Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-14
https://doi.org/10.22049/cco.2024.29123.1848

CCO
Commun. Comb. Optim.

Research Article

On the rainbow connection number of the connected inverse graph of a finite group

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Received: 31 November 2023; Accepted: 26 March 2024
Published Online: 4 October 2024

Abstract: Let Γ be a finite group with $T_\Gamma = \{t \in \Gamma \mid t \neq t^{-1}\}$. The inverse graph of Γ, denoted by $IG(\Gamma)$, is a graph whose vertex set is Γ and two distinct vertices, u and v, are adjacent if $u * v \in T_\Gamma$ or $v * u \in T_\Gamma$. In this paper, we study the rainbow connection number of the connected inverse graph of a finite group Γ, denoted by $rc(IG(\Gamma))$, and its relationship to the structure of Γ. We improve the upper bound for $rc(IG(\Gamma))$, where Γ is a group of even order. We also show that for a finite group Γ with a connected $IG(\Gamma)$, all self-invertible elements of Γ is a product of r non-self-invertible elements of Γ for some $r \leq rc(IG(\Gamma))$. In particular, for a finite group Γ, if $rc(IG(\Gamma)) = 2$, then all self-invertible elements of Γ is a product of two non-self-invertible elements of Γ. The rainbow connection numbers of some inverse graphs of direct products of finite groups are also observed.

Keywords: rainbow connection number, inverse graph, finite group.

AMS Subject classification: 05C15, 05C25

1. Introduction

The rainbow connection number of a graph was introduced by Chartrand *et al.* in 2008 [3]. In an edge-colored graph G, where adjacent edges may have the same color, a path is called a rainbow path if no two edges within the path share the same color.

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The rainbow connection number of G, denoted by $rc(G)$, is the minimum number of colors that are needed to color the edges of G such that every two distinct vertices of G are linked by a rainbow path. Chakraborty et al. [2] proved that determining the rainbow connection number of a graph is an NP-hard problem.

Several studies have been conducted to study the rainbow connection number of some graphs. For example, Chartrand et al. [3] studied the rainbow connection number of complete graphs, trees, wheel graphs, bipartite graphs, and multipartite graphs. The rainbow connection number of line graphs was studied by Li et al. [11]. The rainbow connection number of amalgamation of some graphs was studied by Fitriani et al. [7]. Fitriani et al. [8] also studied the rainbow connection number of comb product of graphs. The rainbow connection numbers of the other graphs classes have also been studied, such as rainbow connection numbers of dense graphs and sparse graphs [12], rainbow connection numbers of graphs with diameter 3 [10], and rainbow connection numbers of Cayley graphs ([9], [13]).

The inverse graph of a finite group $(\Gamma, *)$, a group with a binary operation $*$ on a set Γ of finite cardinality, was introduced by Alfuraidan and Zakariya in [1]. For a finite group $(\Gamma, *)$ with $S_\Gamma = \{s \in \Gamma | s = s^{-1}\}$ and $T_\Gamma = \{t \in \Gamma | t \neq t^{-1}\}$, the inverse graph of the group, denoted by $IG(\Gamma)$, is a graph whose set of vertices is Γ and two distinct elements $g_1, g_2 \in \Gamma$ are adjacent if $g_1 * g_2 \in T_\Gamma$ or $g_2 * g_1 \in T_\Gamma$. In recent years, the inverse graphs of finite groups have garnered significant interest. For example, Ejima et al. [6] studied the energy of inverse graphs of dihedral and symmetric groups. The $L(h, k)$-labeling index of inverse graphs associated with finite cyclic groups was studied by Mageshwaran et al. in [14]. Murni et al. [15] determined the spectrum of the anti-adjacency and Laplacian matrices of the inverse graph a group of integers modulo n.

In [17], we studied the rainbow connection number of the inverse graph of some finite abelian groups. In [16], we proved that for a finite group $(\Gamma, *)$ of even order, $2 \leq rc(IG(\Gamma)) \leq |T_\Gamma| + m + 2$, where m is the number of $s \in S_\Gamma$ that satisfy $s * t = t^{-1} * s$ for all $t \in T_\Gamma$. In this paper, we improve the upper bound of $rc(IG(\Gamma))$ for a finite group $(\Gamma, *)$ of even order, so that the upper bound becomes $4 + m$. This is a better upper bound because $4 + m \leq |T_\Gamma| + m + 2$ since $|T_\Gamma|$ is always even when T_Γ is not empty. We also show in this paper that for a finite group $(\Gamma, *)$ and $k \geq 2$, if $rc(IG(\Gamma)) = k$, then every self-invertible element of Γ is a product of r non-self-invertible elements of Γ, where $r \leq k$. In particular, we show that if $rc(IG(\Gamma)) = 2$, then every self-invertible element of Γ is a product of two non-self-invertible elements of Γ.

This paper is organized as follows. In Section 2, we write some definitions and preliminary results from some previous studies. In Section 3, we discuss the rainbow connection number of the connected inverse graph of a finite group. In Section 4, we give some conclusions.

