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Research Article

On connected graphs with integer-valued Q-spectral radius

Jesmina Pervin[†], **Lavanya Selvaganesh**[*], **Smrati Pandey**[‡]

Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi-221005, India
[†]jesminapervin.rs.mat18@iitbhu.ac.in
[*]lavanyas.mat@iitbhu.ac.in
[‡]smratipandey.rs.mat20@iitbhu.ac.in

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Abstract: The Q-eigenvalues are the eigenvalues of the signless Laplacian matrix $Q(G)$ of a graph G, and the largest Q-eigenvalue is known as the Q-spectral radius $q(G)$ of G. The edge-degree of an edge is defined as the number of edges adjacent to it. In this article, we characterize the structure of simple connected graphs having integral Q-spectral radius. We show that the necessary and sufficient condition for such graphs to contain either a double star $\mathcal{S}_r^2$ or its variation $\mathcal{S}_r^{2,1}$ (having exactly one common neighbor between the central vertices) as a subgraph is that the maximum edge-degree is $2r$, where $r = q(G) - 3$. In particular, we characterize all graphs that contain only double star as a subgraph when $q(G)$ equals 8 and 9. Further, we characterize all the connected edge-non-regular graphs with a maximum edge-degree equal to 4 whose minimum Q-eigenvalue does not belong to the open interval $(0, 1)$ and has an integral Q-spectral radius.

Keywords: edge-degree, integral graph, signless Laplacian matrix, Q-integral graph, Q-spectral radius.

AMS Subject classification: 05C50, 05C07

1. Introduction

All the graphs considered in this article are simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We call a graph G as *H-free* if H is not a subgraph of G. For a vertex $x \in V(G)$, the *degree*, $d_G(x)$, is the number of vertices adjacent to x in G, and $d_G^{\max}$ is used to denote the *maximum degree of* G. We use $N(x)$ to denote the *neighborhood of* x. An edge in G with incident vertices x, y is denoted by xy.

* *Corresponding Author*

The *cartesian product* $G_1 \Box G_2$ of two graphs G_1 and G_2 is defined by $V(G_1 \Box G_2) = V(G_1) \times V(G_2)$ and $(x_1, y_1)(x_2, y_2) \in E(G_1 \Box G_2)$ if and only if $x_1 = x_2$ and $y_1 y_2 \in E(G_2)$ or, $y_1 = y_2$ and $x_1 x_2 \in E(G_1)$. We define a *double star* $\mathcal{S}_r^2$ obtained by taking two disjoint copies of star graph $K_{1,r}$ and adding an edge between the vertices of degree r. Consider $\mathcal{S}_r^{2,1}$ as a *variation of* $\mathcal{S}_r^2$, where the vertices of degree $r+1$ in S_r^2 have exactly one common neighbor, see Figure 3. We call the vertices of degree $r+1$ in $\mathcal{S}_r^2$ and $\mathcal{S}_r^{2,1}$ as the central vertices. We use $\lambda_{\min}(B)$ to denote the *minimum eigenvalue* of a real symmetric square matrix B.

Let the adjacency matrix of G be $A(G)$. G is called an *integral graph* if the spectrum of $A(G)$ consists entirely of integers. The question about which graphs are integral dates back to Harary and Schwenk (1974) [7], who remarked that the general problem appeared intractable. For some results on integral graphs, see [2, 5].

Let $D(G)$ be the diagonal matrix with $D(G)_{xx} = d_G(x)$, for any $x \in V(G)$. The matrix $D(G) + A(G)$ is called the *signless Laplacian matrix* of G and is denoted by $Q(G)$. The matrix $Q(G)$ is positive semidefinite and irreducible. The eigenvalues of $Q(G)$ are known as the *Q-eigenvalues* of the graph G. The *Q-spectral radius* $q(G)$ of G is the largest Q-eigenvalue. A graph is called *Q-integral* if the spectrum of $Q(G)$ consists entirely of integers. Several studies on signless Laplacian matrix of graphs and Q-integral graphs can be found in [1, 4, 6, 9, 13–18, 20, 21]. The *edge-degree* e-$deg_G(e')$ of an edge $e' = xy \in E(G)$ is $|N(x)| + |N(y)| - 2$. We denote the *maximum edge-degree* of a graph G by e-$deg_G^{\max}$. A graph G is called *edge-regular* if for all $e' \in E(G)$, e-$deg_G(e')$ are equal and is denoted by e-deg_G, the *edge-degree of* G. If a graph is not edge-regular, we call it as *edge-non-regular graph.*

In 2008, Simić and Stanić [19] studied the connected Q-integral graphs with e-$deg_G^{\max} \leq 5$. In 2019, the connected Q-integral graphs with e-$deg_G^{\max} \leq 6$ was studied by Park and Sano [11]. They gave a structural classification for such graphs G when $q(G) = 6$. It is interesting that any connected Q-integral graph with e-$deg_G^{\max} = q(G) = 6$ always contains a double star $\mathcal{S}_3^2$ as a subgraph. Though, it was proved in [10] that there is no connected Q-integral bipartite graph having $\mathcal{S}_3^2$ as an induced subgraph.

Recently, in 2023 [12], the authors studied connected Q-integral graphs with e-$deg_G^{\max} \leq 8$ and gave a structural classification under the restriction $q(G) = 7$. They showed that $\mathcal{S}_4^2$ is a subgraph of the connected Q-integral graph with e-$deg_G^{\max} = 8$. Besides, they also gave an upper bound and a lower bound for e-$deg_G^{\max}$ in terms of $q(G)$ for Q-integral graphs and proved that there does not exist any connected edge-non-regular Q-integral graph with $q(G) \leq 4$.

Moreover, it is quite surprising to observe that the double star $\mathcal{S}_r^2$ is always a subgraph of connected Q-integral graph with e-$deg_G^{\max} = 2r$, where $r = q(G) - 3$; $q(G) \in \{5, 6, 7\}$, see [11, 12, 19]. Eventually, a question arises about the existence of such a double star in a connected Q-integral graph for any value of $q(G)$. Also, it is quite interesting to analyze, whether the condition of integral Q-spectrum can be relaxed. If so, then what conditions on its Q-spectrum are required for a graph to have $\mathcal{S}_r^2$ as a subgraph.

With the quest to answer the above questions, we study connected graphs with integral Q-spectral radius $q(G)$ and e-$deg_G^{\max} = 2q(G) - 6$. We give a necessary and sufficient condition for such a connected graph G to contain $\mathcal{S}_r^2, \mathcal{S}_r^{2,1}$, where $r = q(G) - 3$, as a subgraph. Using this condition, we also characterize connected graphs having $q(G) \in \{8, 9\}$ and $\lambda_{\min}(Q(G)) \notin (0, 1)$ to contain only double star $\mathcal{S}_5^2, \mathcal{S}_6^2$ as a subgraph, respectively.

In 2008, Simić and Stanić [19] showed that the only connected edge-non-regular Q-integral graph with e-$deg_G^{\max} = 4$ is $\mathcal{H}^*$ and $K_{1,2} \square K_2$, see Figure 1. In this article, we extend this result by characterizing all such edge-non-regular connected graph G, when it is not Q-integral and instead have only integral $q(G)$ and $\lambda_{\min}(Q(G)) \notin (0, 1)$.

2. Preliminaries

The principal submatrix $Q_p(H)$ of the signless Laplacian matrix $Q(G)$, corresponding to a subset $H \subseteq V(G)$ is defined by

$$Q_p(H)_{xy} = \begin{cases} d_G(x) & x = y \\ 1 & xy \in E(G) \\ 0 & xy \notin E(G). \end{cases}$$

Let M be a complex matrix of order n described in the following block form

$$M = \begin{bmatrix} M_{11} & \dots & M_{1t} \\ \vdots & \ddots & \vdots \\ M_{t1} & \dots & M_{tt} \end{bmatrix}$$

where the blocks M_{ij} are $n_i \times n_j$ matrices for any $1 \leq i, j \leq t$ and $n = n_1 + \cdots + n_t$. For $1 \leq i, j \leq t$, let r_{ij} denote the average row sum of M_{ij}, i.e., r_{ij} is the sum of all entries in M_{ij} divided by the number of rows. Then $\mathcal{E}_M = (r_{ij})$ is called the *quotient matrix* of M. If, in addition, for each pair i, j, M_{ij} has a constant row sum, then $\mathcal{E}_M$ is called the *equitable quotient matrix* of M.

