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Research Article

The monophonic pebbling number of neural networks with generalized algorithm and their applications

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Abstract: Consider a graph σ(V, E) with nodes V and edges E is a connected graph with some pebbles scattered over its nodes V. By removal of two pebbles from one node and placing one pebble to an adjacent node is a pebbling move. A monophonic pebbling number, $\lambda_M(\sigma, v)$, of a node v of a graph σ is the least number m such that minimum of one pebble could be shifted to v by a sequence of pebbling shifts for any distribution of $\lambda_M(\sigma, v)$ pebbles on the nodes of σ using monophonic path. A link between the nodes x and y is an x-y path which consists of no chords and is monophonic. The monophonic pebbling number of a graph σ is the highest $\lambda_M(\sigma, v)$ among all the nodes notated as $\lambda_M(\sigma)$. For the first time, we calculate the monophonic pebbling number on families of neural networks such as probabilistic neural networks(PNNs), convolutional neural networks(CVNNs), modular neural networks(MNNs), generalized regression neural networks(GRNNs) and Hopfield neural networks(HNNs) and discuss their applications. We give the generalized algorithm to find the monophonic pebbling number of any graph σ.

Keywords: Monophonic pebbling number, (PNNs), (CVNNs), (MNNs), (GRNNs) and (HNNs).

AMS Subject classification: 05C12, 05C25, 05C38, 05C76

1. Introduction

Pebbling is a recent development in graph theory and combinatorics. The pebbling game was first established by Lagarias and Saks as a device to solve a specific problem in number theory. In 1989 Chung [3] established the concept of pebbling into the

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literature. Hurlbert [7] gave an overview of graph pebbling. The concept has grown in two decades as a network optimization model for the transportation of consumable resources. When two pebbles are deleted from one node, one of the pebbles will be shifted to the adjacent node and the other pebble is lost, similar to a toll. The monophonic length between two points x and y is the length of the largest x-y monophonic path, denoted by $d_M(x, y)$ in σ. For any two nodes x and y, in a connected graph σ, the $x - y$ path is said to be a monophonic path if it does not contain any chords [20]. The line segment which joins two nodes x and y in a curve is said to have a chord.

In graph theory approach neurons are called nodes/vertices and edges are called connections between them. The term neural network was coined by neurophysiologist McCulloch in 1943. A neural network is a series of algorithms that behave the way the human brain operates in a set of data through a process. A neural network refers to either a neural circuit of biological neurons or a network of artificial neurons or nodes in the case of an artificial neural network. A neural network works similarly to the human brain's neural networks. A neural network is applied in artificial intelligence, deep learning, machine learning, chemistry etc. We refer to topological indices of probabilistic neural network by Nilanjan de et al. [21] showed the generalized multiplicative version of Zagreb indices and computed it for probabilistic neural network. Kdenka kuncic et al. [12] introduced the topological properties of Neuromorphic nanowires network using the concept of graph theory.

Among several types of neural networks, we consider probabilistic neural networks(PNNs), convolutional neural networks(CVNNs), modular neural networks(MNNs), generalized regression neural networks(GRNNs) and Hopfield neural networks (HNNs). Firstly, Probabilistic neural networks (PNNs) were introduced by Specht in 1990. PNNs is a feedforward neural network, which is widely used in pattern recognition process problems and classification. Figure 1 shows the graphical representation of 3-layered PNNs. There are several layers in PNNs such

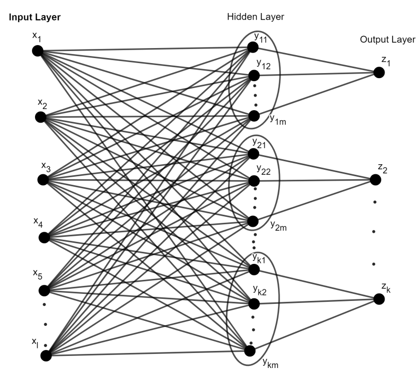

Figure 1. **Diagram of the 3-layered PNNs (l, m, k)**

as the Input layer, Pattern layer, summation layer, and output layer. In Fig.1, the input layer is named l-nodes, the hidden layer has m-nodes and the output layer has k-nodes. PNNs are used by machine learning engineers in classification and pattern recognition tasks. PNNs are used in ship identification. PNNs are used in the prediction of Leukemia and Embryonal tumours of the central nervous system. PNNs are faster than multilayer perceptron networks. PNNs are more accurate than multilayer perceptron.

Similarly, we have 4-layered PNNs which have 4 layers of neurons. The initial layer is the input layer which has q-nodes, the second layer is the pattern layer which has r-nodes, the third layer is the summation layer which has s-nodes, the fourth layer is the output layer which has one node named as T. The first layer is adjacent to all the nodes in the pattern layer. The particular class of pattern layer is adjacent to the respective class of summation layer nodes. All the summation nodes are adjacent to the output layer. Figure 2 shows the 4-layered PNNs which will be useful in the future sections.

