$ to compute M_1 and M_2. Both C_1 and C_2 can be computed in $O(|V(B_i)|)$ time. Thus, the computation in Algorithm 2 costs $O(|V(B_i)|)$ time in total. Hence, we deduce that Algorithm 2 can be computed in linear time.

In Algorithm 4, the computation involves executing STATE-FOR-PATH, which is illustrated in Algorithm 3. In Algorithm 3, it takes $O(C_1)$ time to initialize the parameters $P_{ij}(G_{k-1,k}, v_{k-1}, v_k)$ for $i \in \{0, 1, 2, 0^-, 0^+\}$ and $j \in \{0, 1, 2, 0^-, 0^+\}$, where C_1 is a constant. The statements in Theorem 7 can be computed in $O(C_2)$, where C_2 is a constant. The for-loop in lines 2-7 requires $O(|V(B_i)|)$ time. Therefore, we can obtain the parameters $P_{ij}(G_{k-1,k}, v_{k-1}, v_k)$ of $G_{2,k}$ for $i \in \{0, 1, 2, 0^-, 0^+\}$ and $j \in \{0, 1, 2, 0^-, 0^+\}$ by executing STATE-FOR-PATH in $O(|V(B_i)|)$ time. Clearly, the statements in Theorem 3 can be computed in $O(C_3)$, where C_3 is a constant. Hence, STATES-FOR-CYCLE-BLOCK can be computed in linear time.

Since each block is considered exactly once, the for-loop takes at most $O(|V|)$ time in total. Therefore, we can conclude that the total running time of Algorithm 5 is $O(|V| + |E|)$. The following theorem holds.

Theorem 9. *The independent Roman* $\{2\}$*-domination problem can be solved in linear time on block-cactus graphs.*

Corollary 2. *The independent Roman* $\{2\}$*-domination problem can be solved in linear time on cactus graphs.*

index	block	specific vertex	type	S	γ^0	γ^1	γ^2	γ^{0^-}	γ^{0^+}
1	B_1	v_{16}	cycle	$S \cup \{v_1, v_2, v_{16}, v_{17}\}$	2	2	3	∞	2
2	B_2	v_3	clique	$S \cup \{v_3, v_{16}\}$	3	3	4	2	2
3	B_3	v_5	cycle	$S \cup \{v_5, v_6, v_7, v_8, v_9, v_{10}\}$	3	3	4	∞	4
4	B_4	v_{11}	clique	$S \cup \{v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$	2	∞	2	∞	∞
5	B_5	v_4	clique	$S \cup \{v_3, v_4\}$	4	3	4	3	3
6	B_6	v_4	clique	$S \cup \{v_4, v_5, v_{11}\}$	8	8	9	8	8

Table 3. The process of the for-loop in Algorithm 5 to compute $i_{\{R2\}}(G)$.

Example 2. Consider the connected block-cactus graph $G = (V, E)$ illustrated in Figure 4(a) again. By performing a depth-first search, we obtain the cut-tree T_G of G, shown in Figure 4(b). Take v_4 as the root of T_G. Search T_G using level-order traversal and reverse it to obtain a block order $\sigma = (B_1, B_2, B_3, B_5, B_6, B_4)$. Let S be a set of vertices in G that have already been visited. Initialize $S = \emptyset$ and the parameters $\gamma^i(v)$ for $i \in \{0, 1, 2, 0^-, 0^+\}$ of all vertices $v \in V$ by Lemma 6. The process of the for-loop in Algorithm 5 to compute the independent Roman {2}-domination number of G is shown in Table 3. Using Algorithm 5, the independent Roman {2}-domination number of G is 8.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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