We use $B_{m \times n}$ to denote a matrix B of order $m \times n$ and B_n to denote a square matrix B of order n. The spectral radius of a square matrix B is denoted by $\rho(B)$ and the spectrum $\sigma(B)$ is the set of all eigenvalues of B. For any two non-negative matrices $B_m = (b_{ij})$ and $C_m = (c_{ij})$, we say B_m dominates C_m if $B_m \geq C_m$ (i.e., $b_{ij} \geq c_{ij}$ for all $i, j = 1, \dots, m$). Note that, if B_m dominates C_m, then $\rho(B_m) \geq \rho(C_m)$.

We use J to mean a matrix with all entries equal to 1 and I to denote identity matrix. $K_{1,n}$ is a complete bipartite graph with 1 (resp. n) vertex in the first (resp. second) partite set. C_n is a cycle of order n and P_n denotes a path on n vertices.

We will use the well known theorems, namely Perron-Frobenius Theorem [[8], Theorem 8.4.4] and Interlacing Theorem [[8], Theorem 4.3.17] on eigenvalues to prove several results in this article. Now we state some important results that we require for our proofs.

Theorem 1. *([22], Theorem 2.3). Let $\mathcal{E}_M$ be the equitable quotient matrix of a complex square matrix M. Then $\sigma(\mathcal{E}_M) \subseteq \sigma(M)$.*

Theorem 2. *([22], Theorem 2.5). Let $\mathcal{E}_M$ be the equitable quotient matrix of a non-negative square matrix M. Then $\rho(\mathcal{E}_M) = \rho(M)$.*

Theorem 3. *([3], Proposition 1.3.9). The number of connected bipartite components of G is equal to the multiplicity of the Q-eigenvalue 0 in G.*

Theorem 4. *([11], Proposition 2.7). A connected graph G has $d_G(v) \leq \lceil q(G) - 1 \rceil$ for any $v \in V(G)$, where $q(G)$ is the Q-spectral radius of G. If G has a vertex v having $d_G(v) = q(G) - 1$ and $q(G) \in \mathbb{Z}^+$, then $G = K_{1,q(G)-1}$.*

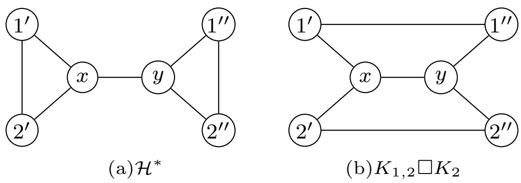


Figure 1. Edge-non-regular connected graphs G having $q(G), \lambda_{\min}(Q(G)) \in \mathbb{Z}$ and e-$deg_G^{\max} = 4$

Theorem 5. *([19], Theorem 3.2). If G is a connected edge-non-regular Q-integral graph with maximum edge-degree 4, then G is one of the two graphs: $\mathcal{H}^*$ and $K_{1,2}\square K_2$ (of Figure 1).*

The following results give the bounds for the maximum edge-degree of a graph.

Theorem 6. *([12], Remark 3.2). For a connected edge-regular graph G, e-$deg_G = q(G) - 2$.*

Theorem 7. *([12], Lemma 3.3, Remark 3.6). Let G be a connected edge-non-regular graph with $q(G) \in \mathbb{Z}$, then $q(G) - 1 \leq$ e-$deg_G^{\max} \leq 2q(G) - 6$.*

Theorem 8. *([12], Remark 3.4, Remark 3.6). There does not exist any connected edge-non-regular graph with integral $q(G) \leq 4$. Moreover, if $q(G) = 5$, then e-$deg_G^{\max} = 4$.*

3. Main Result

In this section, we study the structure of the graphs G with integral Q-spectral radius $q(G)$ and maximum edge-degree $2q(G)-6$. Thus from now, we consider connected graph G having $q(G) \in \mathbb{Z}$.

For $q(G) \geq 5$, it can be observed from Theorem 4, Theorem 6 and Theorem 7 that e-$deg_G^{\max} = 2q(G)-6$ if and only if G contains at least two adjacent vertices x and y with vertex degree $d_G(x) = d_G(y) = q(G)-2$. For any two distinct vertices i, j of G, we use a_{ij} to denote the (i,j)-th entry of the adjacency matrix $A(G)$. We use $a_{..}$ and $d_G(\cdot)$ to mean a_{xy} and $d_G(z)$ for suitable vertices x, y, and z.

Lemma 1. *Let G be a connected graph with integral Q-spectral radius $q(G) \geq 5$. If e-$deg_G^{\max} = 2q(G)-6$, then the incident vertices on any edge with edge-degree $2q(G)-6$ can have at most one common neighbor.*

Proof. Let $xy \in E(G)$ be any arbitrary edge with e-$deg(xy) = 2q-6$, where $q = q(G)$. Thus $d_G(x) = d_G(y) = q-2$. Let $N(x) = \{y, 1', 2', \ldots, r'\}$ and $N(y) = \{x, 1'', 2'', \ldots, r''\}$, where $r = q-3$, be the neighborhood sets of x and y, respectively.

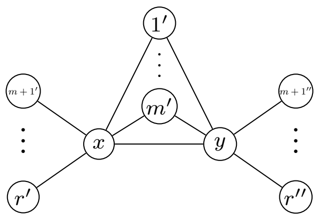


Figure 2. **x and y with m common neighbors, $r = q-3$**

Suppose x and y have exactly m common neighbors say, $1' = 1'', \ldots, m' = m''$, where $2 \leq m \leq r$. The principal submatrix $Q_p(H)$ of the signless Laplacian matrix $Q(G)$ corresponding to the vertex set $H = N(x) \cup N(y) = \{x, y, 1', 2', \ldots, m', m+1', \ldots, r', m+1'', \ldots, r''\}$ is given by

$$Q_p(H) = \begin{bmatrix} q-2 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & q-2 & 1 & \cdots & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & 1 & d_G(1') & \cdots & a_{1'm'} & a_{1'm+1'} & \cdots & a_{1'r'} & a_{1'm+1''} & \cdots & a_{1'r''} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & a_{1'm'} & \cdots & d_G(m') & a_{m'm+1'} & \cdots & a_{m'r'} & a_{m'm+1''} & \cdots & a_{m'r''} \\ 1 & 0 & a_{1'm+1'} & \cdots & a_{m'm+1'} & d_G(m+1') & \cdots & a_{m+1'r'} & a_{m+1'm+1''} & \cdots & a_{m+1'r''} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & a_{1'r'} & \cdots & a_{m'r'} & a_{m+1'r'} & \cdots & d_G(r') & a_{r'm+1''} & \cdots & a_{r'r''} \\ 0 & 1 & a_{1'm+1''} & \cdots & a_{m'm+1''} & a_{m+1'm+1''} & \cdots & a_{r'm+1''} & d_G(m+1'') & \cdots & a_{m+1''r''} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & a_{1'r''} & \cdots & a_{m'r''} & a_{m+1'r''} & \cdots & a_{r'r''} & a_{m+1''r''} & \cdots & d_G(r'') \end{bmatrix}$$

where $1 \le d_G(\cdot) \le r+1$, $d_G(i') \ge 2$ for $i = 1, \ldots, m$, and $a_{\cdot\cdot} \in \{0, 1\}$. By Interlacing Theorem [[8], Theorem 4.3.17] and Perron-Frobinius Theorem [[8], Theorem 8.4.4], $\rho(Q_p(H)) \le \rho(Q(G)) = q$. Then for any possible choices of $d_G(\cdot)$ and $a_{\cdot\cdot}$, the matrix $Q_p(H)$ dominates the following matrix

$$M = \begin{bmatrix} q-2 & 1 & 1 & \ldots & 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \\ 1 & q-2 & 1 & \ldots & 1 & 0 & \ldots & 0 & 1 & \ldots & 1 \\ 1 & 1 & 2 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 0 & \ldots & 2 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \ldots & 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

The equitable quotient matrix of M is

$$\mathcal{E}_M = \begin{bmatrix} q-1 & m & q-m-3 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The characteristics polynomial of $\mathcal{E}_M$ is

$$\mathcal{P}_{\mathcal{E}_M}(x) = x^3 - (q+2)x^2 + (2q - m + 2)x - 4.$$

Note that when $x = q$, we get

$$\mathcal{P}_{\mathcal{E}_M}(q) = -(m-2)q - 4 < 0, \qquad \text{since } m \ge 2.$$

Also when $x = q+1$, we have for $2 \le m \le q-3$,

$$\mathcal{P}_{\mathcal{E}_M}(q+1) = (q^2 + 2q - 3) - m(q+1) \ge 4q > 0.$$

Thus we observe that $\mathcal{P}_{\mathcal{E}_M}(x)$ has a root in $(q, q+1)$ and hence $\rho(\mathcal{E}_M) > q$. Since M is a non-negative matrix, by Theorem 2, we have $\rho(M) = \rho(\mathcal{E}_M) > q$. Further, $\rho(Q_p(H)) \ge \rho(M) > q$, which is a contradiction to $\rho(Q_p(H)) \le \rho(Q(G)) = q(G) = q$. Therefore, x and y can not have $m (2 \le m \le q-3)$ common neighbors in G. Thus, x and y can have at most 1 common neighbor in G. Hence the lemma holds. □

Let $\mathcal{S}_r^2$ be the double star, as shown in Figure 3(a), with $V(\mathcal{S}_r^2) = \{x, y, 1', \ldots, r', 1'', \ldots, r''\}$ and $E(\mathcal{S}_r^2) = \{xy, x1', \ldots, xr', y1'', \ldots, yr''\}$. Also, consider $\mathcal{S}_r^{2,1}$, as shown in Figure 3(b), with $V(\mathcal{S}_r^{2,1}) = \{x, y, 1', 2', \ldots, r', 2'', \ldots, r''\}$ and $E(\mathcal{S}_r^{2,1}) = \{xy, x1', x2', \ldots, xr', y1', y2'', \ldots, yr''\}$.