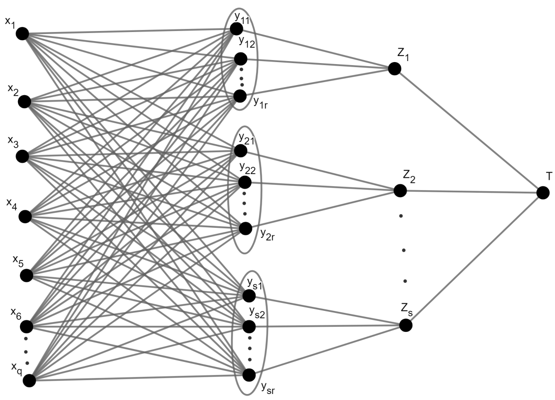

Figure 2. **Diagram of the 4-layered PNNs (q, r, s, T)**

Secondly, a Convolutional neural network(CVNNs) was introduced by LeCun in 1980. Convolutional neural network(CVNNs) is a type of artificial neural network architecture for any computer vision and image processing-related AI tasks. CVNNs are a type of deep learning neural network architecture. CVNNs consist of an input, hidden and output layer. Figure 3 shows the Convolutional neural network. The first layer is the input layer which has a-nodes, the second layer is the hidden layer which has b-nodes, third layer is the output layer which has c-nodes. Zhou et al. [22] showed the embedding of topological features into convolutional neural networks.

Modular neural network(MNNs) was introduced by Widrow and Hoff in 1959. A modular neural network is an artificial neural network characterized by some inde-

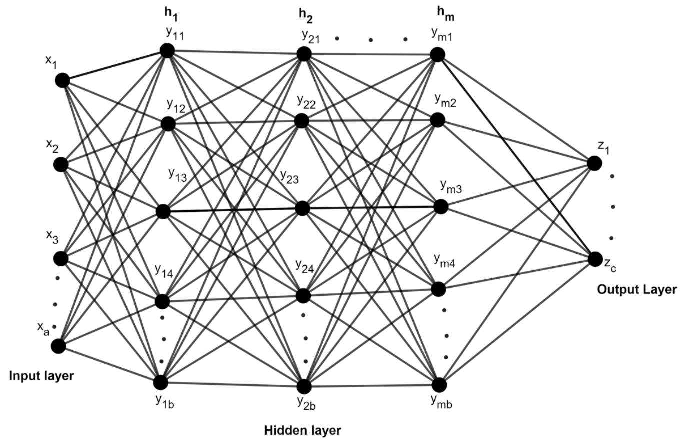

Figure 3. **Diagram of the CVNNs(a, b, c)**

pendent neural network moderated by some intermediary. Modular neural network is a trending topic for researchers in various domains like Biology, Computer science, Mathematics, Chemistry etc. Figure 4 shows the modular neural network.

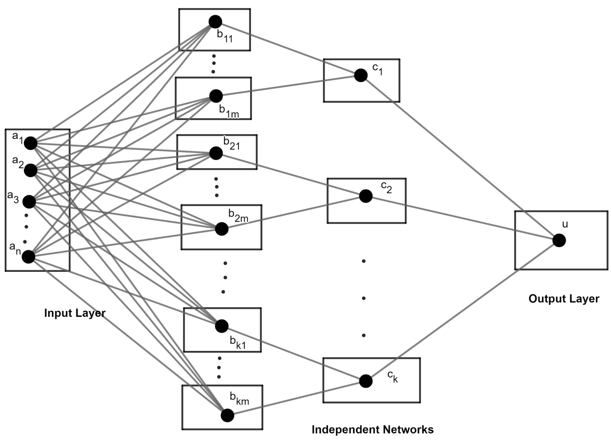

Figure 4. **Diagram of the MNNs (n, m, k, u)**

The architecture of a modular neural network consists of one or more input layers, one or more intermediate layers and an output layer. The initial layer is the input layer which has n-nodes, the second layer is the hidden layer which has k-nodes, and the third layer is the output layer which has one node which is named as u. Audal et al. [1] showed the different motivations for creating MNNs in biological, psychological,

hardware and computational.
Generalized Regression neural network (GRNNs) was introduced by Specht in 1991. A regression neural network is an artificial neural network used in machine learning for solving regression problems. The architecture of a regression neural network consists of an l-input layer, m-pattern layer, w-summation layer and r-output layer. The input layer accepts input data features. Each neuron represents each input feature. The input layer takes in input data. The pattern layer processes the data by learned patterns. The output data generates predictions based on the learned patterns. The applications of generalized regression neural networks are used for building mathematical models of dynamic systems from measurements of systems inputs and output. Figure 5 shows the generalized regression neural network.

