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Research Article

Combinations without specified separations

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Abstract: We consider the restricted subsets of $\mathbb{N}_n = \{1, 2, \dots, n\}$ with $q \geq 1$ being the largest member of the set $\mathcal{Q}$ of disallowed differences between subset elements. We obtain new results on various classes of problem involving such combinations lacking specified separations. In particular, we find recursion relations for the number of k-subsets for any $\mathcal{Q}$ when $|\mathbb{N}_q - \mathcal{Q}| \leq 2$. The results are obtained, in a quick and intuitive manner, as a consequence of a bijection we give between such subsets and the restricted-overlap tilings of an $(n+q)$-board (a linear array of $n+q$ square cells of unit width) with squares (1×1 tiles) and combs. A $(w_1, g_1, w_2, g_2, \dots, g_{t-1}, w_t)$-comb is composed of t sub-tiles known as teeth. The i-th tooth in the comb has width w_i and is separated from the $(i+1)$-th tooth by a gap of width g_i. Here we only consider combs with $w_i, g_i \in \mathbb{Z}^+$. When performing a restricted-overlap tiling of a board with such combs and squares, the leftmost cell of a tile must be placed in an empty cell whereas the remaining cells in the tile are permitted to overlap other non-leftmost filled cells of tiles already on the board.

Keywords: restricted combination, combinatorial proof, tiling, directed pseudograph.

AMS Subject classification: 05A15, 05A19, 05B45

1. Introduction

The problem of enumerating combinations with disallowed separations is as follows. We wish to find the number S_n of subsets of $\mathbb{N}_n = \{1, 2, \dots, n\}$ satisfying the condition $|x - y| \notin \mathcal{Q}$ where x, y is any pair of elements in the subset and $\mathcal{Q}$ is a given nonempty subset of $\mathbb{Z}^+$. We also wish to find the number $S_{n,k}$ of such subsets of $\mathbb{N}_n$ that are of size k. For $\mathcal{Q} = \{1\}$ it is well known that $S_n = F_{n+2}$ where F_j is the j-th Fibonacci number defined by $F_j = F_{j-1} + F_{j-2} + \delta_{j,1}$ with $F_{j<1} = 0$, where $\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise, and $S_{n,k} = \binom{n+1-k}{k}$ [12]. The quantity $\binom{n+1-k}{k}$ is also the number of ways of tiling an $(n+1)$-board (i.e., an $(n+1) \times 1$ board of unit square cells) using unit squares and k dominoes (2×1 tiles) [6]. This correspondence can be regarded as a result of the bijection given in [3], generalized in [1], and that we will extend to the most general case here. It also appears to be well known that if $\mathcal{Q} = \{1, \ldots, q\}$ then $S_n = S_{n-1} + S_{n-q-1} + \delta_{n+q,0}$ with $S_{n+q<0} = 0$. Expressions for the number of combinations when $\mathcal{Q} = \{2\}$, $\mathcal{Q} = \{q\}$ for $q \geq 2$, and $\mathcal{Q} = \{m, 2m, \ldots, jm\}$ for $j, m \geq 1$ are derived in [14], [15, 16, 19], and [1, 17], respectively.

$S_{n,k}$ also has links with graph theory. A *path scheme* $P(n, \mathcal{Q})$ is an undirected graph with vertex set $V = \mathbb{N}_n$ and edge set $\{(x, y) : |x - y| \in \mathcal{Q}\}$ [13]. A subset $\mathcal{S}$ of V is said to be an *independent set* (or a stable set) if no two elements of $\mathcal{S}$ are adjacent. The number of independent sets of path scheme $P(n, \mathcal{Q})$ of size k is then clearly $S_{n,k}$ (and the total number is S_n). The elements q_i for $i = 1, \ldots, |\mathcal{Q}|$ of set $\mathcal{Q}$ are said to form a *well-based sequence* if, when ordered so that $q_j > q_i$ for all $j > i$, then $q_1 = 1$ and for all $j > 1$ and $\Delta = 1, \ldots, q_j - 1$, there is some q_i such that $q_j = q_i + \Delta$ [13, 21]. Equivalently, the sequence of elements of $\mathcal{Q}$, the largest of which is q, is well based if $a = |\mathbb{N}_q - \mathcal{Q}|$ is zero or if for all $i, j = 1, \ldots, a$ (where i and j can be equal), $p_i + p_j \notin \mathcal{Q}$ where the p_i are the elements of $\mathbb{N}_q - \mathcal{Q}$. E.g., the only well-based sequences of length 3 are the elements of the sets $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$, and $\{1, 3, 5\}$. By considering $P(n, \mathcal{Q})$, Kitaev obtained an expression for the generating function of S_n when the elements of $\mathcal{Q}$ are a well-based sequence [13]. We will show the result via combinatorial proof and also obtain a recursion relation for $S_{n,k}$.

We define a $(w_1, g_1, w_2, g_2, \ldots, g_{t-1}, w_t)$*-comb* as a linear array of t sub-tiles (which we refer to as *teeth*) of dimensions $w_i \times 1$ separated by $t-1$ gaps of width g_i. The *length* of a comb is $\sum_{i=1}^{t-1}(w_i + g_i) + w_t$. When the t teeth are all of width w and the $t-1$ gaps are all of width g, it is referred to as a $(w, g; t)$-comb [4]; such combs can be used to give a combinatorial interpretation of products of integer powers of two consecutive generalized Fibonacci numbers [5]. Evidently, a $(w, g; 1)$-comb (or w-comb) is just a w-omino (and a $(w, 0; n)$-comb is an nw-omino). A $(w, g; 2)$-comb is also known as a (w, g)*-fence*. The fence was introduced in [7] to obtain a combinatorial interpretation of the tribonacci numbers as the number of tilings of an n-board using just two types of tiles, namely, squares and $(\frac{1}{2}, 1)$-fences. $(\frac{1}{2}, g)$-fences have also been used to obtain results on strongly restricted permutations [8].

In this paper, we start in Section 2 by giving the bijection between combinations with disallowed separations and the restricted-overlap tilings of boards with squares and combs. Counting these types of tilings requires knowledge of all permissible minimal gapless configurations of square-filled and/or restricted overlapping combs known as metatiles which are introduced in Section 3. In Section 4 we derive results from which recursion relations for $S_{n,k}$ (or S_n) for various classes of the set of disallowed differences $\mathcal{Q}$ can be obtained.

2. The bijection between combinations with disallowed separations and restricted-overlap tilings with squares and combs

In order to formulate the bijection we first introduce the concept of the *restricted-overlap tiling* of an n-board. In this type of tiling, any cell but the leftmost cell of a tile is permitted to overlap any non-leftmost cell of another tile. The C^2, C^2S, and C^3S metatiles in Figure 1 are examples where such overlap occurs (note that in that figure and in Figure 3, for clarity, some of the overlapping tiles are shown displaced downwards a little from their final position). It is readily seen that when tiling an n-board, only tiles with gaps can overlap in this sense and so restricted-overlap tiling of n-boards with just squares and other w-ominoes is the same as ordinary tiling.