2. Preliminaries

2.1. Graph and the rainbow connection number of a graph

We follow [4] for definitions and notations in Graph Theory that are not described in this text. A *graph* is a pair of sets $G = (V(G), E(G))$, where the elements of $E(G)$ are 2-elements subsets of $V(G)$. The elements of $V(G)$ are called the *vertices* of G and the elements of $E(G)$ are called the *edges* of G. If $|V(G)| = 0$ or 1, then G is called a *trivial* graph. Any edge in $E(G)$ is in the form of $\{x, y\}$, where $x, y \in V(G)$. In this situation, $\{x, y\}$ is called an *edge* between x and y, and the vertices x and y are called the *endvertices* of the edge $\{x, y\}$. For simplicity, the edge $\{x, y\}$ will be written as xy. The conventional method of representing a graph is to depict a point for each vertex and connect two of these points with a line if the corresponding vertices form an edge.

For a graph G, if $u, v \in V(G)$ and $uv \in E(G)$, then u and v are said to be *adjacent* in G. If every two distinct vertices of a graph G are adjacent, then G is called a *complete* graph. A *path* of size k, $k \geq 1$, is a graph $P_k = (V(P_k), E(P_k))$, whose vertices can be ordered into $v_0, v_1, \ldots, v_k$ such that $E(P_k) = \{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\}$. The vertices v_0 and v_k are called the *ends* of P_k and the vertices $v_1, v_2, \ldots, v_{k-1}$ are called the *inner vertices* of P_k. In this case, P_k is called a path between v_0 and v_k, and the vertices v_0 and v_k are linked by P_k. The *length* of the path P_k is the number of edges of P_k. The path between two vertices v_0 and v_k can also be written as $v_0v_1...v_k$. The *distance* between two vertices u and v is the length of the shortest path between u and v. A graph G is said to be *connected* if every two distinct vertices of G are linked by a path. The largest distance between any two distinct vertices of G is called the *diameter* of G, denoted by $diam(G)$.

In 2008, Chartrand et al. proposed the concept of the rainbow connection number of a graph [3]. Let k be a natural number and $c : E(G) \to \{1, 2, \ldots, k\}$ be a coloring of the edges of a non-trivial connected graph G that allows the adjacent edges of G to have the same color. A path P is called a *rainbow path* if every two distinct edges of P have different colors. A graph G is called *rainbow-connected* (under the coloring c) if every two distinct vertices of G are linked by a rainbow path. In this case, c is called a *rainbow k-coloring* of G, where k is the number of colors of c. The smallest value of k for which a rainbow k-coloring of the edges of G exists is called the *rainbow connection number* of G, denoted by $rc(G)$. A rainbow coloring of a graph G that utilizes $rc(G)$ colors is referred to as the *minimum rainbow coloring* of G. It has been shown in [3] that $rc(G) = 1$ if and only if G is a complete graph, and $diam(G) \leq rc(G) \leq |E(G)|$.

2.2. Group and the inverse graph of a finite group

We follow [5] for definitions and notations in Group Theory that are not described in this paper. The definition of a group is as follows.

Definition 1. A *group* is an ordered pair $(\Gamma, *)$, where Γ is a nonempty set Γ and

$* : \Gamma \times \Gamma \to \Gamma$ is a binary operation, satisfying the following axioms:

1. $a * (b * c) = (a * b) * c$ for all a, b, and c in Γ.
2. There is an element $e \in \Gamma$ such that $e * a = a * e = a$ for every $a \in \Gamma$. The element e is called the *identity* element.
3. For each element $a \in \Gamma$, there is an element $a^{-1} \in \Gamma$ such that $a * a^{-1} = a^{-1} * a = e$. The element a^{-1} is called the *inverse* of a.

For simplicity, in the remaining parts of this paper, we use Γ as a notation for a group $(\Gamma, *)$ if the operation $*$ is clear from the context. The number of elements of a group Γ is called the *order* of Γ, denoted by $|\Gamma|$. If $|\Gamma|$ is finite, then Γ is called a *finite* group. If $|\Gamma| = 1$, then Γ is called a *trivial* group. An element $a \in \Gamma$ is called *self-invertible* if $a^{-1} = a$. If a, b, and c are elements of a group Γ that satisfy $a*b = a*c$ or $b*a = c*a$, then $b = c$. This is called the *cancellation law*. For a group Γ, a subset $X \subseteq \Gamma$ is called a *set of generators* or a *generating set* of Γ if every element of Γ is a product (under the binary operation of the group) of finitely many elements of X or their inverses. If X is a set of generators of a group Γ and X does not contain any other set of generators of Γ, then X is called a *minimal set of generators* of Γ. A group Γ can be partitioned into two subsets, $S_\Gamma = \{s \in \Gamma | s = s^{-1}\}$ and $T_\Gamma = \{t \in \Gamma | t \neq t^{-1}\}$, which are the set of all self-invertible elements and the set of all non-self-invertible elements of Γ, respectively. It is obvious that $S_\Gamma \cap T_\Gamma = \emptyset$. For any group Γ, if t is an element of T_Γ, then t^{-1} is also in T_Γ. Hence, $|T_\Gamma|$ is always even.