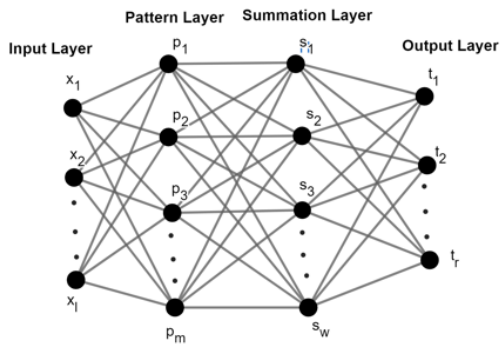

Figure 5. **Diagram of the GRNNs (q, r, s, T)**

Hopfield neural network (HNNs) was introduced by Hopfield in 1982. The architecture of the Hopfield neural network is a fully connected neural network. Hopfield neural network is used for associative memory and pattern recognition. There are $v_1, v_2, \cdots, v_m$ nodes in hopfield neural network. In the hopfield neural network, each node is connected to every other node. The structure of the Hopfield neural network is similar to a complete graph. The applications of the Hopfield neural network are used in combinatorial optimization and communications. Figure 6 shows the structure of hopfield neural network. Fang et al. [10] computed the results on clique number, chromatic number, independence number, matching ratio and the domination number of 3 and 4-layered PNNs, Cellular neural networks and tickysym spiking neural networks. Also, showed that the clique and chromatic number are equal for all NNs. Nelson et al. [17] used graph theory for microscopic, functional networks of neurons recorded by calcium imaging. For topological indices in neural networks, we refer to Kashif Shafiq et al. [11] computed the topological properties on m^{th} chain silicates. Defferrard et al. [4] computed the convolutional neural networks from low-dimensional regular grids where image, video and speech are represented to high-dimensional irregular domains.

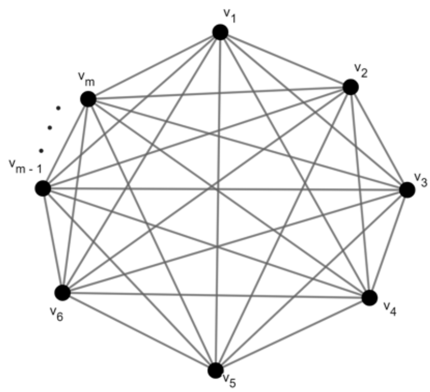

Figure 6. **Diagram of the HNNs (q, r, s, T)**

Hurlbert et al. [6] gave the survey of graph pebbling. He has given the updated development on graph pebbling. Isaak et al. [8] showed the game played on Powers of Paths. Singh et al. [2] have simulated the fingerprint pattern analysis mathematically using Graph isomorphism, graph dominance and graph pebbling. Santhakumaran A. P et al. [20] introduced the concept of monophonic numbers and computed the monophonic length in some graphs. Dhivviyanandam et al. [13] introduced the concept of the monophonic pebbling number and monophonic t-pebbling number of path graph, the square of even and odd paths, Jahangir graphs. Sadiquali et al. [19] computed the monophonic domination number of special graph structures like K-dimension cube, triangle-free graph, tree, middle graph and edge deleted graphs. Lourdusamy et al. [15] computed the monophonic pebbling number of families of cycles. Lourdusamy et al. [14] computed the monophonic pebbling number of some standard graphs. Arul Sudhahar et al. [18] established the concept of edge monophonic domination number of a graph. Kavitha et al. [9] computed the monophonic rubbling number of some standard graphs. Lourdusamy et al. [16] computed the monophonic pebbling number of some network related graphs like sun graph, $(C_n \times P_2) + K_1$ graph, the spherical graph, the anti-prism graphs, and an n-crossed prism graph. Table 1 explains the nomenclatures used in the article.

Motivation

With the help of the related work survey, here we outline the restrictions which were observed in the related work section.

(i) Computational and topological properties of neural networks using graph-theoretic parameters do not explain the recent trends in graph theory.

(ii) Methodology used by Loeffler et al. [12] is restricted to neuromorphic neural networks.

(iii) So far only theoretical analysis was restricted on graph pebbling numbers and

Table 1. **Methods with graph-theoretic tools**

Method	Description
λ_M	Monophonic pebbling number
q	Target node for neural networks
α	Target node for silicate networks
p(P)	Pebbles on the monophonic path P
$p(P^{`})$	Pebbles on the nodes not on the monophonic path P
PNNs	Probabilistic neural networks
CVNNs	Convolutional neural networks
MNNs	Modular neural networks
GRNNs	Generalized regression neural networks
HNNs	Hopfield neural networks
$\mid (V(\sigma)) \mid$	Cardinality of the nodes of the graph σ

application on the neural networks was not discussed. (iv) A Game using the pebbling concept on powers of Paths has been restricted from using the concept of monophonic pebbling numbers.

In this paper, we made the following contributions using the monophonic pebbling number:

- We compute some graph pebbling parameters such as the monophonic pebbling number of PNNs, CVNNs, MNNs, GRNNs, and HNNs.
- These computation provides the general results of PNNs, CVNNs, MNNs, RNNs, and HNNs.
- These results provide the number of pebbles required to transmit one pebble to the destination using a monophonic path which relates to the cost of the resources required to reach the destination.
- The monophonic distance of 3-layered PNNs, 4-layered PNNs and MNNs are constant.
- The generalized result of a monophonic pebbling number of the Hopfield neural network is isomorphic to the generalized result of a monophonic pebbling number of the complete graph.

2. Graph-Pebbling Preliminaries

We consider simple and connected graph $\sigma(V, E)$.