Let q be the largest element in the set $\mathcal{Q}$ of disallowed differences. The comb corresponding to $\mathcal{Q}$ is of length $q+1$, has cells numbered from 0 to q, and is constructed as follows. By definition, cells 0 and q are filled (as they are the end teeth or parts of them). For the cells in between, cell i (for $i = 1, \dots, q-1$) is filled if and only if $i \in \mathcal{Q}$. For example, $\mathcal{Q} = \{1, 2, 4\}$ corresponds to a $(3, 1, 1)$-comb. One could also regard it as a $(1, 0, 2, 1, 1)$-comb but for simplicity we insist that all teeth and gaps in a comb corresponding to $\mathcal{Q}$ are of positive width. This ensures that there is only one comb that corresponds to a given $\mathcal{Q}$.

Theorem 1. *There is a bijection between the k-subsets of $\mathbb{N}_n$ each pair x, y of elements of which satisfy $|x-y| \notin \mathcal{Q}$ and the restricted overlap-tilings of an $(n+q)$-board using squares and k combs corresponding to $\mathcal{Q}$ as described above.*

Proof. The $(n+q)$-board associated with a subset $\mathcal{S}$ satisfying the conditions regarding disallowed differences has cells numbered from 1 to $n+q$. It is obtained by placing a comb at cell i (with the leftmost tooth of the comb occupying cell i) if and only if $i \in \mathcal{S}$ and then, after all the comb tiles have been placed, filling any remaining empty cells with squares. $\mathcal{S} = \varnothing$ therefore corresponds to a board tiled with squares only. The n singletons correspond to each of the possible places to put a tile of length $q+1$ on an $(n+q)$-board. If $\mathcal{S}$ contains more than one element, then there will be a tiling corresponding to $\mathcal{S}$ iff, for any two elements $x < y$ in $\mathcal{S}$, the comb representing x (which has its cell 0 at cell x of the board) has a gap at its cell $y - x$ (which is cell y on the board). This will be the case if $y - x \notin \mathcal{Q}$. □

To illustrate the bijection, we return to the $\mathcal{Q} = \{1, 2, 4\}$ example. The first case of an allowed subset of $\mathbb{N}_n$ with more than one element is $\{1, 4\}$ which can only occur if $n \geq 4$. This subset corresponds to the restricted-overlap tiling of an $(n+4)$-board with the first comb (corresponding to the element 1) placed at the start (cell number 1) of the board and the second comb (corresponding to the element 4) placed with its start in the gap of the first comb which is at cell number 4 of the board. As the length of a $(3, 1, 1)$-comb is 5, a board of length at least 8 is required for such a tiling. The remaining empty cells on the board (cell 7 and any cells beyond cell 8) are filled with squares.

The bijection also applies in the $\mathcal{Q} = \varnothing$ case if we set $q = 0$. The comb corresponding to $\mathcal{Q}$ is then a square (but we still call it a comb to distinguish it from the ordinary squares) so $S_{n,k}$ is the number of tilings of an n-board using k square combs (and $n-k$ ordinary squares) which is $\binom{n}{k}$.

Let B_n be the number of ways to restricted-overlap tile an n-board with squares and combs corresponding to $\mathcal{Q}$, and let $B_{n,k}$ be the number of such tilings that use k combs. We choose to set $B_0 = B_{0,0} = 1$.

Theorem 2. *$S_n = B_{n+q}$ and $S_{n,k} = B_{n+q,k}$.*

Proof. This follows immediately from Theorem 1. □

Lemma 1. *If the number of tilings of an n-board with squares and length-$(q+1)$ combs is given by*

$$B_n = \delta_{n,0} + \sum_{m>0} (\alpha_m \delta_{n,m} + \beta_m B_{n-m}), \quad B_{n<0} = 0,$$

then the generating function $G(x)$ for S_n can be written as

$$G(x) = \frac{1 + \sum_{m>0} \left(\alpha_{m+q} + \sum_{j=1}^{q} \beta_{m+j} \right) x^m}{1 - \sum_{m>0} \beta_m x^m}. \tag{2.1}$$

Proof. The generating function for B_n is $(1 + \sum_{m>0} \alpha_m x^m)/(1 - \sum_{m>0} \beta_m x^m)$. The first $q+1$ terms of the expansion of this must be $1 + x + \cdots + x^q$ since there is only one way to tile an n-board with squares and combs when $0 \le n \le q$, namely, the all-square tiling. From Theorem 2, $S_n = B_{q+n}$, and so

$$G(x) = \frac{1}{x^q} \left(\frac{1 + \sum_{m>0} \alpha_m x^m}{1 - \sum_{m>0} \beta_m x^m} - 1 - x - \cdots - x^{q-1} \right),$$

which simplifies to (2.1) after first putting the terms inside the parentheses over a common denominator and then in the numerator discarding the x^r terms for $1 \le r < q$ since they must sum to zero. □

It can be seen that restricted-overlap tiling with squares and just one type of $(1, g; t)$-comb, where $g = 0, 1, 2 \ldots$, will result in no overlap of the combs and so the number of such tilings is the same as for ordinary tiling. For other types of comb, there will be some tilings where overlap occurs. The case $\mathcal{Q} = \{m, 2m, \ldots, jm\}$ with $j, m \ge 1$ as studied in [1, 17], which is a generalization of all the other cases for which results for $S_{n,k}$ were obtained previously, corresponds to tiling with squares and $(1, m-1; j+1)$-combs. Thus any results for $S_{n,k}$ we obtain for nonempty $\mathcal{Q}$ not of this form (and so overlap does occur) will be new ones.

3. Metatiles

We extend the definition of a metatile given in [7, 8] to the case of restricted-overlap tiling. A *metatile* is a minimal arrangement of tiles with restricted overlap that exactly covers a number of adjacent cells without leaving any gaps. It is minimal in the sense that if one or more tiles are removed from a metatile then the result is no longer a metatile.

When tiling with squares (denoted by S) and combs corresponding to $\mathcal{Q}$ (denoted by C), the two simplest metatiles are the free square (i.e., a square which is not inside a comb), and a comb with all the gaps filled with squares. The symbolic representation of the latter metatile is CS^g where $g = \sum_i g_i$. If a comb has no gaps then it is a $(q+1)$-omino and is therefore a metatile by itself.

If the comb contains a gap, we can initiate the creation of at least one more metatile by placing the start of another comb in the gap. For instance, in the case of an $(l,1,r)$-comb where $r \geq l$, this is the only other possible metatile and so there are three metatiles in total (Figure 1(a)). However, if $r < l$, adding a comb leaves a gap which can be filled either with a square to form a completed metatile (C^2S) or with another comb which will leave a further gap, and so on (Figure 1(b)). Thus the possible metatiles in this case are C^mS for $m = 0, 1, 2, \ldots$. More generally, we have the following lemma.