If $(\Gamma_1, *_1), (\Gamma_2, *_2), \ldots, (\Gamma_n, *_n)$ are finite groups, where $n \geq 2$, the *direct product* of the groups is a group $(\Gamma, *)$ with the set of elements $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n = \{(g_1, \ldots, g_n) : g_i \in \Gamma_i \text{ for every } i \in \{1, \ldots, n\}\}$ whose binary operation $*$ is defined componentwise as $(g_1, \ldots, g_n) * (h_1, \ldots, h_n) = (g_1 *_1 h_1, \ldots, g_n *_n h_n)$. For simplicity, we will use $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ as a notation for a direct product of n finite groups if the operation is clear from the context.

In 2017, Alfuraidan and Zakariya proposed the inverse graph of a finite group [1]. The definition of the graph is as follows.

Definition 2. [1] Given a finite group Γ with $T_\Gamma = \{t \in \Gamma | t \neq t^{-1}\}$. The *inverse graph* of Γ, denoted by $IG(\Gamma)$, is a graph whose vertices are the elements of Γ such that two distinct vertices u and v are adjacent if $u * v \in T_\Gamma$ or $v * u \in T_\Gamma$.

Based on the definition, for any group Γ with $T_\Gamma \neq \emptyset$, every element $t \in T_\Gamma$ is adjacent to the identity element e and every element $s \in S_\Gamma$ is not adjacent to e, since $t * e = e * t = t \in T_\Gamma$ and $s * e = e * s = s \in S_\Gamma$. Some other properties of the inverse graph of a finite group are written in the following theorems.

Theorem 1. *[1] There is no inverse graph that is complete for any non-trivial finite group.*

Theorem 2. *[16] Let Γ be a group of finite order with $T_\Gamma \neq \emptyset$. The inverse graph $IG(\Gamma)$ is connected if and only if T_Γ is a set of generators of Γ.*

The rainbow connection number of the inverse graph of a finite group Γ, or $rc(IG(\Gamma))$, has been studied in [16]. The exact value of $rc(IG(\Gamma))$ for a finite group Γ of odd order has been determined. The other properties of $rc(IG(\Gamma))$ for a finite group Γ of even order have also been studied.

Theorem 3. *[16] If Γ is a group of odd order with $T_\Gamma \neq \emptyset$, then $rc(IG(\Gamma)) = 2$.*

Theorem 4. *[16] Let Γ be a group of even order and $T_\Gamma \neq \emptyset$ be a set of generators of Γ. Then*

$$2 \leq rc(IG(\Gamma)) \leq |T_\Gamma| + m + 2$$

where m is the number of $s \in S_\Gamma$ that satisfy $s * t = t^{-1} * s$ for all $t \in T_\Gamma$. Furthermore, the lower bound is tight.

Theorem 5. *[16] Let Γ be a group of even order with $T_\Gamma \neq \emptyset$. If $s * t = t * s$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$, then $rc(IG(\Gamma)) = 2$.*

3. Rainbow connection numbers of the connected inverse graphs of finite groups

As mentioned in the previous section, for a group Γ, the sets S_Γ and T_Γ are partitions of Γ. Hence, $S_\Gamma \cup T_\Gamma = \Gamma$ and $S_\Gamma \cap T_\Gamma = \emptyset$. However, some elements of S_Γ may be adjacent to some elements of T_Γ in $IG(\Gamma)$. The adjacency between some elements of S_Γ and some elements of T_Γ results in several properties of $IG(\Gamma)$, as shown in the following lemmas.

Lemma 1. *Let Γ be a finite group with $T_\Gamma \neq \emptyset$. An element $t \in T_\Gamma$ is adjacent to an element $s \in S_\Gamma$ in $IG(\Gamma)$ if and only if t^{-1} is adjacent to s in $IG(\Gamma)$.*

Proof. Let $t \in T_\Gamma$ and $s \in S_\Gamma$ be adjacent in $IG(\Gamma)$. Then, there exists t_1 or t_2 in T_Γ such that $t * s = t_1$ or $s * t = t_2$. Because $s = s^{-1}$, we get $s * t^{-1} = t_1^{-1}$ or $t^{-1} * s = t_2^{-1}$. Since the inverse of an element of T_Γ is also in T_Γ, we get that t^{-1} is adjacent to s. Conversely, let t^{-1} be adjacent to s in $IG(\Gamma)$. Then, there exists t_3 or t_4 in T_Γ such that $t^{-1} * s = t_3$ or $s * t^{-1} = t_4$. Therefore, we get $s * t = t_3^{-1}$ or $t * s = t_4^{-1}$. Since the inverse of an element of T_Γ is also in T_Γ, we get that t is adjacent to s. □