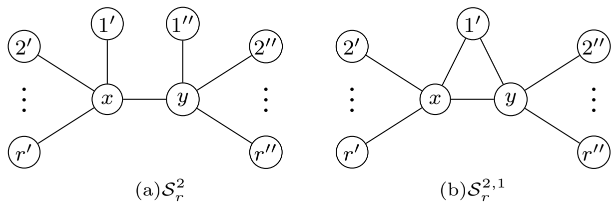


Figure 3. Possible subgraphs of connected graphs G having $q(G) \in \mathbb{Z}$ and e-$deg_G^{\max} = 2r$, where $r = q(G) - 3$

Remark 1. Now consider the connected graph G with maximum edge-degree e-$deg_G^{\max}$ equal to $2q(G) - 6$. It can be verified from Theorem 4, Theorem 6 and Theorem 8 that there is no such graphs having $q(G) = 1, 2, 3$. Also, $q(G) = 4$ if and only if $G = K_{1,3}$ or $G = C_n$, $n \geq 3$.

Theorem 9. *Let $G(\neq K_{1,3})$ be a connected graph with integral Q-spectral radius $q(G) \geq 4$. Then e-$deg_G^{\max} = 2r$ if and only if G contains $\mathcal{S}_r^2$ or $\mathcal{S}_r^{2,1}$ as a subgraph, where $r = q(G) - 3$.*

Proof. By Remark 1, the theorem holds when $q(G) = 4$. Let $q(G) \geq 5$ and $xy \in E(G)$ be an edge with edge-degree $2r$, where $r = q(G) - 3$. Let the neighborhood sets of x and y in G be $N(x) = \{y, 1', 2', \dots, r'\}$ and $N(y) = \{x, 1'', 2'', \dots, r''\}$, respectively. By Lemma 1, we have $i' \neq j''; i, j = 2, 3, \dots, r$. Therefore, G contains at least one of $\mathcal{S}_r^2$, $\mathcal{S}_r^{2,1}$ as a subgraph.
Conversely, if G contains either S_r^2 or $S_r^{2,1}$, where $r = q(G) - 3$, as a subgraph, then e-$deg_G^{\max} \geq 2r$. From Theorem 6 and Theorem 7 for $q(G) \geq 4$, we have e-$deg_G^{\max} \leq 2q(G) - 6 = 2r$. Hence the theorem holds. □

Here with the help of Theorem 9, we give a necessary and sufficient conditions for connected graphs having $q(G) \in \{5, 6, 7, 8, 9\}$ and $\lambda_{\min}(Q(G)) \notin (0, 1)$ to contain $\mathcal{S}_{q(G)-3}^2$ as a subgraph but not $\mathcal{S}_{q(G)-3}^{2,1}$.

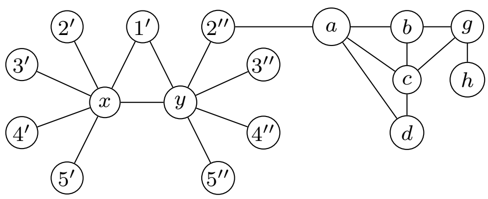
Lemma 2. *Let G be a connected graph having $q(G) = 5$ and $\lambda_{\min}(Q(G)) \notin (0, 1)$. Then e-$deg_G^{\max} = 4$ if and only if G is $\mathcal{S}_2^{2,1}$-free and contains $\mathcal{S}_2^2$ as a subgraph.*

Proof. Let e-$deg_G^{\max} = 4$. On the contrary, suppose G has either $\mathcal{S}_2^{2,1}$ as a subgraph or is $\mathcal{S}_2^2$-free. By Theorem 9, in both the cases $\mathcal{S}_2^{2,1}$ is a subgraph of G. Let $V(\mathcal{S}_2^{2,1}) = \{x, y, 1', 2', 2''\}$, with $d_G(x) = d_G(y) = 3$, $N(x) = \{y, 1', 2'\}$ and $N(y) = \{x, 1', 2''\}$. Clearly, G is non-bipartite as it contains a triangle and hence by Theorem 3, we have $\lambda_{\min}(Q(G)) \geq 1$. Now the principal submatrix $Q_p(V(\mathcal{S}_2^{2,1}))$ of $Q(G)$ is given by

$$Q_p(V(\mathcal{S}_2^{2,1})) = \begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 & 1 \\ 1 & 1 & d_G(1') & a_{1'2'} & a_{1'2''} \\ 1 & 0 & a_{1'2'} & d_G(2') & a_{2'2''} \\ 0 & 1 & a_{1'2''} & a_{2'2''} & d_G(2'') \end{bmatrix}, \tag{3.1}$$

where $1 \leq d_G(\cdot) \leq 3, d_G(1') \geq 2$ and $a_{..} \in \{0,1\}$. Using MATLAB computation, we find that the only possible values in (3.1) are $d_G(1') = d_G(2') = d_G(2'') = 2, a_{1'2'} = a_{1'2''} = a_{2'2''} = 0$, to have $\rho(Q_p(V(\mathcal{S}_2^{2,1}))) \leq 5$ and $\lambda_{\min}(Q_p(V(\mathcal{S}_2^{2,1}))) \geq 1$. Further, in this case, the spectral radius of $Q_p(V(\mathcal{S}_2^{2,1}))$ is equal to 5 and hence by Perron-Frobenius Theorem, we have $Q_p(V(\mathcal{S}_2^{2,1})) = Q(G)$, which is not true since $Q_p(V(\mathcal{S}_2^{2,1}))$ is not a signless Laplacian matrix. Therefore G is $\mathcal{S}_2^{2,1}$-free and thus by Theorem 9, $\mathcal{S}_2^2$ is a subgraph of G.
Conversely, if G is $\mathcal{S}_2^{2,1}$-free containing $\mathcal{S}_2^2$ as a subgraph, then e-$deg_G^{\max} \geq 4$. Also, e-$deg_G^{\max} \leq 4$ when $q(G) = 5$. Thus, e-$deg_G^{\max} = 4$. Hence, the lemma holds. □

Remark 2. For $q(G) = 6$ and 7, the above theorem was proved for Q-integral graphs in [11] and [12], respectively. But if we relax the condition on the graph to be Q-integral and instead having only the maximum Q-eigenvalue to be an integer and $\lambda_{\min}(Q(G)) \notin (0,1)$, we get an analogous version of the above theorem. That is, for a connected graph G having $q(G) = 6$ and $\lambda_{\min}(Q(G)) \notin (0,1)$, the same proof given in ([11], Lemma 3.13) can be used to show that if e-$deg_G^{\max} = 6$, then $\mathcal{S}_3^2$ is a subgraph of G. The converse can be easily verified by using Theorem 6 and Theorem 7.
Similarly, when $q(G) = 7$, the same proof given in ([12], Lemma 4.3, Lemma 4.4), will work to show that e-$deg_G^{\max} = 8$ if and only if $\mathcal{S}_4^2$ is a subgraph of G.