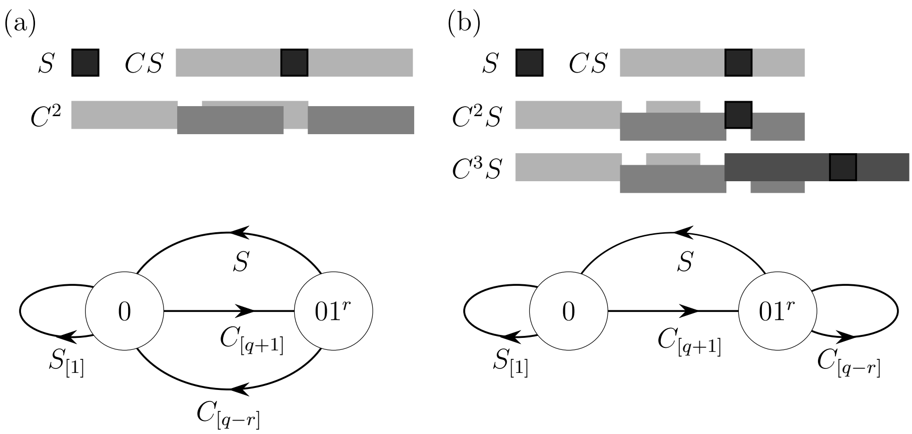

Figure 1. Metatiles, their symbolic representations, and digraphs for generating them when restricted-overlap tiling an n-board with squares and $(l,1,r)$-combs for (a) $r \geq l$ (b) $r < l$.

Lemma 2. *Let r be the length of the final tooth in the comb which has at least one gap. The set of possible metatiles when restricted-overlap tiling a board of arbitrary length with squares and combs is finite if and only if $2r \geq q$.*

Proof. We can reuse the depiction of tilings in Figure 1 but with the left tooth of each comb now replaced by arbitrary teeth and gaps (but starting with a tooth).

There is no possibility of a 'chain' of combs if, when a second comb is placed with the first cell of its first tooth just before the start of the right tooth of the first comb, the start of the right tooth of the second comb is before or aligned with the end of the right tooth of the first comb. This occurs if $q - r \le r$. If $q - r > r$, a third comb can be placed so that its first cell is immediately to the left of the right tooth of the second comb and this can be continued indefinitely. □

As with fence tiling [8], we can systematically construct all possible metatiles with the aid of a directed pseudograph (henceforth referred to as a digraph). As before, the *0 node* corresponds to the empty board or the completion of the metatile. The other nodes represent the state of the incomplete metatile. The occupancy of a cell in it is represented by a binary digit: 0 for empty, 1 for filled. We label the node by discarding the leading 1s and trailing zeros and so the label always starts with a 0 and ends with a 1, with 1^r denoting 1 repeated r times. Each arc represents the addition of a tile and any walk beginning and ending at the 0 node without visiting it in between corresponds to a metatile. With our restricted-overlap tiling, all nodes have an out-degree of 2 as a gap may always be filled by a square or the start of a comb. The destination node is obtained by performing a bitwise OR operation on the bits representing the added tile and the label of the current node, and then discarding the leading 1s. Figure 1 illustrates this for the metatiles involved when tiling with squares and $(l, 1, r)$-combs.

The most important property of a metatile is its length. This is obtained by summing the contribution to the length associated with each arc in the walk representing the metatile. The contribution to the length associated with an arc is zero if it corresponds to the addition of a square (except in the case of the trivial S metatile) and for a comb arc equals $q + 1 - d$ where d is the number of digits in the node label from which the arc emanates except when that node is the 0 node in which case $d = 0$. In the digraphs, the contribution to the length associated with each arc is given as a subscript in square brackets if it is not zero. For example, from the digraph in Figure 1(b), for tiling with squares and $(l, 1, r < l)$-combs we see that the length of a $C^m S$ metatile where $m > 0$ is $q + 1 + (m - 1)(q - r) = ml + r + 1$ as in this case $q = l + r$.

4. Counting restricted-overlap tilings

For brevity, we just give results for B_n and $B_{n,k}$ as these are easily converted to recursion relations for S_n and $S_{n,k}$ and the generating function for S_n using Theorem 2 and Lemma 1. As with ordinary (non-overlapping) tiling, the following lemma is the basis for obtaining recursion relations for B_n and $B_{n,k}$.

Lemma 3. *For all integers n and k,*

$$B_n = \delta_{n,0} + \sum_{i=1}^{N_\mathrm{m}} B_{n-l_i}, \tag{4.1a}$$

$$B_{n,k} = \delta_{n,0}\delta_{k,0} + \sum_{i=1}^{N_\mathrm{m}} B_{n-l_i,k-k_i}, \tag{4.1b}$$

where N_m is the number of metatiles, l_i is the length of the i-th metatile and k_i is the number of combs it contains, and $B_{n<0} = B_{n,k<0} = B_{n<k,k} = 0$.

Proof. As in [6, 8], we condition on the last metatile on the board. To obtain (4.1b) we note that if an n-board tiled with squares and k combs ends with a metatile of length l_i that contains k_i combs then there are $B_{n-l_i,k-k_i}$ ways to tile the rest of the board. The $\delta_{n,0}\delta_{k,0}$ term is from the requirement that $B_{0,0} = 1$. This term arises in the sum when a particular metatile containing k combs completely fills the board; there is only one tiling where this occurs. The derivation of (4.1a) is analogous but we ignore the number of combs. Alternatively, each term in (4.1a) is obtained from the corresponding term in (4.1b) by summing over all k. □

Theorem 3. *If $\mathcal{Q} = \{1, \ldots, l-1, q-r+1, \ldots, q-1, q\}$ (or $\mathcal{Q} = \{q-r+1, \ldots, q-1, q\}$ if $l = 1$) where $l \geq 1$, $2r \geq q$, and $q+1 \geq l+r$, then*

$$B_n = \delta_{n,0} + B_{n-1} + B_{n-q-1} + \sum_{j=0}^{q-l-r} f_j^{(l)} B_{n-l-q-1-j},$$

$$B_{n,k} = \delta_{n,0}\delta_{k,0} + B_{n-1,k} + B_{n-q-1,k-1} + \sum_{j=0}^{q-l-r} \sum_{i=0}^{\lfloor j/l \rfloor} \binom{j-(l-1)i}{i} B_{n-l-q-1-j,k-2-i},$$

where the $(1,l)$-bonacci number $f_j^{(l)} = f_{j-1}^{(l)} + f_{j-l}^{(l)} + \delta_{j,0}$, $f_{j<0}^{(l)} = 0$. The sums are omitted if $q+1 = l+r$.