Lemma 2. *Let Γ be a finite group with $T_\Gamma \neq \emptyset$. An element $s \in S_\Gamma$ is adjacent to an element $t \in T_\Gamma$ in $IG(\Gamma)$ if and only if there exists t' or t'' in T_Γ such that $s = t * t'$ or $s = t'' * t$.*

Proof. Let an element $s \in S_\Gamma$ be adjacent to an element $t \in T_\Gamma$ in $IG(\Gamma)$. Then there exists t_1 or t_2 in T_Γ such that $s * t = t_1$ or $t * s = t_2$. Since $s = s^{-1}$, we get $s = t * t_1^{-1}$ or $s = t_2^{-1} * t$. Write $t_1^{-1} = t'$ and $t_2^{-1} = t''$. Since t_1 or t_2 is in T_Γ, t' or t'' is also in T_Γ. Thus, there exists t' or t'' in T_Γ such that $s = t * t'$ or $s = t'' * t$.
Conversely, let $s = t * t'$ or $s = t'' * t$ for an element $s \in S_\Gamma$ and an element $t \in T_\Gamma$, where t' or t'' is in T_Γ. Hence, $s * t = (t')^{-1}$ or $t * s = (t'')^{-1}$. Since t' or t'' is in T_Γ, $(t')^{-1}$ or $(t'')^{-1}$ is also in T_Γ. Thus, s and t are adjacent in $IG(\Gamma)$. □

Lemma 3. *Let Γ be a finite group with $T_\Gamma \neq \emptyset$. If $s = t_1 * t_2$ for an element $s \in S_\Gamma$ and for some t_1 and t_2 in T_Γ, then s is adjacent to both t_1 and t_2 in $IG(\Gamma)$.*

Proof. Let s be an element of S_Γ that satisfies $s = t_1 * t_2$ for some t_1 and t_2 in T_Γ. Since $s^{-1} = s$, we get $s * t_1 = t_2^{-1}$ and $t_2 * s = t_1^{-1}$. Because t_1^{-1} and t_2^{-1} are also elements of T_Γ, we conclude that s is adjacent to both t_1 and t_2. □

Lemma 4. *Let Γ be a finite group with $|T_\Gamma| \geq 4$. If $s * t = u \in T_\Gamma$ for an element $s \in S_\Gamma$ and element $t \in T_\Gamma$, where $t \neq u$, then s is adjacent to t, t^{-1}, u, and u^{-1}.*

Proof. Let Γ be a finite group with $|T_\Gamma| \geq 4$, s be an element of S_Γ, t be an element of T_Γ, and $s * t = u$, where $u \in T_\Gamma$ and $u \neq t$. Since $s * t = u \in T_\Gamma$, s is adjacent to t and also $s * u = t$. Hence, s is also adjacent to u. According to Lemma 1, s is also adjacent to t^{-1} and u^{-1}. □

The properties related to the adjacency between some elements of S_Γ and some elements of T_Γ in the previous lemmas are then used to determine the rainbow connection number of a connected $IG(\Gamma)$. In [16], it has been shown that for a finite group Γ of odd order, $rc(IG(\Gamma)) = 2$. Also in [16], the lower and upper bounds of $rc(IG(\Gamma))$ for a finite group Γ of even order have been determined, as mentioned in Theorem 4. However, the upper bound is too large because it depends on the cardinality of T_Γ. In Theorem 6, we improve the upper bound so that it no longer depends on the cardinality of T_Γ.

Theorem 6. *Let Γ be a finite group of even order and $T_\Gamma \neq \emptyset$ be a set of generators of Γ. Then*

$$2 \leq rc(IG(\Gamma)) \leq 4 + m,$$

where m is the number of $s \in S_\Gamma$ that satisfy $s * t = t^{-1} * s$ for all $t \in T_\Gamma$. Moreover, the lower bound is tight.

Proof. Let Γ be a finite group of even order with T_Γ be nonempty and be a set of generators of Γ. According to Theorem 2, since T_Γ is a set of generators of Γ, the inverse graph $IG(\Gamma)$ is connected. According to Theorem 4, $rc(IG(\Gamma)) \geq 2$, and this lower bound is tight. Because $\Gamma = S_\Gamma \cup T_\Gamma$ and $|T_\Gamma|$ is even, $|S_\Gamma|$ is also even, and hence $|S_\Gamma| > 1$. Recall that every element of T_Γ is adjacent to the identity element

e and every element of S_Γ is not adjacent to e in $IG(\Gamma)$. Given that $IG(\Gamma)$ is a connected graph, there are some $s \in S_\Gamma$ which are adjacent to some $t \in T_\Gamma$. Hence, there are some elements $s \in S_\Gamma$ such that $s * t \neq (s * t)^{-1}$ for some elements $t \in T_\Gamma$. Note that $(s * t)^{-1} = t^{-1} * s$. According to Lemma 1, an element $s \in S_\Gamma$ is adjacent to an element $t \in T_\Gamma$ if and only if s is adjacent to t^{-1}.