Definition 1. [13] A monophonic pebbling number, $\lambda_M(\sigma, v)$, of a node v of a graph σ is the least integer $\lambda_M(\sigma, v)$ in which minimum of one pebble could be shifted to v through a monophonic path by an arrangement of pebbling shifts for any distribution of $\lambda_M(\sigma, v)$ pebbles on the nodes of σ. An x-y path which contains no chords between them is known as a monophonic path. The highest $\lambda_M(\sigma, v)$ among all the nodes of σ is known as the monophonic pebbling number of a graph, notated as $\lambda_M(\sigma)$.

Definition 2. [5] A transmitting subgraph of a graph σ is a path $v_0, v_1, v_2, \cdots,$ v_m in which one pebble is shifted from v_0 to v_m with the splitting of minimum of two pebbles in v_0 and minimum of one pebble on each of the other nodes in the path, except possibly v_m. So we can transfer a pebble from v_0 to v_m using this orientation. With this orientation, one can transfer a pebble from v_0 to v_m.

Theorem 1. *[14] The monophonic pebbling number of complete graph K_m is $\lambda_M(K_m) = m$.*

3. Main results

3.1. The algorithm for finding the monophonic pebbling number for any given graph $\sigma(V, E)$

Here the generalized algorithm for finding the monophonic pebbling number is introduced.
Step 1.Firstly, the destination to be fixed.
Step 2. For a given graph $\sigma(V, E)$, the monophonic distance d_M needs to be calculated for all vertices.
Step 3. Compare the monophonic distances of all the nodes. Among which choose the maximum monophonic length d_M which is chordless.
Step 4. The monophonic path (P) needs to be fixed based on the longest monophonic distance d_M.
Step 5. Find the monophonic pebbling number for each node considering the pebbling move along the monophonic path which is chordless.
Step 6. The node which has the least upper bound number of pebbles will be the monophonic pebbling number of the graph $\sigma(v, E)$.
Step 7. Let us assume the monophonic path (P_1) which has the maximum monophonic distance d_M.
Step 8. Placing some pebbles on the nodes which are not on the monophonic path (P_1) in such a way that no pebbles will reach the path (P_1).
Step 9. Let us denote the nodes which are not on the monophonic path P_1 as $P_1^{'}$. Now based on the monophonic distance d_M we get 2^{d_M}.
Step 10. Now adding pebbles on the nodes of the monophonic path and pebbles on the nodes which are not on the monophonic path.
Hence, $\lambda_M(\sigma(v, E)) = 2^{d_M} + p(V(P_1^{'}))$.
Step 11. This is the least upper-bound monophonic pebbling number when we compare it with the rest of the nodes. Thus, we arrive at a monophonic pebbling number of any graph for all the distributions.

Theorem 2. *For any 3-layered PNNs, the monophonic pebbling number is given by $\lambda_M(PNN) = 2^{n+1} + (k-2)(m+1)$, where n is the 3 layers, $\forall\ l > 2$, $k \geq 2$.*

Proof. Let the node set of the 3-layered PNNs be $\{x_1, x_2, \cdots, x_l, y_{11}, y_{12}, \cdots,$

$y_{1m}, y_{21}, y_{22}, \cdots, y_{2m}, y_{31}, y_{32}, \cdots, y_{3m}, \cdots, y_{k1}, y_{k2}, \cdots, y_{km}, z_1, z_2, \cdots, z_k\}$.
Let q = z_k. The monophonic length from q to any other node is (n + 1). Assume the monophonic path $P_1 : z_1 \to y_{1m} \to x_l \to y_{km} \to q$. The remaining nodes are not on the monophonic path P_1. On placing $(2^{n+1} - 1)$ pebbles on P_1 and placing one pebble each on the nodes of induced sub-graph $<(PNN - \{x_1, x_2, \cdots, x_l\} \cup \{y_{11}, y_{12}, \cdots, y_{1m}, \cdots y_{k1}, y_{k2}, \cdots, y_{km}\} \cup \{z_1, z_k\})>$ which sums to (k - 2)(m + 1) pebbles, q is not reached. Therefore, the monophonic pebbling number is $\lambda_M(PNN) \geq 2^{n+1}$ + (k-2)(m + 1).

Let D be any distribution of $2^{n+1} + (k-2)(m+1)$ pebbles on the nodes of PNNs to prove for sufficient condition. For our convenience, let X = x_k where k =$\{1, \ldots, l\}$, Y = y_{ij} where i = $\{1, \ldots, k\}$ j = $\{1, \cdots, m\}$ and Z = z_m where m = $\{1, \cdots, k\}$.