Proof. We use Lemma 3. If $q+1 = l+r$, C is just a $(q+1)$-omino and the results follow immediately. Otherwise, C is an $(l, q+1-l-r, r)$-comb. There are two trivial metatiles: S and $CS^{q+1-l-r}$ which have lengths of 1 and $q+1$, respectively. For the remaining metatiles, since $2r \geq q$, as described in the proof of Lemma 2, the final comb in the metatile must start within the gap of the first comb. Number the cells in this gap from $j = 0$ to $q-l-r$. If the start of the left tooth of the final comb in the metatile lies in cell j of this gap, the length of the metatile is $l+q+1+j$. Cells 0 to $j-1$ of the gap can have either an S or the left tooth of a C and we end up with a metatile of this length. The number of ways this can be done is simply the number of ways to tile a j-board using squares and l-ominoes which is $f_j^{(l)}$ [6] (when $l = 1$, the l-ominoes are regarded as being distinguishable from the ordinary squares). Hence there are $f_j^{(l)}$ metatiles of length $l+q+1+j$ of which $\binom{j-li+i}{i}$ have $2+i$ combs. □

As an example of the application of the above theorem, we consider the case $\mathcal{Q} = \{2\}$. Then $l = r = 1$ and $q = 2$ and we obtain the recursion relation $B_{n,k} = \delta_{n,0}\delta_{k,0} + B_{n-1,k} + B_{n-2,k-1} + B_{n-4,k-2}$. Using Theorem 2 gives $S_{n,k} = \delta_{n,-2}\delta_{k,0} + S_{n-1,k} + S_{n-2,k-1} + S_{n-4,k-2}$ which is in agreement with the recursion relation first obtained by Konvalina [14]. Note that the metatiles in this case are the square (S), a comb with its gap filled by a square (CS), and two interlocking combs (C^2). No overlapping occurs in this case and so it also an ordinary tiling which has been examined extensively [9]. For instances where the largest element of $\mathbb{N}_q - \mathcal{Q}$ (the set of allowed differences less than q) is $q - r$ and $2r \geq q$ but the other conditions in Theorem 3 do not hold, as the number of possible metatiles is finite (by Lemma 2), it is straightforward to find the length and number of combs in each of them and then use (4.1) to obtain recursion relations for B_n and $B_{n,k}$.

To enable us to tackle some cases where there are an infinite number of possible metatiles, we begin by reviewing some terminology describing features of the digraphs used to construct metatiles [8]. A *cycle* is a closed walk in which no node or arc is repeated aside from the starting node. We refer to cycles by the arcs they contain. For example, the digraph in Figure 1(a) has 3 cycles: $S_{[1]}$, $C_{[q+1]}S$, and $C_{[q+1]}C_{[q-r]}$. An *inner cycle* is a cycle that does not include the 0 node. For example, the digraph in Figure 1(b) has a single inner cycle, namely, $C_{[q-r]}$. If a digraph has an inner cycle, there are infinitely many possible metatiles as, once reached, the cycle can be traversed an arbitrary number of times before the walk returns to the 0 node. If all of the inner cycles of a digraph have one node (or more than one node) in common, that node (or any one of those nodes) is said to be the *common node*. In the case of a digraph with one inner cycle, any of the nodes of the inner cycle can be chosen as the common node. A *common circuit* is a simple path from the 0 node to the common node followed by a simple path from the common node back to the 0 node. For example, in the digraph in Figure 1(b), the common node is 01^r and the common circuit is $C_{[q+1]}S$. If a digraph has a common node, members of an infinite family of metatiles can be obtained by traversing the first part of the common circuit from the 0 node to the common node and then traversing the inner cycle(s) an arbitrary number of times (and in any order if there are more than one) before returning to the 0 node via the second part of the common circuit. An *outer cycle* is a cycle that includes the 0 node but does not include the common node. Thus any metatile which is not a member of an infinite family of metatiles has a symbolic representation derived from an outer cycle. E.g., the only outer cycle in the digraph in Figure 1(b) is $S_{[1]}$.

The following theorem is a restatement of Theorem 5.4 and Identity 5.5 in [8] but with more compact expressions for B_n and $B_{n,k}$ and improved proofs. Note that the length of a cycle or circuit is simply the total contributions to the length of the arcs it contains.

Theorem 4. *For a digraph possessing a common node, let $l_{\mathrm{o}i}$ be the length of the i-th outer cycle ($i = 1, \dots, N_{\mathrm{o}}$) and let $k_{\mathrm{o}i}$ be the number of combs it contains, let L_r be the length of the r-th inner cycle ($r = 1, \dots, N$) and let K_r be the number of combs it contains,* *and let $l_{\mathrm{c}i}$ be the length of the i-th common circuit ($i = 1, \dots, N_{\mathrm{c}}$) and let $k_{\mathrm{c}i}$ be the number of combs it contains. Then for all integers n and k,*

$$B_n = \delta_{n,0} + \sum_{r=1}^{N}(B_{n-L_r} - \delta_{n,L_r}) + \sum_{i=1}^{N_{\mathrm{o}}}\left(B_{n-l_{\mathrm{o}i}} - \sum_{r=1}^{N} B_{n-l_{\mathrm{o}i}-L_r}\right) + \sum_{i=1}^{N_{\mathrm{c}}} B_{n-l_{\mathrm{c}i}}, \tag{4.2a}$$

$$\begin{aligned} B_{n,k} &= \delta_{n,0}\delta_{k,0} + \sum_{r=1}^{N}(B_{n-L_r,k-K_r} - \delta_{n,L_r}\delta_{k,K_r}) \\ &\quad + \sum_{i=1}^{N_{\mathrm{o}}}\left(B_{n-l_{\mathrm{o}i},k-k_{\mathrm{o}i}} - \sum_{r=1}^{N} B_{n-l_{\mathrm{o}i}-L_r,k-k_{\mathrm{o}i}-K_r}\right) + \sum_{i=1}^{N_{\mathrm{c}}} B_{n-l_{\mathrm{c}i},k-k_{\mathrm{c}i}}, \end{aligned} \tag{4.2b}$$

where $B_{n<0} = B_{n,k<0} = B_{n<k,k} = 0$.

