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Short Note

Disproof of two conjectures on proper 2-dominating sets in graphs

Mustapha Chellali[1,*], Lutz Volkmann[2]
[1]LAMDA-RO Laboratory, Department of Mathematics, University of Blida, Blida, Algeria
*m_chellali@yahoo.com

[2]Lehrstuhl II für Mathematik, RWTH Aachen, 52056 Aachen, Germany
volkm@math2.rwth-aachen.de

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Abstract: In this note, we disprove two conjectures recently stated on proper 2-dominating sets in graphs. We recall that a proper 2-dominating set of a graph $G = (V, E)$ is a subset D of V such that every vertex in $V - D$ has at least two neighbors in D except for at least one vertex which must have exactly two neighbors in D.

Keywords: 2-dominating sets, proper 2-dominating sets, proper 2-domination number.

AMS Subject classification: 05C69

1. Introduction

In 1985, Fink and Jacobson [2] gave a generalization of the concept of domination in graphs. For a positive integer k, a subset S of vertices in a graph $G = (V, E)$ is *k-dominating* if every vertex of $V - S$ is adjacent to at least k vertices in S. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality of a k-dominating set of G, and a k-dominating set of cardinality $\gamma_k(G)$ is called a $\gamma_k(G)$-set. Thus for $k = 1$, a 1-dominating set is the classical dominating set. For more details on k-domination, we refer the reader to the book chapter of Hansberg and Volkmann [3].
Noting that any 3-dominating set of a graph G is 2-dominating, Bednarz and Pirga [1] were interested in the study of 2-dominating sets that are not 3-dominating and called them proper 2-dominating sets. Therefore, a proper 2-dominating set D is 2-dominating for which there is at least one vertex in $V - D$ having exactly two neighbors in D. The minimum cardinality of a proper 2-dominating set of G is the *proper 2-domination number* denoted by $\gamma_{\overline{2}}(G)$.

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* *Corresponding Author*

2. Conjectures and counterexamples

Bednarz and Pirga [1] concluded their paper with the following two conjectures. Recall that an *independent set* is a set of vertices such that no two of which are adjacent. Also, a *leaf* in a graph G is a vertex of degree one.

Conjecture 1. If the graph G has no leaves, then $\gamma_{\overline{2}}(G) = \gamma_2(G)$.

Conjecture 2. If $\gamma_{\overline{2}}(G) = \gamma_2(G)+1$, then the graph has a unique independent $\gamma_2(G)$-set.

To disprove Conjectures 1 and 2, consider the complete bipartite graph $K_{3,n}$, with $n \geq 3$, and let X and Y denote the partite sets of $K_{3,n}$ such that $|X| = 3$ and $|Y| = n$. First, it is clear that $\gamma_2(K_{3,n}) = 3$, and that X is a $\gamma_2(K_{3,n})$-set which is additionally unique when $n \geq 4$. Also, if $n = 3$, then X and Y are the only $\gamma_2(K_{3,3})$-sets that are, in addition, disjoint and independent. Moreover, we can see in any case that every $\gamma_2(K_{3,n})$-set is also a minimum 3-dominating set, and thus it cannot be a proper 2-dominating set. Therefore $\gamma_{\overline{2}}(K_{3,n}) \geq 4$ for every $n \geq 3$, which disproves Conjecture 1. It is worth noting that adding an edge between two vertices of the partite set of size n for $n \geq 4$ also provides another counterexample to Conjecture 1, showing that this conjecture is far from being valid. On the other hand, since two vertices of each partite set of $K_{3,3}$ together form a proper 2-dominating set, we have $\gamma_{\overline{2}}(K_{3,3}) \leq 4$ and hence $\gamma_{\overline{2}}(K_{3,3}) = 4 = \gamma_2(K_{3,3}) + 1$. Now, in the aim of obtaining a higher-order graph G disproving conjecture 2, we consider a disjoint union of p graphs $K_{3,3}$ with $p \geq 1$. One can easily check that $\gamma_2(G) = 3p$, $\gamma_{\overline{2}}(G) = 3p + 1$ and G has two disjoint independent $\gamma_2(G)$-sets.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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