Let $S'_\Gamma = \{s \in S_\Gamma | s * t \neq t^{-1} * s \text{ for some } t \in T_\Gamma\}$ and $S''_\Gamma = S_\Gamma \setminus S'_\Gamma = \{s \in S_\Gamma | s * t = t^{-1} * s \text{ for all } t \in T_\Gamma\}$. Based on its definition, each element of S'_Γ is adjacent to some elements of T_Γ in $IG(\Gamma)$ and each element of S''_Γ is not adjacent to all elements of T_Γ in $IG(\Gamma)$. Since $IG(\Gamma)$ is connected, each element of S'_Γ is linked by a path to e in $IG(\Gamma)$ and each element of S''_Γ is linked by a path to each element of S'_Γ in $IG(\Gamma)$. Let T_Γ be partitioned into two subsets $\bar{T}_\Gamma = \{\bar{t}_1, ..., \bar{t}_{|T|/2}\}$ and $\bar{\bar{T}}_\Gamma = \{\bar{\bar{t}}_1, ..., \bar{\bar{t}}_{|T|/2}\}$ such that $\bar{\bar{t}}_i = \bar{t}_i^{-1}$ for every $i \in \{1, ..., |T|/2\}$. According to Lemma 1 and Lemma 4, every element of S'_Γ is adjacent to at least one element of $\bar{T}_\Gamma$ and one element of $\bar{\bar{T}}_\Gamma$. Next, we color the edges of $IG(\Gamma)$ using $4 + m$ colors, where $m = |S''_\Gamma|$, as follows:

1. For each $i \in \{1, 2, \ldots, |T_\Gamma|/2\}$, the edge $e\bar{t}_i$ is colored with color 1 and the edge $e\bar{\bar{t}}_i$ is colored with color 2.

2. For each $s \in S'_\Gamma$ and for $i \in \{1, 2, \ldots, |T_\Gamma|/2\}$, if s is adjacent to $\bar{t}_i$, we color the edge $s\bar{t}_i$ with color 3 and the edge $s\bar{\bar{t}}_i$ with color 4.

3. For $\alpha \in \{1, 2, \ldots, m\}$, in any path P between an $s_\alpha \in S''_\Gamma$ and an $s \in S'_\Gamma$, the edge $\bar{s}s_\alpha$ in P is colored with color $4 + \alpha$, where $\bar{s}$ can be either s or another element of S''_Γ whose distance from s on P is $d_P(s_\alpha, s)$–1.

4. The remaining edges in $IG(\Gamma)$ are colored with color 1.

Using this edge coloring, the rainbow paths in $IG(\Gamma)$ between any two elements of Γ are as follows:

1. For any $s \in S'_\Gamma$, a rainbow path between s and the identity e is the path ets, where t is an element of T_Γ which is adjacent to s.

2. If two elements $\bar{t}_i$ and $\bar{t}_j$ in $\bar{T}_\Gamma$ are non-adjacent elements, a rainbow path between the two elements is $\bar{t}_i e \bar{\bar{t}}_j s \bar{t}_j$, where s is a non-identity element of S'_Γ such that $s = \bar{t}_i * \bar{t}_j$ and $\bar{\bar{t}}_j \in \bar{\bar{T}}_\Gamma$ is the inverse of $\bar{t}_j$.

3. If two elements $\bar{t}_i \in \bar{T}_\Gamma$ and $\bar{\bar{t}}_j \in \bar{\bar{T}}_\Gamma$ are non-adjacent elements, a rainbow path between the two elements is $\bar{t}_i e \bar{\bar{t}}_j$.

4. If two elements $\bar{\bar{t}}_i$ and $\bar{\bar{t}}_j$ in $\bar{\bar{T}}_\Gamma$ are non-adjacent elements, a rainbow path between the two elements is $\bar{\bar{t}}_i e \bar{t}_j s \bar{\bar{t}}_j$, where s is a non-identity element of S'_Γ such that $s = \bar{\bar{t}}_i * \bar{\bar{t}}_j$ and $\bar{t}_j \in \bar{T}_\Gamma$ is the inverse of $\bar{\bar{t}}_j$.

5. If two non-identity elements s_i and s_j in S'_Γ are non-adjacent elements, a rainbow path between the two elements is $s_i \bar{t}_k e \bar{\bar{t}}_l s_j$, where $\bar{t}_k$ is an element of $\bar{T}_\Gamma$ which is adjacent to s_i and $\bar{\bar{t}}_l$ is an element of $\bar{\bar{T}}_\Gamma$ which is adjacent to s_j.

6. If an element $\bar{t}_i \in \bar{T}_\Gamma$ and a non-identity element $s_j \in S'_\Gamma$ are non-adjacent elements, a rainbow path between the two elements is $\bar{t}_i e \bar{\bar{t}}_k s_j$, where $\bar{\bar{t}}_k$ is an element of $\bar{\bar{T}}_\Gamma$ which is adjacent to s_j.