Case 1. Let $q = x_k$.
The monophonic length from q to any other node is $(n-1)$. Assume the monophonic path $P_2 : x_1 \to y_{11} \to q$. By placing 2^{n-1} pebbles on any one of the nodes of the set x_k where $k \neq l$ or z_m, we can move a pebble to q. By placing $(2^{n-1} - 1)$ pebbles on any one of the nodes of set x_k where k $\neq$ l and placing one pebble on any one of the nodes of the set y_{ij}, we can move a pebble to q. By placing 2^{n-2} pebbles on any one of the nodes of the set y_{ij}, we can move a pebble to q. By placing 2^{n-2} pebbles on any one of the nodes of the set x_i where i $\neq$ l and placing 2^{n-3} pebbles on any one of the nodes of the set y_{ij} or placing 2^{n-2} pebbles on any one of the nodes of the set z_m, we can move a pebble to q. By placing 2^{n-2} pebbles on any one of the nodes of the set x_k and placing 2^{n-3} pebbles adjacent to the node where the pebble is placed on the nodes of the set y_{ij}, we can move a pebble to q.

Case 2. Let $q = y_{ij}$.
The monophonic length from y_{11} to any other node is n. Assume the monophonic path $P_3 : z_k \to y_{km} \to x_l \to q$. If the monophonic path $p(P_3)$ has 2^n pebbles, we can move a pebble to q. If the monophonic path $p(P_3)$ has $(2^n - 1)$ pebbles and placing one pebble each on non-monophonic nodes $P_3^{‘}$ pebbles, we can move a pebble to q. Let anyone of the nodes of the set x_k has 2^{n-2} pebbles or y_{ij} has 2^{n-1} pebbles or z_m adjacent to the target node has 2^{n-2} pebbles, we can move a pebble to q. If anyone of the nodes of the set x_k has 2^{n-3} pebbles and any one of the nodes of the set z_m has 2^{n-1} pebbles, we can move a pebble to q.

Case 3: Let $q = Z$.
The monophonic length from q to any other node is $(n+1)$. Assume the monophonic path $P_4 : z_1 \to y_{1m} \to x_l \to y_{km} \to q$. On placing 2^{n+1} pebbles on the path P_4, we can move one pebble to q. On placing $(2^{n+1} - 1)$ pebbles on z_m where $m \neq k$ and placing one pebble on any one of the nodes of the set x_k, we can move a pebble to q. By placing 2^n pebbles on any one of the nodes of the set z_m where $m \neq k$ and placing one pebble on any one of the nodes of the set y_{ij} adjacent to the target node, we can move a pebble to q. By placing 2^n pebbles on any one of the nodes of the set y_{ij}, apart from the vertices adjacent to the target node, we can move a pebble to q. By placing 2^{n-2} pebbles on the nodes adjacent to the target node, we can move a pebble to q. By placing 2^{n-1} pebbles on the set x_k, we can move a pebble to q. On placing

$(2^{n+1}-2^{n-1})$ pebbles on any of the nodes of the set z_m where $m \neq k$, and placing one pebble on any one of the vertices of set x_k, we can move a pebble to q. On placing 2^{n-1} pebbles on the set z_m where m $\neq$ k and placing $2^{n-2}+1$ pebbles on the set x_k, the destination is reached. Thus, for all the placement of pebbles used on the nodes of 3-layered PNNs is $\leq 2^{n+1}+(k-2)(m+1)$. Therefore, the monophonic pebbling number is $\lambda_M(PNN) = 2^{n+1}+(k-2)(m+1)$. This completes the proof. □

Theorem 3. *For any 4-Layered PNNs, the monophonic pebbling number is* $\lambda_M(PNN) = 2^n+(s-2)(r+1)+T$*, where n is the 4 layers,* $\forall\ q > 2$*,* $s \geq 2$*.*

Proof. Let the node set of the 4-layered PNNs be $\{x_1, x_2, \ldots, x_q, y_{11}, y_{12}, \ldots, y_{1r}, y_{21}, y_{22}, \ldots, y_{2r}, \ldots, y_{k1}, y_{k2}, \ldots, y_{sr}, z_1, z_2, \ldots, z_s, T\}$. Let $q = z_1$. The monophonic distance from q to any other node is n. Consider the monophonic path $P_1 : z_s \to y_{sr} \to x_q \to y_{1r} \to q$. The remaining nodes apart from the path P_1 is given by $P_1^{'}$. By placing 2^n-1 pebbles on the path P_1 and placing one pebble each on the adjacent nodes of z_d where d $= \{2, 3, \ldots, s-1\}$ and the node T which sums up to $(s-2)(r+1)+T$, the destination is not reached. Therefore, the monophonic pebbling number is $\lambda_M(PNN) \geq 2^n+(s-2)(r+1)+T$. Let D be any distribution of $2^n+(s-2)(r+1)+T$ pebbles on the nodes of PNNs to prove for sufficient condition. For our convenience, let X $= \{x_1, x_2, \ldots, x_q\}$, Y $= \{y_{11}, y_{12}, \ldots, y_{1r}, \ldots, y_{s1}, y_{s2}, \ldots, y_{sr}\}$ and Z $=\{z_1, z_2, \ldots, z_s\}$ and T.