Proof. From Lemma 3,

$$B_n = \delta_{n,0} + \sum_{i=1}^{N_{\mathrm{o}}} B_{n-l_{\mathrm{o}i}} + \sum_{i=1}^{N_{\mathrm{c}}} \sum_{j_1,\dots,j_N\ge 0} \binom{j_1+\cdots+j_N}{j_1,\dots,j_N} B_{n-\lambda_i}, \tag{4.3a}$$

$$B_{n,k} = \delta_{n,0}\delta_{k,0} + \sum_{i=1}^{N_{\mathrm{o}}} B_{n-l_{\mathrm{o}i},k-k_{\mathrm{o}i}} + \sum_{i=1}^{N_{\mathrm{c}}} \sum_{j_1,\dots,j_N\ge 0} \binom{j_1+\cdots+j_N}{j_1,\dots,j_N} B_{n-\lambda_i,k-\kappa_i} \tag{4.3b}$$

with $B_{n<0} = B_{n,k<0} = B_{n<k,k} = 0$, where $\lambda_i = l_{\mathrm{c}i} + \sum_{s=1}^{N} j_s L_s$ and $\kappa_i = k_{\mathrm{c}i} + \sum_{s=1}^{N} j_s K_s$. The multinomial coefficient (which counts the number of arrangements of the inner cycles) results from the fact that changing the order in which the inner cycles are traversed (after the common node is reached via the outgoing path of a common circuit) gives rise to distinct metatiles of the same length. The sum of terms over $j_1, \dots, j_N$ in (4.3a) may be re-expressed as

$$\sum_{j_1,\dots,j_N\ge 0} M(\varnothing) B_{n-\lambda_i} = B_{n-l_{\mathrm{c}i}} + \sum_{m=0}^{N-1} \sum_{\mathcal{R}_m} \sum_{\substack{j_{s\notin\mathcal{R}_m}\ge 1,\\ j_{t\in\mathcal{R}_m}=0}} M(\mathcal{R}_m) B_{n-\lambda_i}$$

where $M(\mathcal{R})$ denotes the multinomial coefficient $\binom{j_1+\cdots+j_N}{j_1,\dots,j_N}$ with $j_{t\in\mathcal{R}} = 0$, and $\mathcal{R}_m$ denotes a set of m numbers drawn from $\mathbb{N}_N$. For example, if $N > 2$ an instance of $\mathcal{R}_2$ is $\{1, 2\}$ in which case $M(\mathcal{R}_2) = \binom{j_3+\cdots+j_N}{j_3,\dots,j_N}$. Replacing n by $n - L_r$ in (4.3a) gives

$$B_{n-L_r} = \delta_{n,L_r} + \sum_{i=1}^{N_{\mathrm{o}}} B_{n-l_{\mathrm{o}i}-L_r} + \sum_{i=1}^{N_{\mathrm{c}}} \sum_{j_1,\dots,j_N\ge 0} M(\varnothing) B_{n-\lambda_i-L_r}.$$

After changing j_r to $j_r - 1$, the sum of terms over $j_1, \dots, j_N$ may be re-expressed as

$$\sum_{\substack{j_r\ge 1,\\ j_{t\ne r}\ge 0}} M_r(\varnothing) B_{n-\lambda_i} = \sum_{m=0}^{N-1} \sum_{\mathcal{R}_m^{[r]}} \sum_{\substack{j_{s\notin\mathcal{R}_m^{[r]}}\ge 1,\\ j_{t\in\mathcal{R}_m^{[r]}}=0}} M_r(\mathcal{R}_m^{[r]}) B_{n-\lambda_i},$$

where $M_r(\mathcal{R})$ denotes the multinomial coefficient $M(\mathcal{R})$ with j_r replaced by $j_r - 1$ (so, for example, $M_3(\{1,2\}) = \binom{j_3+\cdots+j_N-1}{j_3-1,\ldots,j_N}$) and $\mathcal{R}_m^{[r]}$ is a set of m numbers none of which equal r drawn from $\mathbb{N}_N$. After subtracting $\sum_{r=1}^{N} B_{n-L_r}$ from (4.3a) and using the result for multinomial coefficients that $M(\mathcal{R}) = \sum_{r\notin R} M_r(\mathcal{R})$, we obtain (4.2a). Similarly, subtracting $\sum_{r=1}^{N} B_{n-L_r,k-K_r}$ from (4.3b) gives (4.2b). □

The rest of the theorems in this section concern families of $\mathcal{Q}$ whose corresponding digraphs each have a common node. Once the lengths of and the number of combs in each cycle and common circuit have been determined, the recursion relations for B_n and $B_{n,k}$ follow immediately from Theorem 4.

It can be seen from (4.2) that the recursion relation for B_n can be obtained from that for $B_{n,k}$ by replacing $B_{n-\nu,k-\kappa}$ and $\delta_{n,\nu}\delta_{k,\kappa}$ by $B_{n-\nu}$ and $\delta_{n,\nu}$, respectively, for any ν and κ. For the remaining theorems we therefore give the recursion relation for $B_{n,k}$ and only also show the one for B_n if we use the expression for B_n elsewhere.

For the more generally applicable theorems that follow, B_n and $B_{n,k}$ are given most simply in terms of the elements p_i of $\mathbb{N}_q - \mathcal{Q}$, the set of allowed differences less than q. We order the p_i so that $p_i < p_{i+1}$ for all $i = 1, \ldots, a-1$, where $a = |\mathbb{N}_q - \mathcal{Q}|$, the number of allowed differences less than q. Note that if the comb corresponding to $\mathcal{Q}$ has leftmost and rightmost teeth of widths l and r, respectively, then $p_1 = l$ and $p_a = q-r$. In digraphs, we let σ_i (for $i = 1, \ldots, a$) denote the bit string corresponding to filling the first $i-1$ empty cells of a comb with squares and discarding the leading 1s. Thus σ_a is always 01^r. It is also easily seen that the comb arc leaving the σ_i node is $C_{[p_i]}$.

Theorem 5. *If the elements of $\mathcal{Q}$ are a well-based sequence then*

$$B_n = \delta_{n,0} + B_{n-1} + B_{n-q-1} + \sum_{i=1}^{a}(B_{n-p_i} - B_{n-p_i-1} - \delta_{n,p_i}), \tag{4.4a}$$

$$B_{n,k} = \delta_{n,0}\delta_{k,0} + B_{n-1,k} + B_{n-q-1,k-1} + \sum_{i=1}^{a}(B_{n-p_i,k-1} - B_{n-p_i-1,k-1} - \delta_{n,p_i}\delta_{k,1}). \tag{4.4b}$$

If $\mathcal{Q} = \mathbb{N}_q$ (and so $a = 0$) then the sums over i are omitted.

Proof. If $\mathcal{Q} = \mathbb{N}_q$ the comb is a $(q+1)$-omino and the results follow immediately. Otherwise, we first need to establish that the comb leaving the σ_i node for any $i = 1, \ldots, a$ in the digraph (Figure 2; Figure 1(b) shows the $a = 1$ instance of the digraph) takes us back to the σ_1 node. This is equivalent to there being a gap at position $p_i + p_j$ (for any $i, j = 1, \ldots, a$) of the first comb (or that position being beyond the end of the comb) where p_j can be viewed as the position of the j-th empty cell in the comb added at cell p_i in the first comb. This must be the case by the definition of a well-based sequence; if there were no gap it would mean $p_i + p_j \in \mathcal{Q}$ which is impossible. The digraph has a inner cycles, namely, $S^{i-1}C_{[p_i]}$ for $i = 1, \ldots, a$, which have lengths p_i,

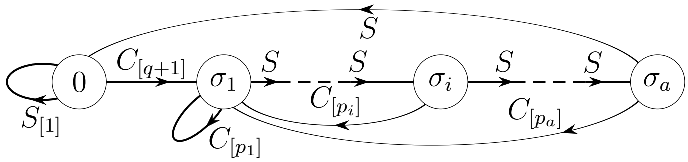

Figure 2. Digraph for tiling a board with squares and combs corresponding to a well-based sequence of disallowed differences.

respectively. The common node is σ_1. There is one common circuit ($C_{[q+1]}S^a$), which is of length $q+1$, and one outer cycle ($S_{[1]}$), which is of length 1. □

The following corollary was established via graph theory and results concerning bit strings in [13].