(a) I^*

Figure 4. **Graph I^* having $e - deg_{I^*}^{\max} = 10$ and $\lambda_{\min}(Q(I^*)) = 0.2192$, where $q(I^*) = 8$**

Remark 3. Note that in the above lemma and Remark 2, the condition $\lambda_{\min}(Q(G)) \notin (0,1)$ can not be relaxed. For example, let I^* be as given in Figure 4. Here, we have e-$deg_{I^*}^{\max} = 10$, $q(I^*) = 8$ and $\lambda_{\min}(Q(I^*)) = 0.2192$ while $\mathcal{S}_5^2$ is not its subgraph but $\mathcal{S}_5^{2,1}$ is a subgraph. Similarly, one can construct graphs for $q(G) = 5, 6$ and 7.

With the above remark, we next prove that $\mathcal{S}_5^{2,1}$ cannot be a subgraph of G for which $q(G) = 8$ and $\lambda_{\min}(Q(G)) \notin (0,1)$.

Lemma 3. *Let G be a connected graph having $q(G) = 8$ and $\lambda_{\min}(Q(G)) \notin (0,1)$. Then e-$deg_G^{\max} = 10$ if and only if G is $\mathcal{S}_5^{2,1}$-free and contains $\mathcal{S}_5^2$ as a subgraph.*

Proof. Let e-$deg_G^{\max} = 10$. On the contrary, assume that either G is $\mathcal{S}_5^2$-free or contains $\mathcal{S}_5^{2,1}$ as a subgraph. In both cases, $\mathcal{S}_5^{2,1}$ is a subgraph of G by Theorem 9.

Let $V(\mathcal{S}_5^{2,1}) = \{x, y, 1', 2', 3', 4', 5', 2'', 3'', 4'', 5''\}$, with $N(x) = \{y, 1', 2', 3', 4', 5'\}$ and $N(y) = \{x, 1', 2'', 3'', 4'', 5''\}$. The principal submatrix $Q_p(V(\mathcal{S}_5^{2,1}))$ of $Q(G)$ is given by

$$Q_p(V(\mathcal{S}_5^{2,1})) = \begin{bmatrix} 6 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & d_G(1') & a_{1'2'} & a_{1'3'} & a_{1'4'} & a_{1'5'} & a_{1'2''} & a_{1'3''} & a_{1'4''} & a_{1'5''} \\ 1 & 0 & a_{1'2'} & d_G(2') & a_{2'3'} & a_{2'4'} & a_{2'5'} & a_{2'2''} & a_{2'3''} & a_{2'4''} & a_{2'5''} \\ 1 & 0 & a_{1'3'} & a_{2'3'} & d_G(3') & a_{3'4'} & a_{3'5'} & a_{3'2''} & a_{3'3''} & a_{3'4''} & a_{3'5''} \\ 1 & 0 & a_{1'4'} & a_{2'4'} & a_{3'4'} & d_G(4') & a_{4'5'} & a_{4'2''} & a_{4'3''} & a_{4'4''} & a_{4'5''} \\ 1 & 0 & a_{1'5'} & a_{2'5'} & a_{3'5'} & a_{4'5'} & d_G(5') & a_{5'2''} & a_{5'3''} & a_{5'4''} & a_{5'5''} \\ 0 & 1 & a_{1'2''} & a_{2'2''} & a_{3'2''} & a_{4'2''} & a_{5'2''} & d_G(2'') & a_{2''3''} & a_{2''4''} & a_{2''5''} \\ 0 & 1 & a_{1'3''} & a_{2'3''} & a_{3'3''} & a_{4'3''} & a_{5'3''} & a_{2''3''} & d_G(3'') & a_{3''4''} & a_{3''5''} \\ 0 & 1 & a_{1'4''} & a_{2'4''} & a_{3'4''} & a_{4'4''} & a_{5'4''} & a_{2''4''} & a_{3''4''} & d_G(4'') & a_{4''5''} \\ 0 & 1 & a_{1'5''} & a_{2'5''} & a_{3'5''} & a_{4'5''} & a_{5'5''} & a_{2''5''} & a_{3''5''} & a_{4''5''} & d_G(5'') \end{bmatrix}, \tag{3.2}$$

where $1 \leq d_G(\cdot) \leq 6, d_G(1') \geq 2, a_{..} \in \{0, 1\}$. Note that $d_G(1') \in \{2, 3\}$, otherwise $\rho(Q_p(V(S_5^{2,1}))) > 8$. Suppose $d_G(1') = 3$, then we find by computation, the following holds:

(i) $a_{1'i'} = a_{1'i''} = 0; \forall i = 2, 3, 4, 5;$

(ii) $a_{i'j'} = a_{i''j''} = a_{i'l''} = 0; \forall i, j, l = 2, 3, 4, 5; i \neq j;$

(iii) $d_G(i'), d_G(i'') \leq 2; \forall i = 2, 3, 4, 5.$

For each of the possible choices of $d_G(\cdot)$, we have either $\lambda_{\min}(Q_p(V(S_5^{2,1}))) < 1$ or $\rho(Q_p(V(S_5^{2,1}))) > 8$, which is a contradiction to the fact that G is non-bipartite, $q(G) = 8$, and $\lambda_{\min}(Q(G)) \notin (0, 1)$. Thus $d_G(1') = 2$.

Now, the edge set of the induced subgraph $G[N(x) \setminus \{y\}]$ is either empty or contains exactly one edge, namely $\{2'3'\}$ (up to isomorphism), otherwise $\rho(Q_p(V(S_5^{2,1}))) > 8$. Suppose $E(G[N(x) \setminus \{y\}]) = \{2'3'\}$, then $E(G[V(S_5^{2,1}) \setminus \{x, y\}]) = \{2'3'\}$. Now for each possible choices of $d_G(\cdot)$, either $\lambda_{\min}(Q_p(V(S_5^{2,1}))) < 1$ or $\rho(Q_p(V(S_5^{2,1}))) > 8$, which is a contradiction to the fact that G is a non-bipartite graph with $q(G) = 8, \lambda_{\min}(Q(G)) \notin (0, 1)$. Due to the symmetric structure of $S_5^{2,1}$, we have $E(G[N(x) \setminus \{y\}]) = E(G[N(y) \setminus \{x\}]) = \{\phi\}$.

Computationally, one can find that $|E(G[\{N(x) \cup N(y)\} \setminus \{x, y\}])| \leq 1$ otherwise the spectral radius of the corresponding principal submatrix in (3.2) is greater than 8. Suppose without loss of generality, $E(G[\{N(x) \cup N(y)\} \setminus \{x, y\}]) = \{5'5''\}$. Observe that, for the spectral radius of the corresponding $Q_p(V(S_5^{2,1}))$ in (3.2) to be 8, the admissible values of $d_G(\cdot)$ are $d_G(5') \leq 3, d_G(5'') = 2$, and $d_G(i'), d_G(i'') \leq 2$ for $i = 2, 3, 4$. However, for these choices of values of $d_G(\cdot)$, the least eigenvalue of $Q_p(V(S_5^{2,1}))$ is less than 1, which is a contradiction. Thus we have $E(G[\{N(x) \cup N(y)\} \setminus \{x, y\}]) = \{\phi\}$. Therefore, $Q_p(V(S_5^{2,1}))$ in (3.2) becomes

$$Q_p(V(\mathcal{S}_5^{2,1})) = \begin{bmatrix} 6 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & d_G(2') & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & d_G(3') & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & d_G(4') & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & d_G(5') & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & d_G(2'') & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & d_G(3'') & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_G(4'') & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_G(5'') \end{bmatrix}, \quad (3.3)$$

where $1 \le d_G(\cdot) \le 6$. The only possible choice of $d_G(\cdot)$ for which $\lambda_{\min}(Q_p(V(\mathcal{S}_5^{2,1}))) \ge 1$ and $\rho(Q_p(V(\mathcal{S}_5^{2,1}))) \le 8$ for the matrix in (3.3) is $d_G(i') = d_G(i'') = 2$ $(i = 2, 3, 4, 5)$. However, in this case, the spectral radius of this matrix equals 8. By Perron-Frobenius Theorem, $Q(G) = Q_p(V(\mathcal{S}_5^{2,1}))$, which is a contradiction since the matrix in (3.3) is not a signless Laplacian matrix.

Thus G is $\mathcal{S}_5^{2,1}$-free and hence G contains $\mathcal{S}_5^2$ as a subgraph.