Case 1. Let $q = X$ or T.
The monophonic length from q to any other node is $(n-1)$. Assume the monophonic path $P_2 : x_1 \to y_{11} \to z_1 \to T$. Let $P_2^{`}$ be the nodes not on the path $P_2$. On placing $2^{n-1}$ pebbles on the path $P_2$, we can move a pebble to $q$. On placing $(2^{n-1}-1)$ pebbles on the path $P_2$ and placing one pebble each on the non-monophonic nodes $P_2^{`}$, we can move a pebble to q. If anyone of the nodes of the set X has 2^{n-2} pebbles and anyone of the nodes of the set Z has a pebble, the destination is reached. If any of the nodes of the set Y has 2^{n-3} pebbles and anyone of the nodes of the set Z has a pebble, the destination is reached. If anyone of the nodes of the set Z has 2^{n-3} pebbles, the destination is reached. If anyone of the nodes of the set X has 2^{n-3} pebbles and anyone of the nodes of the set Y has one pebble or any one of the nodes of the set Z has 2^{n-3} pebbles or the node T has 2^{n-2} pebbles, we can move a pebble to q. If anyone of the nodes of the set Y has 2^{n-3} pebble and if anyone of the nodes of the set Z adjacent to the pebbles placed on the set Y has 2^{n-4} pebbles, we can move a pebble to q.

Case 2. Let $q = Y$ or Z. The monophonic length from q to any other node is n. Assume the monophonic path $P_3 : z_1 \to y_{11} \to x_q \to y_{sr} \to q$. On placing 2^n pebbles on the path P_3, we can move a pebble to q. On placing (2^n-1) pebbles on the path P_3 and placing one pebble each on the nodes adjacent to the nodes of the set z_d where $d = \{2, 3, \cdots, s-1\}$, placing a pebble on the vertex T and placing a pebble on anyone of the node of the set X, the destination is reached. If anyone of the nodes of the set X has 2^{n-3} pebble and anyone of the nodes of the set Y adjacent to the target node has 2^{n-4} pebbles, we can move a pebble to q. If anyone of the nodes of the set X has 2^{n-2} pebbles, we can move a pebble to q. If anyone of the nodes of the set Y or the node T has 2^{n-3} pebbles, we can move a pebble to q. Thus, for all the placement of pebbles used on the nodes of 4-layered PNN is $\leq\mid V(PNN)\mid$. Therefore, the monophonic pebbling number is $\lambda_M(PNN) =\mid V(PNN)\mid$. This completes the proof. □

Theorem 4. *For any CVNNs, the monophonic pebbling number is* $\lambda_M(CVNN)$ $= 2^{m+1}$*+(a - 1)+(c - 1), where m is the number of hidden layers.*

Proof. Let the node set of the CVNN be V(CVNN) = $\{x_1, x_2, \ldots, x_a, y_{11}, y_{12}, \ldots, y_{1b}, y_{21}, y_{22}, \ldots, y_{2b}, \ldots, y_{m1}, y_{m2}, \ldots, y_{mb}, z_1, \ldots z_c\}$. Let $q = x_1$. The monophonic length from q to any other node is $(m + 1)$. Consider the monophonic path P_1 be $z_1 \to y_{m1} \to \cdots, \to y_{21} \to y_{11} \to q$. On placing $(2^{m+1} - 1)$ pebbles on the path P_1 and placing (a - 1) pebbles each on the input layer except the target node and $(c-1)$ pebbles each on the output layer except the node on the path P_1, we cannot reach the destination. Therefore, $\lambda_M(CVNN) \geq 2^{m+1} + (a-1) + (c-1)$.
Let D be any destination of $2^{m+1} + (a-1) + (c-1)$ pebbles on the nodes of CVNNs to prove for sufficient condition. For our convenience, let $X = \{x_1, x_2, \ldots, x_a\}$, H = y_{ij} where i = $\{1, \ldots, m\}$ and j = $\{1, \ldots, b\}$ and Z = $\{z_1, \ldots, z_c\}$.

Case 1. Let $q = X$ or Z.
The monophonic length from q to any other node is $(m+1)$. Assume the monophonic path $P_2 : x_1 \to y_{11} \to y_{21} \to y_{31} \to q$. On placing 2^{m+1} pebbles on the path P_2, one pebble is reached to q. If anyone of the nodes of the set X has 2^{m+1} pebbles, one pebble is reached to q. By placing $(m-l+1)$ pebbles from q to any one of the nodes of h_l where $l = \{1, \ldots, m\}$, we can reach q. If anyone of the nodes of the set X has 2^m pebbles and anyone of the nodes of the set h_m has one pebble, we can move one pebble to q.

Case 2. Let $q = h_l$ be the target node, where $l = \{1, \ldots, m\}$.
The monophonic distance from h_l to h_i where $l <$ i $\leq m$ is at most $(m-l)$. By Case 1, if $p(<PNN - \{h_l, h_{l+1}, \ldots, h_m\}>)$ consists of at least 2^{m-l} pebbles, then we can transfer a pebble to q. Otherwise, we can transfer a pebble to q if $<h_1, h_2, \cdots, h_l>$ contains at least 2^l pebbles. We can use 2^l pebbles to reach the target. Because the monophonic length from h_l to h_j is at most l, where $1 \leq j \leq l$. Thus, for all the placement of pebbles used on the nodes of CVNNs is $\leq 2^{m+1}$+(a - 1)+(c - 1). Therefore, the monophonic pebbling number is $\lambda_M(CVNN) = 2^{m+1}+(a-1)+(c-1)$. This compeltes the proof. □

Theorem 5. *For any (MNNs), the monophonic pebbling number is* $\lambda_M(MNN) = |V(MNN)|$.