Corollary 1. *The generating function for S_n when the elements q_i of $\mathcal{Q}$ form a well-based sequence is given by*

$$G(x) = \frac{c}{(1-x)c - x}, \tag{4.5}$$

where $c = 1 + \sum_{i=1}^{|\mathcal{Q}|} x^{q_i}$.

Proof. Applying Lemma 1 to (4.4a) it can be seen that the numerator of (2.1) reduces to

$$1 + \sum_{m=1}^{q}\left(\sum_{j=1}^{q-m}\left(\sum_{i=1}^{a}(\delta_{m+j,p_i} - \delta_{m+j,p_i+1})\right) + 1\right)x^m,$$

where the $+1$ inside the brackets results from the fact that β_{q+1} appears as a term in the sum over j for every m up to q. Note also that we must have $p_1 > 1$ and $p_a < q$. When summed over j, $\delta_{m+j,p_i} - \delta_{m+j,p_i+1}$ cancels (thus leaving just the $+1$ multiplying the x^m) except if $p_i = m$ in which case δ_{m+j,p_i} is always zero and the $-\delta_{m+j,p_i+1}$ when $j = 1$ cancels the $+1$. Hence the numerator simplifies to c. The denominator of (2.1) is, in the present case, $1-x-x^{q+1}-(1-x)\bar{c}$, where $\bar{c} = \sum_{i=1}^{a} x^{p_i}$. Using the result that $c + \bar{c} = \sum_{i=0}^{q} x^i = (1-x^{q+1})/(1-x)$ it is then easily shown that the denominator can be re-expressed as $(1-x)c - x$. □

We consider the case $\mathcal{Q} = \{1, 3, 5\}$ as an example for the application of Theorem 5. Then $q = 5$, $a = 2$, $p_1 = 2$, and $p_2 = 4$. Hence $B_{n,k} = \delta_{n,0}\delta_{k,0} - \delta_{n,2}\delta_{k,1} - \delta_{n,4}\delta_{k,1} + B_{n-1,k} + B_{n-2,k-1} - B_{n-3,k-1} + B_{n-4,k-1} - B_{n-5,k-1} + B_{n-6,k-1}$. Since in this case $c = 1 + x + x^3 + x^5$, Corollary 1 gives the generating function for S_n of $(1 + x + x^3 + x^5)/(1 - x - x^2 + x^3 - x^4 + x^5 - x^6)$ as found by Kitaev [13].

The proofs of some of the results that follow (starting with the next lemma which is used in the proof of Theorem 6) require combs where the number of gaps depends on the particular instance of $\mathcal{Q}$. We therefore extend our notation: an $(l, [g], r)$-comb is a comb of length $l+g+r$ whose left and right teeth have widths of l and r, respectively.

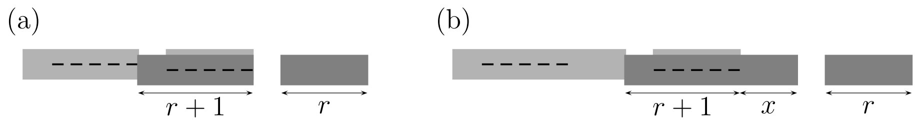

Figure 3. The origin of the 1-arc inner cycle at the 01^r node when restricted-overlap tiling with squares and (a) $(r+2-g,[g],r)$-combs where $1 \le g \le r+1$ (b) $(w_1, g_1, \dots, w_t)$-combs such that $w_t = r$, $g_{t-1} = 1$, $w_{t-1} \ge q-2r-1 = x$, and $t \ge 2$.

Lemma 4. *When restricted-overlap tiling an n-board with S and C, where C contains at least one gap and the width of the final tooth is r, there is a 1-arc inner cycle $C_{[q-r]}$ containing the 01^r node iff (a) $q = 2r+1$ or (b) the final gap is of unit width and the penultimate tooth has a width of at least $q-2r-1$.*

Proof. If $q < 2r+1$, by Lemma 2, there can be no inner cycles. If $q = 2r+1$ and so the length of C is $2(r+1)$, we have the situation depicted in Figure 3(a). In the figure, the cells of each comb marked with the dashed line could be parts of teeth or gaps. For the first (paler) comb depicted in each case, any such cells which are parts of gaps are filled with other tiles leaving the final gap cell empty (which corresponds to the 01^r node). The start of the next comb is placed in that cell (and we return to the 01^r node). If $q > 2r+1$ the final gap in the comb must be of unit width and the width w_{t-1} of the penultimate tooth cannot be less than $x = q+1-2(r+1)$ (Figure 3(b)). □

Theorem 6. *Let θ be the bit string representation of $\mathcal{Q}$ whereby the j-th bit from the right of θ is 1 if and only if $j \in \mathcal{Q}$. By $\lfloor \theta/2^b \rfloor$ we mean discarding the rightmost b bits in θ and shifting the remaining bits to the right b places. Using | to denote the bitwise OR operation, if $\theta \mid \lfloor \theta/2^{p_i-1} \rfloor$ for each $i = 1, \dots, a-1$ is all ones after discarding the leading zeros, $a \ge 2$, and $p_a = q-r$ (which implies that $r \ge 1$), then if (a) $q = 2r+1$ or (b) $q > 2r+1$ and $1 \le p_{a-1} \le r$, then*

$$B_{n,k} = \delta_{n,0}\delta_{k,0} - \delta_{n,q-r}\delta_{k,1} + B_{n-1,k} + B_{n-q+r,k-1} - B_{n-q+r-1,k-1} + B_{n-q-1,k-1} + \sum_{i=1}^{a-1}(B_{n-q-1-p_i,k-2} - B_{n-2q+r-1-p_i,k-3}). \tag{4.6}$$

Proof. The condition $p_a = q-r$ means that the final tooth is of width r. The conditions (a) and (b) correspond to those in Lemma 4 and thus guarantee a single-comb inner cycle at the 01^r node. The condition on θ means that placing a comb at an empty cell (other than the final empty cell) will result in all gaps in the combs to the right of this point being filled. On the digraph this means that there is an arc from the σ_i node, where $i = 1, \dots, a-1$, to the 0 node. Tiling with squares and combs corresponding to $\mathcal{Q}$ leads to the digraph shown in Figure 4. There is one inner cycle ($C_{[q-r]}$) and one common circuit ($C_{[q+1]}S^a$). The outer cycles are $S_{[1]}$ and $C_{[q+1]}S^{i-1}C_{[p_i]}$ for $i = 1, \dots, a-1$ and their respective lengths are 1 and $q+1+p_i$. □

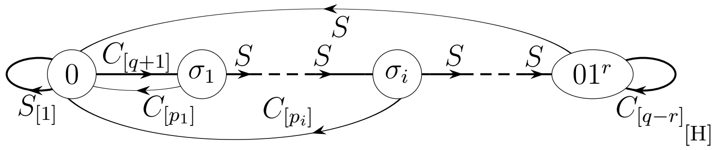

Figure 4. Digraph for tiling a board with squares and combs corresponding to $\mathcal{Q}$ specified in Theorem 6.