Conversely, if G is $\mathcal{S}_5^{2,1}$-free containing $\mathcal{S}_5^2$ as a subgraph, then e-$deg_G^{\max} \ge 10$. Since $q(G) = 8$, we have e-$deg_G^{\max} \le 10$. Hence the lemma holds. □

Lemma 4. *Let G be a connected graph with $q(G) = 9$ and $\lambda_{\min}(Q(G)) \notin (0, 1)$. Then e-$deg_G^{\max} = 12$ if and only if G is $\mathcal{S}_6^{2,1}$-free and contains $\mathcal{S}_6^2$ as a subgraph of G.*

Proof. Suppose e-$deg_G^{\max} = 12$. Assume that G is either $\mathcal{S}_6^2$-free or contains $\mathcal{S}_6^{2,1}$ as a subgraph. By Theorem 9, $\mathcal{S}_6^{2,1}$ is a subgraph of G in both the cases. Suppose $V(\mathcal{S}_6^{2,1}) = \{x, y, 1', 2', 3', 4', 5', 6', 2'', 3'', 4'', 5'', 6''\}$, with $N(x) = \{y, 1', 2', 3', 4', 5', 6'\}$ and $N(y) = \{x, 1', 2'', 3'', 4'', 5'', 6''\}$.

Let $H = V(\mathcal{S}_6^{2,1}) = N(x) \cup N(y)$ and $\Gamma = H \setminus \{x, y\}$. Suppose $Q_p(H)$ is a principal submatrix of $Q(G)$ corresponding to $H = \{x, y, 1', 2', 3', 4', 5', 6', 2'', 3'', 4'', 5'', 6''\}$. Since G contains a triangle, G is non-bipartite and thus least eigenvalue of $Q_p(H)$ is at least 1 by Theorem 3. Then $1'$ is not adjacent to any vertices of Γ and $d_G(1') \in \{2, 3\}$ otherwise $\rho(Q_p(H)) > 9$. If $d_G(1') = 3$, then $a_{i'j''} = 0; i, j = 2, \ldots, 6$, for $\rho(Q_p(H))$ to be at most 9. Computationally, we observed that for every admissible choices of $1 \le d_G(\cdot) \le 7$ in $Q_p(H)$, either $\rho(Q_p(H)) > 9$ or $\lambda_{\min}(Q_p(H)) < 1$, which is a contradiction to the fact that G is a non-bipartite graph having $q(G) = 9$. Therefore, $d_G(1') = 2$.

Now we have the following claims.

Claim (i). All the edges in $G[\Gamma]$ are disjoint.

If the above claim is not true, then $G[\Gamma]$ must contain P_3 as a subgraph. Due to the symmetric structure of $\mathcal{S}_6^{2,1}$, we have the following choices for P_3 as a subset of $E(G[\Gamma])$: (i) $\{2'3', 3'4'\}$, (ii) $\{2'2'', 2'3'\}$, (iii) $\{2'2'', 2'3''\}$. In each of these 3 cases, we have $\rho(Q_p(H)) > 9$, a contradiction to our assumption that $q(G) = 9$. Thus all the edges of $G[\Gamma]$ are disjoint.

Claim (ii). $|E(G[\Gamma])| = \phi$.

We have following choices for the subset of $E(G[\Gamma])$ (up to symmetry):

(i) $\{2'2'', 3'3'', 4'4''\}$, (ii) $\{2'2'', 3'3'', 4'5'\}$, (iii) $\{2'2'', 3'4', 3''4''\}$, (iv) $\{2'3', 4'5', 6'6''\}$, (v) $\{2'3', 4'5', 2''3''\}$,(vi) $\{2'3', 4'5'\}$, (vii) $\{2'3', 2''3''\}$, (viii) $\{2'3', 6'6''\}$, (ix) $\{2'2'', 3'3''\}$, (x) $\{2'3'\}$, (xi) $\{2'2''\}$. For all the admissible choices of $1 \leq d_G(\cdot) \leq 7$, either the spectral radius of the corresponding $Q_p(H)$ is greater than 9 or the least eigenvalue is less than 1, which is a contradiction. Hence $|E(G[\Gamma])| = \phi$.

The matrix $Q_p(H)$ becomes

$$Q_p(H) = \begin{bmatrix} 7 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 7 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & d_G(2') & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & d_G(3') & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & d_G(4') & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & d_G(5') & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & d_G(6') & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & d_G(2'') & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_G(3'') & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_G(4'') & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_G(5'') & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_G(6'') \end{bmatrix} \tag{3.4}$$

where $1 \leq d_G(\cdot) \leq 7$. Now the only admissible choices for which $\rho(Q_p(H)) \leq 9$ and $\lambda_{\min}(Q_p(H)) \geq 1$ of the corresponding $Q_p(H)$ in (3.4) is $d_G(i') = d_G(i'') = 2, i = 2, \ldots, 6$. Moreover, here we have $\rho(Q_p(H)) = 9$ and thus by Perron-Frobenius Theorem, we have $Q_p(H) = Q(G)$ which is a contradiction since the matrix in (3.4) does not represent a signless Laplacian matrix. Therefore G is $\mathcal{S}_6^{2,1}$-free and hence $\mathcal{S}_6^2$ is a subgraph of G by Theorem 9.

Conversely, suppose G is $\mathcal{S}_6^{2,1}$-free containing $\mathcal{S}_6^2$ as a subgraph. Then e-$deg_G^{\max} \geq 12$. From Theorem 7, we have e-$deg_G^{\max} \leq 12$. Hence the lemma holds. □

Now we combine the above results from Lemma 2 to Lemma 4 in the following theorem.

Theorem 10. *Let $G(\neq K_{1,3}, C_3)$ be a connected graph with $q(G) \in \{4, 5, 6, 7, 8, 9\}$ and $\lambda_{\min}(Q(G)) \notin (0, 1)$. Then e-$\deg_G^{\max} = 2q(G) - 6$ if and only if G is $\mathcal{S}_{q(G)-3}^{2,1}$-free and contains $\mathcal{S}_{q(G)-3}^2$ as a subgraph.* □

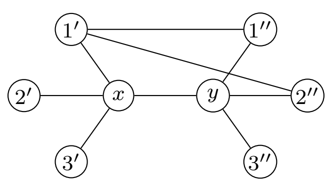


Figure 5. $\mathcal{S}^*$

From now, for simplicity, we use Γ to denote the set of vertices $N(x) \cup N(y) \setminus \{x, y\}$ in $\mathcal{S}_r^2, \mathcal{S}_r^{2,1}$ (given in Figure 3). Now, we will study the structure of the graphs under the condition when either $\mathcal{S}_{q(G)-3}^2$ or, $\mathcal{S}_{q(G)-3}^{2,1}$ is a subgraph of G.

Theorem 11. *Let G be a connected graph with integral Q-spectral radius $q = q(G) \geq 6$. If $\mathcal{S}^2_{q-3}$ is a subgraph of G, then the following hold.*

(1) $G[N(x)], G[N(y)]$ are C_{q-t}-free, $q \geq t+3, 3 \leq t \leq 5$.

(2) Either $G = \mathcal{S}^*$ (in Figure 5) or $d_{G[\mathcal{S}^2_{q-3}]}(v) \leq q-5$, $\forall v \in \Gamma$.

Proof. For $r = q-3$, the principal submatrix $Q_p(V(\mathcal{S}^2_r))$ corresponding to the vertices $V(\mathcal{S}^2_r) = \{x, y, 1', \ldots, r', 1'', \ldots, r''\}$ of $Q(G)$, is

$$Q_p(V(\mathcal{S}^2_r)) = \begin{bmatrix} q-2 & 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \\ 1 & q-2 & 0 & \ldots & 0 & 1 & \ldots & 1 \\ 1 & 0 & d_G(1') & \ldots & a_{1'r'} & a_{1'1''} & \ldots & a_{1'r''} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & a_{1'r'} & \ldots & d_G(r') & a_{r'1''} & \ldots & a_{r'r''} \\ 0 & 1 & a_{1'1''} & \ldots & a_{r'1''} & d_G(1'') & \ldots & a_{1''r''} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & a_{1'r''} & \ldots & a_{r'r''} & a_{1''r''} & \ldots & d_G(r'') \end{bmatrix} \tag{3.5}$$

where $1 \leq d_G(\cdot) \leq q-2, a_{..} \in \{0, 1\}$.

(1) We first prove that $G[N(x)]$ is C_{q-t}-free for $q \geq t+3, 3 \leq t \leq 5$. Analogously, $G[N(y)]$ is C_{q-t}-free follows.