Proof. Let the node set of the MNNs be written as V(MNNs)= $\{a_1, a_2, \ldots, a_n, b_{11}, \ldots, b_{1m}, b_{21}, \ldots, b_{2m}, \ldots, b_{k1}, \ldots, b_{km}, c_1, c_2, \ldots, c_k, u\}$. Let $q = c_k$. The monophonic distance from c_k to any other node is $(l + 1)$ where l is the number of layers in the MNNs. Consider the monophonic path $P_1 : c_1 \rightarrow b_{11} \rightarrow a_n \rightarrow b_{km} \rightarrow q$. On placing $(|V(MNN)| - 1)$ pebbles each on all the nodes, we can move a pebble to q. Therefore, the monophonic pebbling number holds for $\lambda_M(MNN) \geq |V(MNN)|$. Let D be any placement of $|V(MNN)$ pebbles on the nodes of MNNs to prove for sufficient condition. For our convenience, let $X = \{a_1, a_2, \ldots, a_n\}$, $Y = \{b_{11}, \ldots b_{1m}, b_{21}, \ldots b_{2m}, \ldots b_{k1}, \ldots, b_{km}\}$, $Z = \{c_1, c_2, \ldots, c_k\}$ and u.

Case 1. Let $q = X$ or u. The monophonic length from q to any other node is (l - 1). Assume the monophonic path $P_2 : u \rightarrow c_1 \rightarrow b_{11} \rightarrow q$. Let $P_2^{'}$ be the nodes not on the path P_2. On placing 2^{l-1} pebbles on the path P_2, we can move a pebble to q. On placing $(2^{l-1} - 1)$ pebbles on the path P_2 and placing one pebble each on the non-monophonic vertices $P_2^{'}$, we can move a pebble to q. If anyone of the nodes of the set X, has 2^{l-2} pebbles or anyone of the nodes of the set Y has 2^{l-3} pebbles or anyone of the nodes of the set Z has 2^{l-2} pebbles, we can move a pebble to q. If anyone of the nodes of the set X has 2^{l-3} pebbles and anyone of the nodes of the set Y has one pebble, we can move a pebble to q. If anyone of the nodes of the set X has 2^{l-3} pebbles and anyone of the nodes of the set Z has 2^{l-3} pebbles, we can move a pebble to q. If anyone of the nodes of the set X has 2^{l-3} pebbles and the node u has 2^{l-2} pebbles, we can move a pebble to q. If anyone of the nodes of the set Y has 2^{l-4} pebbles and if anyone of the nodes of the set Z adjacent to the pebbles placed on the set Y has 2^{l-3} pebbles, we can move a pebble to q. If anyone of the nodes of the set Y has 2^{l-4} pebbles and set u has 2^{l-2} pebbles, we can move a pebble to q.

Case 2. Let $q = Y$ or Z. The monophonic length from q to any other node is l. Assume the monophonic path $P_3 : c_1 \rightarrow b_{1m} \rightarrow a_n \rightarrow b_{km} \rightarrow q$. On placing 2^l pebbles on the path P_3, we can move a pebble to q. On placing $(2^l - 1)$ pebbles on the path P_3 and placing one pebble each on the non-monophonic nodes, we can move a pebble to q. If anyone of the nodes of the set X has 2^{l-3} pebbles and anyone of the nodes of the set Y adjacent to the target node has 2^{l-4} pebbles, we can move a pebble to q. If anyone of the nodes of the set X has 2^{l-2} pebbles, we can move a pebble to q. If anyone of the nodes of the set Y or the node u has 2^{l-3} pebbles, we can move a pebble to q. Therefore, for all the placement of pebbles used on the nodes of MNNs is $\leq |V(MNN)|$. Therefore, the monophonic pebbling number is $\lambda_M(MNN) = |V(MNN)|$. This completes the proof. □

Theorem 6. *For any (GRNNs), the monophonic pebbling number is* $\lambda_M(GRNN) = 2^{n-1} + l + m + w + r - 4$.

Proof. Let the node set of the generalized regression neural network (GRNNs) be $\{x_1, x_2, \ldots, x_l, p_1, p_2, \ldots, p_m, s_1, s_2, \ldots, s_w, t_1, t_2, \ldots, t_r\}$. Let q $= t_1$. The monophonic length from q to any other node is $(n - 1)$. Assume the monophonic path $P_1 : x_1 \rightarrow p_1 \rightarrow s_1 \rightarrow q$. The remaining nodes are not on the monophonic path P_1. On placing $2^{n+2}-1$ pebbles on P_1 and placing one pebble each on the non-monophonic

nodes ($P_1^{'}$) which sums up to $l+m+w+r-4$, q is not reached. Therefore, the monophonic pebbling number is $\lambda_M(GRNNs) \geq 2^{n-1}+l+m+w+r-4$.
Let D be any distribution of $2^{n-1}+l+m+w+r-4$ pebbles on the nodes of GRNNs to prove for sufficient condition. For our convenience, let $X = \{x_1, x_2, \ldots, x_l\}$, $Y = \{p_1, p_2, \cdots, p_m\}$, $Z = \{s_1, s_2, \ldots, s_q\}$ and $L = \{t_1, t_2, \ldots, t_r\}$.