7. If an element $\bar{\bar{t}}_i \in \bar{\bar{T}}_\Gamma$ and a non-identity element $s_j \in S'_\Gamma$ are non-adjacent elements, a rainbow path between the two elements is $\bar{\bar{t}}_i e \bar{t}_k s_j$, where $\bar{t}_k$ is an element of $\bar{T}_\Gamma$ which is adjacent to s_j.

8. The rainbow path between $s'' \in S''_\Gamma$ and the identity e is in the form of $s''s_1...s_k te$ for some natural number k, where s_1 is an element of S'_Γ if $k = 1$, s_k is an element of S'_Γ, t is an element of T_Γ which is adjacent to s_k, and s_i is an element of S''_Γ for every $i \in \{1, ..., k-1\}$ if $k \geq 2$.

9. If a non-identity element $s' \in S'_\Gamma$ and an element $s'' \in S''_\Gamma$ are non-adjacent elements and there is a path P between the two elements whose all internal vertices are elements of S''_Γ, then P is a rainbow path between the two elements.

10. If a non-identity element $s' \in S'_\Gamma$ and an element $s'' \in S''_\Gamma$ are non-adjacent elements and there is no path between the two elements whose all internal vertices are elements of S''_Γ, then a rainbow path between the two elements is in the form of $s''s_1...s_k \bar{t} e \bar{\bar{t}} s'$ for some natural number k, where s_1 is an element of S'_Γ if $k = 1$, s_i is an element of S''_Γ for all $i \in \{1, ..., k-1\}$ if $k \geq 2$, s_k is an element of S'_Γ, $\bar{t}$ is an element of $\bar{T}_\Gamma$ which is adjacent to s_k, and $\bar{\bar{t}}$ is an element of $\bar{\bar{T}}_\Gamma$ which is adjacent to s'.

11. A rainbow path between an element $s'' \in S''_\Gamma$ and an element $\bar{t} \in \bar{T}_\Gamma$ is in the form of $s''s_1...s_k \bar{\bar{t}} e \bar{t}$ for some natural number k, where s_1 is an element of S'_Γ if $k = 1$, s_k is an element of S'_Γ, $\bar{\bar{t}}$ is an element of $\bar{\bar{T}}_\Gamma$ which is adjacent to s_k, and s_i is an element of S''_Γ for every $i \in \{1, ..., k-1\}$ if $k \geq 2$.

12. A rainbow path between an element $s'' \in S''_\Gamma$ and an element $\bar{\bar{t}} \in \bar{\bar{T}}_\Gamma$ is in the form of $s''s_1...s_k \bar{t} e \bar{\bar{t}}$ for some natural number k, where s_1 is an element of S'_Γ if $k = 1$, s_k is an element of S'_Γ, $\bar{t}$ is an element of $\bar{T}_\Gamma$ which is adjacent to s_k, and s_i is an element of S''_Γ for every $i \in \{1, ..., k-1\}$ if $k \geq 2$.

13. If two non-adjacent elements s''_1 and s''_2 in S''_Γ are linked by a path P, with at most one of its internal vertices is in S'_Γ, then P is a rainbow path between the two elements.

14. If all paths connecting two non-adjacent elements s''_0 and s''_n in S''_Γ have at least two elements of S'_Γ as their internal vertices, a rainbow path between s''_0 and s''_n is in the form of $s''_0 s_1 \ldots s_k \bar{t} e \bar{\bar{t}} s_{k+1} \ldots s_{n-1} s''_n$, where $k \geq 1$ and $n \geq 3$ are natural numbers, $s_1 \in S'_\Gamma$ if $k = 1$, $s_{n-1} \in S'_\Gamma$ if $n = k+2$, all elements $s_1, \ldots, s_{k-1}, s_{k+2}, \ldots, s_{n-1}$ are in S''_Γ if $k \geq 2$ and $n \geq k+3$, s_k and s_{k+1} are elements of S'_Γ, $\bar{t}$ is an element of $\bar{T}_\Gamma$ which adjacent to s_k, and $\bar{\bar{t}}$ is an element of $\bar{\bar{T}}_\Gamma$ which adjacent to s_{k+1}.

15. The rainbow path between any two adjacent vertices in $IG(\Gamma)$ is the edge between the vertices.

By using the above edge coloring, every two distinct vertices of $IG(\Gamma)$ are linked by a rainbow path. Thus, $rc(IG(\Gamma)) \leq m + 4$. □

The upper bound in Theorem 6 is an improvement to the upper bound in Theorem 4 because the value of $4 + m$ is always less than or equal to $|T_\Gamma| + m + 2$, since $|T_\Gamma|$ is always even for any finite group Γ with $T_\Gamma \neq \emptyset$. Theorem 6 guarantees that $rc(IG(\Gamma))$ does not exceed $4+m$ for a finite group Γ of even order with T_Γ as its set of generators. However, it is difficult for us to find a group Γ such that $rc(IG(\Gamma)) = 4 + m$. Hence, the upper bound cannot yet be stated as a tight upper bound.
Now consider a direct product $\Gamma = \Gamma_1 \times \ldots \times \Gamma_n$ of finite groups, where $n \geq 2$. It is obvious that $S_\Gamma = S_{\Gamma_1} \times \cdots \times S_{\Gamma_n}$ and $T_\Gamma = \Gamma \setminus S_\Gamma$. If $T_{\Gamma_i} \neq \emptyset$ for some $i \in \{1, \ldots, n\}$, then it is clear that $T_\Gamma \neq \emptyset$. The following theorem gives the lower and upper bounds for the rainbow connection number of $IG(\Gamma)$, where Γ is a direct product of finite groups, some of which have even orders.