Suppose $G[N(x)]$ is not C_{q-t}-free, for $q \geq t+3; t \in \{3, 4, 5\}$. The matrix $Q_p(V(\mathcal{S}^2_{q-3}))$ in (3.5) dominates the following matrix

$$M = \begin{bmatrix} q-2 & 1 & J & J & O \\ 1 & q-2 & O & O & J \\ J & O & Q(C_{q-t})+I_{q-t} & O & O \\ J & O & O & I_{t-3} & O \\ O & J & O & O & I_{q-3} \end{bmatrix}. \tag{3.6}$$

Note that when $t = 3$, I_{t-3} becomes I_0 i.e., the rows and columns corresponding to I_0 does not exist in M. Therefore, the equitable quotient matrix of M in (3.6) is given by

$$\mathcal{E}_M = \begin{bmatrix} q-2 & 1 & q-3 & 0 \\ 1 & q-2 & 0 & q-3 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The characteristics polynomial of $\mathcal{E}_M$ is

$$\mathcal{P}_{\mathcal{E}_M}(x) = x^4 - (2q+2)x^3 + (q^2+6q-10)x^2 + (-4q^2+10q-4)x + 4q - 12.$$

Note that when $x = q$, we have

$$\mathcal{P}_{\mathcal{E}_M}(q) = -12 < 0.$$

Also when $x = q+1$, we get

$$\mathcal{P}_{\mathcal{E}_M}(q+1) = 3q^2 - 8q - 27 > 0, \quad \text{for} \quad q \geq 6.$$

Thus $\mathcal{P}_{\mathcal{E}_M}(x)$ has a root in $(q, q+1)$ and therefore by interlacing theorem, the spectral radius $q(G) \geq \rho(Q_p(V(\mathcal{S}^2_{q-3}))) \geq \rho(M) = \rho(\mathcal{E}_M) > q$, which leads to a contradiction. For $t = 4, 5$, using similar techniques, we get $q(G) > q$, a contradiction to our assumption.

Therefore we conclude that $G[N(x)]$ is C_{q-t}-free where $q(G) \geq t+3; t \in \{3, 4, 5\}$. Hence (1) holds.

(2) Suppose there exist a vertex, say $1' \in \Gamma$ such that $d_{G[\mathcal{S}^2_{q-3}]}(1') \geq q-4$. Assume that $1'$ is adjacent to at least m $(0 \leq m \leq q-4)$ vertices of $N(y) \cap \Gamma$, say $1'', 2'', \ldots, m''$ and at least $q-m-4$ vertices of $N(x) \cap \Gamma$, say $2', \ldots, l'$, where $l = q-m-3$. For any admissible choices of $d_G(\cdot), a_{\cdot\cdot}$, the following matrix M is dominated by the principal submatrix $Q_p(V(\mathcal{S}^2_{q-3}))$

$$M = \begin{bmatrix} q-2 & 1 & 1 & J & J & O & O \\ 1 & q-2 & 0 & O & O & J & J \\ 1 & 0 & q-3 & J & O & J & O \\ J & O & J & 2I_{q-m-4} & O & O & O \\ J & O & O & O & I_m & O & O \\ O & J & J & O & O & 2I_m & O \\ O & J & O & O & O & O & I_{q-m-3} \end{bmatrix}. \tag{3.7}$$

Now we have the following cases according to the values of m.

Case 2.1. $m = 0$.

Using similar techniques as in (1), we get that the equitable quotient matrix $\mathcal{E}_M$ of M in (3.7) has an eigenvalue greater than q. Thus $\rho(Q_p(V(\mathcal{S}^2_{q-3}))) \geq \rho(M) = \mathcal{E}_M > q$, which is a contradiction to $\rho(Q(G)) = q$. Hence this case is not possible.

Case 2.2. $1 \leq m \leq q-5$.

The equitable quotient matrix of M given in (3.7) is

$$\mathcal{E}_M = \begin{bmatrix} q-2 & 1 & 1 & q-m-4 & m & 0 & 0 \\ 1 & q-2 & 0 & 0 & 0 & m & q-m-3 \\ 1 & 0 & q-3 & q-m-4 & 0 & m & 0 \\ 1 & 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The characteristics polynomial of $\mathcal{E}_M$ is

$$\begin{aligned} \mathcal{P}_{\mathcal{E}_M}(x) = {} & x^7 + (1-3q)x^6 + (3q^2+q-4)x^5 + (-q^3-5q^2+6q+2m+14)x^4 \\ & + (3q^3-8q-2m^2-14m-36)x^3 + (-2q^3-2mq^2-10q^2+2m^2q \\ & + 12mq+56q+2m^2+16m-8)x^2 + (2mq^2+12q^2-2m^2q-4mq \\ & - 44q-8m^2-44m)x - 8mq-8q+8m^2+40m+32. \end{aligned}$$

When $x = q$,

$$\mathcal{P}_{\mathcal{E}_M}(q) \leq -4q^4+44q^3-104q^2-108q+32 < 0, \quad \text{for} \quad q \geq 6.$$

At $x = q+1$,

$$\mathcal{P}_{\mathcal{E}_M}(q+1) \geq q^4+2mq^3+55q^3+6mq^2+19q^2+4mq-75q > 0, \quad \text{for} \quad q \geq 6.$$

Thus $\mathcal{E}_M$ has an eigenvalue in $(q, q+1)$. Therefore we have $\rho(Q_p(V(\mathcal{S}^2_{q-3}))) > q$, a contradiction to our assumption that $\rho(Q(G)) = q$. Hence Case 2.2 is not a possible choice.

Case 2.3. $m = q-4$.
Similar to Case 2.2, for $q \geq 7$, we arrive at a contradiction to our assumption that $\rho(Q(G)) = q$. When $q = 6$, M represents the matrix $Q(\mathcal{S}^*)$ and here $\rho(M) = \rho(\mathcal{E}_M) = 6$. Since $Q(G), Q_p(V(\mathcal{S}^2_{q-3})), M$ are non-negative real symmetric matrices and $Q_p(V(\mathcal{S}^2_{q-3})) \geq M$, by Perron-Frobenius Theorem, we have $6 = \rho(Q(G)) \geq \rho(Q_p(V(\mathcal{S}^2_{q-3}))) \geq \rho(M) = \rho(\mathcal{E}_M) = 6$. So, this implies $Q(G) = Q_p(V(\mathcal{S}^2_{q-3}))= M= Q(\mathcal{S}^*)$. Therefore $G = \mathcal{S}^*$.

From Cases 2.1-2.3, we conclude that either $d_{G[\Gamma]}(1') \leq q-5$ or $G = \mathcal{S}^*$. Hence (2) holds. $\square$

Theorem 12. *Let G be a connected graph with integral $q = q(G) \geq 10$. If $\mathcal{S}^{2,1}_{q-3}$ is a subgraph of G, then the following hold.*

(1) *$G[N(x) \setminus \{1'\}], G[N(y) \setminus \{1'\}]$ are C_{q-4}-free.*

(2) *$d_{G[\mathcal{S}^{2,1}_{q-3}]}(v) \leq q-4$, $\forall v \in \Gamma$.*

Proof. For $r = q-3$, the principal submatrix $Q_p(\mathcal{S}^{2,1}_r)$ of $Q(G)$ corresponding to the vertex set $V(\mathcal{S}^{2,1}_r) = \{x, y, 1', 2', \ldots, r', 2'', \ldots, r''\}$ is

$$Q_p(V(\mathcal{S}^{2,1}_r)) = \begin{bmatrix} q-2 & 1 & 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \\ 1 & q-2 & 1 & 0 & \ldots & 0 & 1 & \ldots & 1 \\ 1 & 1 & d_G(1') & a_{1'2'} & \ldots & a_{1'r'} & a_{1'2''} & \ldots & a_{1'r''} \\ 1 & 0 & a_{1'2'} & d_G(2') & \ldots & a_{2'r'} & a_{2'2''} & \ldots & a_{2'r''} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & a_{1'r'} & a_{2'r'} & \ldots & d_G(r') & a_{r'2''} & \ldots & a_{r'r''} \\ 0 & 1 & a_{1'2''} & a_{2'2''} & \ldots & a_{r'2''} & d_G(2'') & \ldots & a_{2''r''} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & a_{1'r''} & a_{2'r''} & \ldots & a_{r'r''} & a_{2''r''} & \ldots & d_G(r'') \end{bmatrix}, \tag{3.8}$$

where $1 \leq d_G(\cdot) \leq q-2, d_G(1') \geq 2, a_{\cdot\cdot} \in \{0, 1\}$.