It is straightforward to verify that the following four classes of $\mathcal{Q}$ satisfy the conditions for Theorem 6 to apply: (i) $\mathcal{Q} = \{2, \dots, q-r-1, q-r+1, \dots, q\}$ where $r \geq 1$ and $q \geq \max(2r+1, 4)$ (e.g., for $q \leq 7$: $\{2,4\}$, $\{2,3,5\}$, $\{2,4,5\}$, $\{2,3,4,6\}$, $\{2,3,5,6\}$, $\{2,3,4,5,7\}$, $\{2,3,4,6,7\}$, $\{2,3,5,6,7\}$); (ii) $\mathcal{Q} = \{1, \dots, l-1, l+1, \dots, q-r-1, q-r+1, \dots, q\}$ where $r \geq l \geq 2$ and $q \geq 2r+1$ and $2l \neq q-r$ (e.g., for $q \leq 8$: $\{1,3,4,6,7\}$, $\{1,2,4,6,7,8\}$, $\{1,3,4,5,7,8\}$, $\{1,3,4,6,7,8\}$); (iii) $q = 2r+1$, $p_1 = l$, $p_a = r+1$, and $l \leq r \leq 2l-2$ (e.g., for $q \leq 9$: $\{1,4,5\}$, $\{1,2,5,6,7\}$, $\{1,2,6,7,8,9\}$, $\{1,2,3,6,7,8,9\}$, $\{1,2,4,6,7,8,9\}$); (iv) $\mathcal{Q} = \{2, 4, \dots, 2a, 2a+1, \dots, q\}$ where $a \geq 3$ and $q = 4a-4, 4a-3$ (e.g., for $q \leq 9$: $\{2,4,6,7,8\}$, $\{2,4,6,7,8,9\}$). These classes cover all cases where the theorem applies for $q \leq 9$.

As an example, we consider the case $\mathcal{Q} = \{2,4\}$. Then $q = 4$, $r = 1$, $a = 2$, $p_1 = 1$, and $p_2 = 3$. From (4.6) we get $B_{n,k} = \delta_{n,0}\delta_{k,0} - \delta_{n,3}\delta_{k,1} + B_{n-1,k} + B_{n-3,k-1} - B_{n-4,k-1} + B_{n-5,k-1} + B_{n-6,k-2} - B_{n-9,k-3}$. An explicit formula for $S_{n,k}$ in this case can be obtained in terms of sums of products of binomial coefficients [17]. Summing over k gives us a recursion relation for B_n whose generating function $(1-x^3)/(1-x-x^3+x^4-x^5-x^6+x^9)$ is that of sequence A224809 in the OEIS [20] which does indeed correspond to numbers of subsets with differences not equalling 2 or 4.

Note that, omitting the sum, Theorem 6 holds for the case $a = 1$ if $p_1 = q-r$ and $q \geq 2r+1$. It then coincides with Theorem 5.

Theorem 7. *Suppose $p_1 = l$ and $p_a = 2l$. Then if either (a) $q = 4l-1$ or (b) $p_{a-1} \leq q-2l$ where $q < 4l-1$, then*

$$\begin{aligned} B_{n,k} = {} & \delta_{n,0}\delta_{k,0} - \delta_{n,2l}\delta_{k,1} + B_{n-1,k} + B_{n-2l,k-1} - B_{n-2l-1,k-1} + B_{n-q-1,k-1} \\ & + B_{n-q-l-1,k-2} + B_{n-q-2l-1,k-3} - B_{n-q-3l-1,k-3} - B_{n-q-4l-1,k-4} \\ & + \sum_{i=2}^{a-1} (B_{n-q-p_i-1,k-2} - B_{n-q-2l-p_i-1,k-3}), \end{aligned} \tag{4.7}$$

where the sum is omitted if $a = 2$.

Proof. Tiling with squares and $(l, [l+1], q-2l)$-combs leads to the digraph shown in Figure 5. Note that if $a = 2$, the σ_i nodes are omitted since $i = 1, \dots, a-2$. There is just one inner cycle ($C_{[2l]}$) and one common circuit ($C_{[q+1]}S^a$). Their respective lengths are $2l$ and $q+1$. The outer cycles are $S_{[1]}$, $C_{[q+1]}C_{[l]}\{S, C_{[l]}\}$, and, if $a \geq 3$, $C_{[q+1]}S^{i-1}C_{[p_i]}$ for $i = 2, \dots, a-1$. Their respective lengths are 1, $q+l+1$, $q+2l+1$, and $q+p_i+1$. □

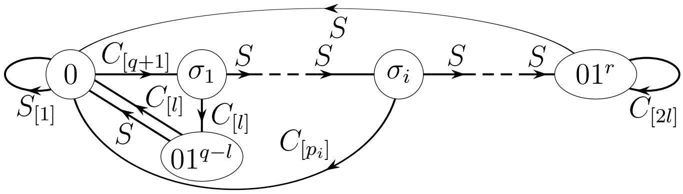

Figure 5. Digraph for tiling a board with squares and $(l,[l+1],r=q-2l)$-combs used in the proof of Theorem 7.

The instances of $\mathcal{Q}$ with $q\le 9$ to which Theorem 7 applies are $\{3\}$, $\{1,3,5,6\}$, $\{1,5,6,7\}$, $\{1,3,5,6,7\}$, and $\{1,2,4,5,7,8,9\}$. As an example, we consider the $\mathcal{Q}=\{3\}$ case. Then $l=1$ and $a=2$ and Theorem 7 yields $B_{n,k}=\delta_{n,0}\delta_{k,0}-\delta_{n,2}\delta_{k,1}+B_{n-1,k}+B_{n-2,k-1}-B_{n-3,k-1}+B_{n-4,k-1}+B_{n-5,k-2}+B_{n-6,k-3}-B_{n-7,k-3}-B_{n-8,k-4}$. For the case $\mathcal{Q}=\{q\}$, Prodinger [19] derived an explicit formula for $S_{n,k}$ involving the sum of a product of four binomial coefficients. Konvalina and Liu also showed that $S_{qm+j}=F_{m+2}^{q-j}F_{m+3}^{j}$ for $m=0,1,2,\dots$ and $j=0,1,\dots,q-1$ [16]. Returning to our $\mathcal{Q}=\{3\}$ result, summing $B_{n,k}$ over all k to obtain a recursion relation for B_n and then applying Lemma 1 gives the generating function for S_n of $(1+x+x^2+3x^3+x^4-x^5-2x^6-x^7)/(1-x-x^2+x^3-x^4-x^5-x^6+x^7+x^8)$ which matches that for the sequence $S_{3m+j}=F_{m+2}^{3-j}F_{m+3}^{j}$ for $m=0,1,2,\dots$ and $j=0,1,2$ (see A006500 in the OEIS [20]).