Theorem 7. *Let $n \geq 2$, $\Gamma = \Gamma_1 \times \ldots \times \Gamma_n$ be a direct product of finite groups, and $|\Gamma_i|$ be even for some $i \in \{1, \ldots, n\}$. If for every $i \in \{1, \ldots, n\}$, T_{Γ_i} is nonempty and generates Γ_i, then $2 \leq rc(IG(\Gamma)) \leq 4$. Moreover, the lower bound is tight.*

Proof. Let $n \geq 2$, $\Gamma = \Gamma_1 \times \ldots \times \Gamma_n$ be a direct product of finite groups, and $|\Gamma_i|$ be even for some $i \in \{1, \ldots, n\}$. Hence, the order of Γ is even. Let T_{Γ_i} be nonempty and generates Γ_i for every $i \in \{1, \ldots, n\}$. Therefore, $IG(\Gamma_i)$ is connected for every $i \in \{1, \ldots, n\}$. Any element of T_Γ is in the form of $(a_1, a_2, \ldots, a_n)$, where a_i is an element of T_{Γ_i} for at least one i in $\{1, \ldots, n\}$. Since T_{Γ_i} is nonempty and generates Γ_i for every $i \in \{1, \ldots, n\}$, T_Γ is nonempty and generates Γ. According to Theorem 6, the lower bound of $rc(IG(\Gamma))$ is 2 and the upper bound of $rc(IG(\Gamma))$ is $4 + m$, where m is the number of $s \in S_\Gamma$ that satisfy $s * t = t^{-1} * s$ for all $t \in T_\Gamma$.
Consider a direct product $\Gamma = Z_n \times \cdots \times Z_n$ of cyclic groups of order n, where n is even. The group Z_n is generated by 1, which is an element of T_{Z_n}. Hence, Z_n is generated by T_{Z_n}. Since Z_n is abelian, Γ is also abelian. According to Theorem 5, $rc(IG(\Gamma)) = 2$. Thus, the lower bound is tight.

Now let $m \neq 0$, $\hat{s} = (\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n)$ be an element of S_Γ such that $\hat{s} * t = t^{-1} * \hat{s}$ for all $t \in T_\Gamma$, and $\tilde{T}_\Gamma = \{\tilde{t} = (\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n) \in T_\Gamma | \tilde{t}_i$ is an element of T_{Γ_i} for one and only one $i \in \{1, 2, ..., n\}\}$. Since $\hat{s}$ is an element of S_Γ, $\hat{s}_i$ must be an element of S_{Γ_i} for every $i \in \{1, \ldots, n\}$. Because $\hat{s} * t = t^{-1} * \hat{s}$ for all $t \in T_\Gamma$, $\hat{s}$ also satisfies $\hat{s} * \tilde{t} = \tilde{t}^{-1} * \hat{s}$ for all $\tilde{t} \in \tilde{T}_\Gamma$. Thus, for every $i \in \{1, \ldots, n\}$, $\hat{s}_i * t_i = t_i^{-1} * \hat{s}_i$ for every $t_i \in T_{\Gamma_i}$ and $\hat{s}_i * s_i = (s_i)^{-1} * \hat{s}_i = s_i * \hat{s}_i$ for every $s_i \in S_{\Gamma_i}$. Since $(\hat{s}_i * t_i)^{-1} = t_i^{-1} * \hat{s}_i$ and $(\hat{s}_i * s_i)^{-1} = s_i * \hat{s}_i$, for every $i \in \{1, \ldots, n\}$, $\hat{s}_i$ is not adjcaent to every $s_i \in S_{\Gamma_i}$ and every $t_i \in T_{\Gamma_i}$ in $IG(\Gamma_i)$. Hence, $\hat{s}_i$ is an isolated vertex in $IG(\Gamma_i)$ for every $i \in \{1, \ldots, n\}$. This is a contradiction since $IG(\Gamma_i)$ is connected for every $i \in \{1, \ldots, n\}$. Thus, m must be 0 and the upper bound for $rc(IG(\Gamma))$ is 4. □

As in Theorem 6, the upper bound in Theorem 7 cannot yet be stated as a tight upper bound since finding a direct product of finite groups $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ such that $rc(IG(\Gamma)) = 4$ is not an easy task. However, the theorem guarantees that the rainbow connection number will not exceed 4.
Now consider a finite group Γ of even order whose every $s \in S_\Gamma$ is adjacent to every $t \in T_\Gamma$ in $IG(\Gamma)$. The following theorem shows that the rainbow connection number of the inverse graph of such a group is exactly 2.