Case 1. Let $q = X$ or L.
The monophonic length from q to any other node is $(n-1)$. Assume the monophonic path $P_2 : x_1 \to p_2 \to s_3 \to q$. By placing 2^{n-1} pebbles on the path P_2, the destination is reached. By placing 2^{n-2} pebbles on the path P_2 and placing one pebble on any one of the nodes, the destination is reached. On placing 2^{n-3} pebbles on any one of the nodes of the set Z, the destination is reached. On placing 2^{n-2} pebbles on the set Y, the destination is reached. On placing 2^{n-1} pebbles on any one of the nodes of the set X, the destination is reached. On placing 2^{n-4} pebbles on any one of the nodes of the set Z and placing 2^{n-3} pebbles on any one of the nodes of the set Y, the destination is reached. On placing 2^{n-4} pebbles on the set Z and placing 2^{n-2} pebbles on the set X, the destination is reached.

Case 2: Let $q = Y$ or Z.
The monophonic length from q to any other node is (n - 2). The monophonic path P_3: $p_1 \to s_1 \to q$. On placing 2^{n-2} pebbles on the path P_3, the destination is reached. On placing 2^{n-3} pebbles on the path P_3 and placing one pebble on any one of the nodes of the set X or Z, the destination is reached. On placing 2^{n-3} pebbles on any one of the nodes of the set X or Z, the destination is reached. On placing 2^{n-2} pebbles on any one of the nodes of the set L, the destination is reached. On placing 2^{n-4} pebbles on any one of the nodes of the set Z and placing 2^{n-3} pebbles on any one of the nodes of the set L, the destination is reached. Therefore, for all the placement of pebbles on the graph $\lambda_M(GRNN)$ is $2^{n-1}+l+m+w+r-4$. This completes the proof. □

Corollary 1. *For any (HNNs), the monophonic pebbling number is $\lambda_M(HNNs) = m$ where m is the number of nodes.*

Proof. Let the node set of λ_M be $\{v_1, v_2, v_3, \cdots, v_m\}$. Let q = v_1. The monophonic distance from q to any other node is 1. Since the Hopfield neural network is isomorphic to complete graph K_m. The proof is followed by Theorem 1. Therefore, $\lambda_M(HNN)$ $= m$. Hence proved. □

4. Conclusion

In this article, we computed the monophonic pebbling number of several neural networks. This approach has been made based on the limitations of the methodology used in the literature. In Table 2, we have given the comparative analysis between

Table 2. **Comparative analysis table**

Article	Limitation	Our Results
Meie Fang et al. (2022)[10]	Limited to computational and topological properties of neural networks using graph-theoretic approach	Our method includes recent trends in graph theory.
Loeffler et al. [12]	Limited to neuromorphic neural networks	Our method includes PNNs,CVNNs, MNNs, GRNNs, HNNs.
A. Lourdusamy et al. [13][15][16]	Limited to theoretical analysis studied on graph pebbling numbers	Our method studied the applications on NNs using $\lambda_M(\sigma, v)$.

the limitation observed in the literature survey and how we have overcome those limitation in our study The general result in this article conveys the minimum number of pebbles required for the transition of one pebble to the destination using the monophonic path. This minimum number of pebbles could also be seen in terms of the minimum resources needed to transfer goods from one place to the destination. Also the monophonic distance of 3-layered PNNs, 4-layered PNNs and MNNs are the same whereas the monophonic distance of other considered networks varies based on their structure.
The limitation of the study are as follows:
(i) There is no well-known algorithm to compute the monophonic pebbling number for any graphs.
(ii) The study is limited only to PNNs, CVNNs, MNNs, GRNNs, HNNs and can be explored to other neural networks.
(iii) The study focuses on theoretical aspects rather than real-world application problems.

The application of determining the monophonic pebbling number of the neural network can be discussed as follows:
(i) For a given path P, which is the longest and chordless, we determine the cost factor, time duration, limited resources through the generalized monophonic pebbling number obtained for the considered networks in the article.
(ii) When there is a blockage in the network and also if a given node has more than one layer, we consider delivering the information to one of the neighbouring nodes which reduces the cost/ less time duration / limited resources and also able to reach the information faster.
(iii) So, we consider chordless whereas detour path information is shared with all the nodes which is time-consuming whereas in the geodesic path, the information is shared with particular nodes.
(iv) The concept of monophonic pebbling number helps in minimizing or optimizing the flow of information through the network layers.

Our future scope will be to compute the monophonic pebbling number for novel networks. Also the network considered in the article can be applied using the other pebbling invariants. For future research monophonic t-pebbling number could be applied to the networks mentioned in this article.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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