Theorem 8. *If $p_1=l$, $p_a=q-r$, $l>r$ and (i) $q=2l$ or (ii) $a=2$, $q\ge 2l$, but $q\ne 2l+r$, then*

$$B_{n,k}=\delta_{n,0}\delta_{k,0}+B_{n-1,k}+B_{n-2l-1,k-1}+B_{n-3l-1,k-2}$$
$$+\sum_{i=2}^{a}(B_{n-p_i,k-1}-B_{n-p_i-1,k-1}+B_{n-l-p_i,k-2}-B_{n-l-p_i-1,k-2}-\delta_{n,p_i}\delta_{k,1}-\delta_{n,l+p_i}\delta_{k,2}). \tag{4.8}$$

Proof. Tiling with (i) squares and $(l,[l-r+1],r)$-combs, where $l>r$, or (ii) squares and $(l,1,m\ne l-1,1,r)$-combs, where $0<l-r\le m+1$, leads to the digraph shown in Figure 6. There are $2(a-1)$ inner cycles: $\{S,C_{[l]}\}S^{i-2}C_{[p_i]}$ for $i=2,\dots,a$. Their lengths are p_i and $l+p_i$. The common node is σ_1 and so the common circuits are $C_{[2l+1]}\{S,C_{[l]}\}S^{a-1}$ which have lengths of $2l+1$ and $3l+1$. □

The instances of $\mathcal{Q}$ for which Theorem 8 applies when $q\le 8$ are $\{1,4\}$, $\{1,2,6\}$, $\{1,2,4,6\}$, $\{1,2,5,6\}$, $\{1,3,4,6\}$, $\{1,2,4,6,7\}$, $\{1,3,4,5,7\}$, $\{1,2,3,8\}$, $\{1,2,3,5,8\}$, $\{1,2,3,6,8\}$, $\{1,2,3,5,6,8\}$, $\{1,2,3,7,8\}$, $\{1,2,3,5,7,8\}$, $\{1,2,3,6,7,8\}$, $\{1,2,4,5,6,8\}$, and $\{1,3,4,5,6,8\}$.

We conclude by showing that all possible $\mathcal{Q}$ such that $a\le 2$ have been covered by the theorems given here. When $a=0$, the comb C is a $(q+1)$-omino and $B_{n,k}$ is given by (4.4b). When $a=1$, C is an $(l,1,r)$-comb (and the two possible cases are

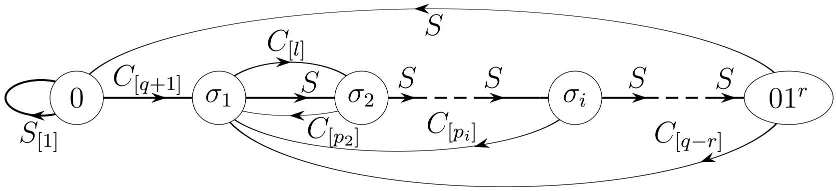

Figure 6. Digraph for tiling a board with squares and $(l, [l-r+1], r)$-combs, where $l > r$, or with squares and $(l, 1, m \neq l-1, 1, r)$-combs, where $0 < l - r \leq m + 1$, used in the proof of Theorem 8.

shown in Figure 1). Then if $r \geq l$, Theorem 3 applies. Otherwise, if $l > r$, Theorem 5 applies since $2p_1 > q$ (as $p_1 = l$ and $q = l + r$) and hence the elements of $\mathcal{Q}$ are a well-based sequence. When $a = 2$, C is either an $(l, 2, r)$-comb or an $(l, 1, m, 1, r)$-comb for some $l, m, r \geq 1$. In the former case, $q = l + r + 1$ and so the condition $2r \geq q$ leads to Theorem 3 applying when $l < r$. When $l = r$, it is covered by the class (iii) instances of $\mathcal{Q}$ that apply to Theorem 6 except when $l = 1$ in which case Theorem 7(a) applies. When $l = r + 1$, we have $q = 2l$ and $p_2 = l + 1 = 2l - r$ and so Theorem 8 applies. The final possibility is if the elements of $\mathcal{Q}$ are a well-based sequence (and the case is then covered by Theorem 5) and this occurs if $2p_1 > q$ which implies that $l > r+1$. For $(l, 1, m, 1, r)$-combs, $q = l+m+r+1$ and so the $2r \geq q$ case (Theorem 3) is when $l < r - m$. Of the $l \geq r - m$ cases we first consider those where $l = m + 1$. This can arise in two ways. If $2p_1 = p_2$ (which implies $l = m + 1$) and $p_1+p_2 > q$ (which implies $l > r$) then the elements of $\mathcal{Q}$ are a well-based sequence and Theorem 5 applies. When $l = m+1$ and $l \leq r$ then Theorem 7(b) applies since these conditions can be re-expressed as $p_2 = 2l$ and $p_1 \leq r$, respectively (and $l \geq r - m$ in this case implies $q \leq 4l - 1$). When $l = 1$ (and $l \geq r - m$), the case falls into class (i) to which Theorem 6 applies. There are three other ways in which $l \neq m + 1$ arises when $l \geq r - m$. If $l \leq r$ then we have class (ii) to which Theorem 6 applies. If $r < l \leq m + r + 1$ then Theorem 8 applies. Finally, if $2p_1 > q$ (which implies $l > m+r+1$) then the elements of $\mathcal{Q}$ are a well-based sequence and Theorem 5 again applies.

5. Discussion

As $a = |\mathbb{N}_q - \mathcal{Q}|$ increases, so, in general, does the number of inner cycles in the digraph and we find more and more instances (e.g., when $\mathcal{Q} = \{1, 5\}$ [2]) where the digraph has inner cycles but no common node. In the simpler of such cases, it is still possible to derive general recursion relations analogous to (4.2) [1, 2]. This enables one to find recursion relations for all the $a = 3$ cases, as we will demonstrate in forthcoming work.

On looking up sequences $(S_n)_{n\geq 0}$ for various choices of $\mathcal{Q}$ in the OEIS, a number of connections between certain classes of $\mathcal{Q}$ and some instances of strongly restricted permutations, combinations, and bit strings were identified. These are described and proved in [2].

Various authors have also considered the number of ways of choosing k objects from n arranged in a circle in such a way that no two chosen objects are certain disallowed separations apart [10–12, 14, 17, 18]. A modified version of our bijection covers such cases if we instead consider restricted-overlap tiling using curved squares and combs of a circular n-board with the n-th cell joined to the first cell. There are, however, subtleties about the rules for overlap which we will address in detail elsewhere.

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Conflict of Interest: The author declares that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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