(1) Suppose $G[N(x) \cap (\Gamma \setminus \{1'\})]$ is not C_{q-4}-free. The principal submatrix $Q_p(V(\mathcal{S}^{2,1}_{q-3}))$ in (3.8) of $Q(G)$ dominates the following matrix M.

$$M = \begin{bmatrix} q-2 & 1 & 1 & J & O \\ 1 & q-2 & 1 & O & J \\ 1 & 1 & 2 & O & O \\ J & O & O & Q(C_{q-4})+I_{q-4} & O \\ O & J & O & O & I_{q-4} \end{bmatrix}.$$

The equivalent quotient matrix of M is

$$\mathcal{E}_M = \begin{bmatrix} q-2 & 1 & 1 & q-4 & 0 \\ 1 & q-2 & 1 & 0 & q-4 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

The characteristics polynomial of $\mathcal{E}_M$ is

$$\mathcal{P}_{\mathcal{E}_M}(x) = x^5 - (2q+4)x^4 + (q^2+10q-6)x^3 + (-6q^2-2q+16)x^2 + (8q^2-18q + 13)x - 12q + 28.$$

For $q \geq 10$, we have $\mathcal{P}_{\mathcal{E}_M}(q) < 0$ and $\mathcal{P}_{\mathcal{E}_M}(q+1) > 0$. Therefore $\mathcal{E}_M$ has an eigenvalue in $(q, q+1)$. Thus $\rho(Q_p(V(\mathcal{S}^{2,1}_{q-3}))) \geq \rho(M) = \rho(\mathcal{E}_M) > q$, which is a contradiction to our assumption that $\rho(Q(G)) = q$. Hence $G[N(x) \cap (\Gamma \setminus \{1'\})]$ is C_{q-4}-free.
Analogously, it can be shown that $G[N(y) \cap (\Gamma \setminus \{1'\})]$ is also C_{q-4}-free, and hence (1) holds.

(2) We first prove the statement for the vertex $1'$ and then prove for any vertex v in $\Gamma \setminus \{1'\}$.

Case 2.1. $d_{G[\Gamma]}(1') \leq q-6$.
Suppose $d_{G[\Gamma]}(1') \geq q-5$, that is, $1'$ is adjacent to at least $m-1$ $(1 \leq m \leq q-5)$ vertices of $N(y) \cap \Gamma$, say $2'', \ldots, m''$, and at least $q-m-4$ vertices of $N(x) \cap \Gamma$, say $2', \ldots, q-m-3'$.
For any admissible choices of $d_G(\cdot), a_{\cdot,\cdot}$, the principal submatrix $Q_p(V(\mathcal{S}^{2,1}_{q-3}))$ in (3.8) corresponding to the vertex set $V(\mathcal{S}^{2,1}_{q-3}) = N(x) \cup N(y) = \{x, y, 1', 2', \ldots, q-m-3', q-m-2', \ldots, q-3', 2'', \ldots, m'', m+1'', \ldots, q-3''\}$ dominates the following matrix

$$M = \begin{bmatrix} q-2 & 1 & 1 & J & J & O & O \\ 1 & q-2 & 1 & O & O & J & J \\ 1 & 1 & q-3 & J & O & J & O \\ J & O & J & 2I_{q-m-4} & O & O & O \\ J & O & O & O & I_m & O & O \\ O & J & J & O & O & 2I_{m-1} & O \\ O & J & O & O & O & O & I_{q-m-3} \end{bmatrix}. \tag{3.9}$$

Now we have the following cases according to the values of m.

Case 2.1.1. $m = 1$.
The equitable quotient matrix of M in (3.9) is given by

$$\mathcal{E}_M = \begin{bmatrix} q-2 & 1 & 1 & q-5 & 1 & 0 \\ 1 & q-2 & 1 & 0 & 0 & q-4 \\ 1 & 1 & q-3 & q-5 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which has an eigenvalue in $(q, q+2)$. Therefore $\rho(Q_p(V(\mathcal{S}^{2,1}_{q-3}))) > q$, a contradiction to $q(G) = q$.

Case 2.1.2. $2 \leq m \leq q-5$.
The equitable quotient matrix of M in (3.9) is

$$\mathcal{E}_M = \begin{bmatrix} q-2 & 1 & 1 & q-m-4 & m & 0 & 0 \\ 1 & q-2 & 1 & 0 & 0 & m-1 & q-m-3 \\ 1 & 1 & q-3 & q-m-4 & 0 & m-1 & 0 \\ 1 & 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The characteristics polynomial of $\mathcal{E}_M$ is

$$\begin{aligned}\mathcal{P}_{\mathcal{E}_M}(x) = {} & x^7 + (1-3q)x^6 + (3q^2+q-3)x^5 + (-q^3-5q^2+3q+19)x^4 + (3q^3+2q^2 \\ & + 2mq - 14q - 2m^2 - 6m - 26)x^3 + (-2q^3 - 2mq^2 - 14q^2 + 2m^2q + 4mq \\ & + 80q + 2m^2 + 6m - 60)x^2 + (2mq^2 + 14q^2 - 2m^2q + 5mq - 63q - 11m^2 \\ & - 33m + 52)x - 12mq - 4q + 12m^2 + 36m + 16.\end{aligned}$$

We have

$$\mathcal{P}_{\mathcal{E}_M}(q) \le -9q^4 + 79q^3 - 166q^2 - 170q + 136 < 0, \quad \text{for} \quad q \ge 10,$$

and

$$\mathcal{P}_{\mathcal{E}_M}(q+2) \ge 9q^4 + 241q^3 + 688q^2 + 280q - 108 > 0.$$

Thus $\mathcal{P}_{\mathcal{E}_M}(x)$ has a root in $(q, q+2)$. Therefore $\rho(Q_p(V(\mathcal{S}^{2,1}_{q-3}))) > q$, which is a contradiction to $q(G) = q$.

From Cases 2.1.1-2.1.2, we conclude that $d_{G[\Gamma]}(1') \le q-6$ and that $1'$ is adjacent to both x and y implies that $d_{G[\mathcal{S}^{2,1}_{q-3}]} \le q-4$ and thus the Case 2.1 holds.

Case 2.2. $d_{G[\mathcal{S}^{2,1}_{q-3}]}(v) \le q-4$, $\forall v \in \Gamma \setminus \{1'\}$.

In fact, we will show that $d_{G[\Gamma\setminus\{1'\}]}(v) \le q-6, \forall v \in \Gamma \setminus \{1'\}$. Assume that there exists a vertex, say $2' \in \Gamma \setminus \{1'\}$ such that $d_{G[\Gamma\setminus\{1'\}]}(2') \ge q-5$, that is, $2'$ is adjacent to at least $m-1$ vertices of $N(y) \cap (\Gamma \setminus \{1'\})$ say, $2'', \ldots, m''$ $(1 \le m \le q-4)$ and at least $q-m-4$ vertices of $N(x) \cap (\Gamma \setminus \{1'\})$ say, $3', \ldots, q-m-2'$.

For any admissible choices of $d_G(\cdot), a_{\cdot,\cdot}$, the following matrix M is dominated by the principal submatrix $Q_p(H)$ in (3.8)

$$M = \begin{bmatrix} q-2 & 1 & 1 & 1 & J & J & O & O \\ 1 & q-2 & 1 & 0 & O & O & J & J \\ 1 & 1 & 2 & 0 & O & O & O & O \\ 1 & 0 & 0 & q-4 & J & O & J & O \\ J & O & O & J & 2I_{q-m-4} & O & O & O \\ J & O & O & O & O & I_{m-1} & O & O \\ O & J & O & J & O & O & 2I_{m-1} & O \\ O & J & O & O & O & O & O & I_{q-m-3} \end{bmatrix}. \quad (3.10)$$

Similar to Case 2.1, for $1 \le m \le q-4$, the equitable quotient matrix of M has an eigenvalue greater than q. Therefore $\rho(Q_p(V(\mathcal{S}^{2,1}_{q-3}))) > q$, a contradiction to $q(G) = q$. Thus we conclude that $d_{G[\Gamma\setminus\{1'\}]}(2') \le q-6$. Also, since $2'$ is adjacent to x and may be adjacent to $1'$, we have that $d_{G[\mathcal{S}^{2,1}_{q-3}]}(2') \le q-4$. Since $2' \in \Gamma \setminus \{1'\}$ is an arbitrary vertex, we have $d_{G[\mathcal{S}^{2,1}_{q-3}]}(v) \le q-4$, $\forall v \in \Gamma \setminus \{1'\}$.

Hence, from Cases **2.1- 2.2**, we have that $d_{G[\mathcal{S}^{2,1}_{q-3}]}(v) \le q-4$, $\forall v \in \Gamma$. □

Finally, we end this section with a result which is an improvement to Theorem 2.5, where we identify all possible graphs when we relax the condition of Q-integrability and restrict that $\lambda_{\min}(Q(G)) \notin (0,1)$ and $q(G)$ to be an integer.

Theorem 13. *If G is a connected edge-non-regular graph with maximum edge-degree equal to 4 and $\lambda_{\min}(Q(G)) \notin (0,1), q(G) \in \mathbb{Z}$, then G is one of the following graphs.*

(a) *$G = \mathcal{H}^*$, shown in Figure 1(a), is the only non-bipartite graph.*

(b) *$G = K_{1,2}\Box K_2$.*

(c) *G is a bipartite graph having $\mathcal{S}_2^2$ or γ_1 (given in Figure 6(a)) as an induced subgraph.*

Proof. From Theorem 3 and e-$deg_G^{\max} = 4$, we have $q(G) = 5$ and thus by Lemma 2, $\mathcal{S}_2^2$ is a subgraph of G. Let $V(\mathcal{S}_2^2) = \{x, y, 1', 2', 1'', 2''\}$, with $d_G(x) = d_G(y) = 3$. Let $N(x) = \{y, 1', 2'\}$ and $N(y) = \{x, 1'', 2''\}$. The principal submatrix $Q_p(V(\mathcal{S}_2^2))$ of $Q(G)$ is

$$Q_p(V(\mathcal{S}_2^2)) = \begin{bmatrix} 3 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 1 \\ 1 & 0 & d_G(1') & a_{1'2'} & a_{1'1''} & a_{1'2''} \\ 1 & 0 & a_{1'2'} & d_G(2') & a_{2'1''} & a_{2'2''} \\ 0 & 1 & a_{1'1''} & a_{2'1''} & d_G(1'') & a_{1''2''} \\ 0 & 1 & a_{1'2''} & a_{2'2''} & a_{1''2''} & d_G(2'') \end{bmatrix}, \tag{3.11}$$

where $a_{\cdot\cdot} \in \{0,1\}, 1 \le d_G(\cdot) \le 3$. If $G[\Gamma]$ contains P_3 as a subgraph, then we have the following two choices for $|E(G[\Gamma])|$ (up to the symmetry $\mathcal{S}_2^2$): (i) $\{1'2', 1'1''\}$; (ii) $\{1'1'', 1'2''\}$. In both of these cases, we have $\rho(Q_p(V(\mathcal{S}_2^2))) > 5$, a contradiction to $q(G) = 5$. Therefore all the edges in $G[\Gamma]$ are disjoint.
The possible choices for $E(G[\Gamma])$ are (up to the symmetry $\mathcal{S}_2^2$): (i) $\{1'2', 1''2''\}$, (ii) $\{1'2'\}$, (iii) $\{1'1'', 2'2''\}$, (iv) $\{1'1''\}$, (iv) $\{\phi\}$.

Case (i). $E(G[\Gamma]) = \{1'2', 1''2''\}$.
Computationally, one can observe that $d_G(i') = d_G(i'') = 2, i = 1, 2$ otherwise the spectral radius of the corresponding $Q_p(V(\mathcal{S}_2^2))$ in (3.11) becomes greater than 5. For this choice of $d_G(\cdot)$, we get $\rho(Q_p(V(\mathcal{S}_2^2))) = 5$. Therefore $Q(G) = Q_p(V(\mathcal{S}_2^2))$ implying $G = \mathcal{H}^*$.

Case (ii). $E(G[\Gamma]) = \{1'2'\}$.
The only possible choice of $d_G(\cdot)$ for which the spectral radius of the corresponding $Q_p(V(\mathcal{S}_2^2))$ in (3.11) is at most 5 and $\lambda_{\min}(Q_p(V(\mathcal{S}_2^2))) \ge 1$ is $d_G(i') = 2, d_G(i'') = 3, i = 1, 2$. Consider a neighbor of $1''$, other than y, in G, say w. Computationally, it can be observed that the spectral radius of the corresponding $Q_p(V(\mathcal{S}_2^2) \cup \{w\})$ is greater than 5, a contradiction to $q(G) = 5$. Thus Case (ii) is not possible.

Case (iii). $E(G[\Gamma]) = \{1'1'', 2'2''\}$.
The principal submatrix $Q_p(V(\mathcal{S}_2^2))$ in (3.11) becomes

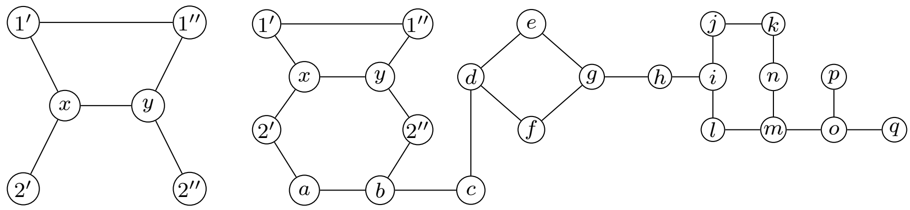
$$Q_p(V(\mathcal{S}_2^2)) = \begin{bmatrix} 3 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 1 \\ 1 & 0 & d_G(1') & 0 & 1 & 0 \\ 1 & 0 & 0 & d_G(2') & 0 & 1 \\ 0 & 1 & 1 & 0 & d_G(1'') & 0 \\ 0 & 1 & 0 & 1 & 0 & d_G(2'') \end{bmatrix},$$

where $d_G(\cdot) \in \{2,3\}$. Computationally, one can observe that $d_G(\cdot) = 2$ for $\rho(Q_p(V(\mathcal{S}_2^2))$ to be at most 5. Moreover, in this case $\rho(Q_p(V(\mathcal{S}_2^2))) = 5$, and thus by Perron-Frobenious Theorem we have $Q(G) = Q_p(V(\mathcal{S}_2^2))$ implying $G = K_{1,2}\Box K_2$.

Case (iv). $E(G[\Gamma]) = \{1'1''\}$.
Computationally, one can observe that for each of possible choices of $d_G(\cdot)$, the spectral radius of the corresponding $Q_p(V(\mathcal{S}_2^2))$ in (3.11) becomes less than 1. Thus, G is bipartite containing γ_1 as an induced subgraph.

Case (v). $E(G[\Gamma]) = \{\phi\}$.
If $d_G(i') = d_G(i'') = 3, i = 1, 2$, then the spectral radius of the corresponding $Q_p(V(\mathcal{S}_2^2))$ in (3.11) is 5, and thus $Q(G) = Q_p(V(\mathcal{S}_2^2))$, which is a contradiction. Therefore at least one $d_G(\cdot) \leq 2$. However, here the least eigenvalue of the corresponding $Q_p(V(\mathcal{S}_2^2))$ in (3.11) becomes less than 1 implying G is bipartite. Also, $\mathcal{S}_2^2$ is an induced subgraph in this case.

From Cases **(i)-(v)**, we conclude that the theorem holds. □



(a)γ_1 (b)J^*

Figure 6. **The graphs γ_1 and J^*.**

Remark 4. There exists at least one bipartite graph as mentioned in Theorem 13(c). We have constructed one such graph J^*, as shown in Figure 6(b), which has maximum edge-degree $= 4$, $q(J^*) = 5$ and whose minimum Q-eigenvalue is 0. Note that this graph in fact contains both $\mathcal{S}_2^2$ and γ_1 as induced subgraphs.

4. Conclusion

In this article, we have studied the structure of simple connected graphs having integral Q-spectral radius. We have shown that the necessary and sufficient condition for such graphs to contain either a double star $\mathcal{S}_{q(G)-3}^2$ or its variation $\mathcal{S}_{q(G)-3}^{2,1}$ as a subgraph is that the maximum edge-degree is $2q(G) - 6$.
However, based on our observations and proofs, we propose the following conjectures:

Conjecture 1. Every connected Q-integral graph having $q(G) \geq 4$ and maximum edge-degree equal to $2q(G) - 6$ is $\mathcal{S}_{q(G)-3}^{2,1}$-free and contains $\mathcal{S}_{q(G)-3}^2$ as a subgraph.

Conjecture 2. Every connected graph with integral Q-spectral radius $q(G) \geq 4$, $\lambda_{\min} \notin (0, 1)$ and maximum edge-degree $2q(G)-6$ is $\mathcal{S}_{q(G)-3}^{2,1}$-free and contains $\mathcal{S}_{q(G)-3}^2$ as a subgraph.

Observe that Conjecture 2 implies Conjecture 1, that is, if Conjecture 2 is true then Conjecture 1 will also be true. In this context, we have thus shown here that the above conjectures are true when $4 \leq q(G) \leq 9$. The above conjectures remain open for $q(G) \geq 10$.
In addition, we have also characterized all the connected edge-non-regular graphs having maximum edge-degree equal to 4 whose Q-spectral radius is an integer and the minimum Q-eigenvalue does not belong to $(0, 1)$.

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