Theorem 8. *Let Γ be a finite group of even order with $T_\Gamma \neq \emptyset$. If $s * t \neq t^{-1} * s$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$, then $rc(IG(\Gamma)) = 2$.*

Proof. Let Γ be a finite group of even order with $T_\Gamma \neq \emptyset$ and $s*t \neq t^{-1}*s$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$. Since $s * t \neq t^{-1} * s = (s * t)^{-1}$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$, we get that $s * t \in T_\Gamma$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$. Therefore, every $s \in S_\Gamma$ is adjacent to every $t \in T_\Gamma$. Since every $t \in T_\Gamma$ is adjacent to the identity e, $IG(\Gamma)$ is a connected graph. Recall that a connected $IG(\Gamma)$ is not a complete graph. Hence, $rc(IG(\Gamma)) \geq 2$.

Since the order of Γ and $|T_\Gamma|$ are even, $|S_\Gamma|$ is also even. Therefore, $|S_\Gamma| > 1$. If $|S_\Gamma|$ is greater than $|T_\Gamma|$, we can find two distinct elements, s_1 and s_2, in S_Γ such that $s_1 * t = s_2 * t$ for some $t \in T_\Gamma$. Applying the cancellation law, we deduce that $s_1 = s_2$, leading to a contradiction. Therefore, $|S_\Gamma|$ cannot exceed $|T_\Gamma|$.

According to Lemma 2, for every $s \in S_\Gamma$ and every $t \in T_\Gamma$, there exists a unique element $\tau \in T_\Gamma$ such that $t * \tau = s$. The uniqueness is guaranteed by the cancellation law. The element τ might be equal to t. From Lemma 3, we know that s is adjacent to both t and τ. Based on these facts, for each $s \in S_\Gamma$, we partition T_Γ into $T_{\Gamma,s} = \{t_{s,1}, t_{s,2}, \ldots, t_{s,|T_\Gamma|/2}\}$ and $T'_{\Gamma,s} = \{t'_{s,1}, t'_{s,2}, \ldots, t'_{s,|T_\Gamma|/2}\}$ such that for every $i \in \{1, 2, \ldots, |T_\Gamma|/2\}$, one of the following conditions holds:

1. If $t_{s,i} * t_{s,i} \neq s$, then $t'_{s,i}$ is an element of T_Γ that satisfies $t_{s,i} * t'_{s,i} = s$.
2. If $t_{s,i} * t_{s,i} = s$, then $t'_{s,i} = t^{-1}_{s,i}$.

Now let the edges of $IG(\Gamma)$ be colored by two different colors. Every edge between two adjacent vertices in T_Γ is colored with color 1 and every edge between two adjacent vertices in S_Γ is colored with color 2. For every $i \in \{1, 2, \ldots, |S_\Gamma|\}$ and every $j \in \{1, 2, \ldots, |T_\Gamma|/2\}$, if the edge $s_i t_{s_i,j}$ is assigned color 1, then the edge $s_i t'_{s_i,j}$ must be assigned color 2, or if the edge $s_i t_{s_i,j}$ is assigned color 2, then the edge $s_i t'_{s_i,j}$ must be assigned color 1.

To show that the edge coloring is a rainbow coloring, assign a $|T_\Gamma|$-tuple $C_{s_i} = (c_{i1}, c_{i2}, \ldots, c_{i|T_\Gamma|})$ for each $s_i \in S_\Gamma$, where c_{ij} is the color of the edge $s_i t_j$, with $t_j \in T_\Gamma$. The edge coloring that is used causes the number of distinct such tuples to be at most $2^{|T_\Gamma|/2}$. Given that $|S_\Gamma|$ and $|T_\Gamma|$ are both even, since $|S_\Gamma|$ is less than or equal to $|T_\Gamma|$, we get $|S_\Gamma|$ is less than or equal to $2^{|T_\Gamma|/2}$. Therefore, we can color the edges of $IG(\Gamma)$ such that $C_{s_i} \neq C_{s_j}$ if $i \neq j$, for $i, j \in \{1, 2, \ldots, |S_\Gamma|\}$. With this edge coloring, the rainbow paths between two different vertices within $IG(\Gamma)$ are as follows:

1. The rainbow path between any $s \in S_\Gamma$ and any $t \in T_\Gamma$ is the edge st.
2. For two distinct elements t_i and t_j in T_Γ which are not adjacent in $IG(\Gamma)$, the rainbow path between t_i and t_j is $t_i s t_j$, where s is an element of S_Γ such that $t_i * t_j$ equals s.
3. For any two distinct elements s_i and s_k in S_Γ which are not adjacent in $IG(\Gamma)$, the rainbow path between s_i and s_k is $s_i t_j s_k$, where t_j is an element of T_Γ such that c_{ij} is not equal to c_{kj}.
4. For any two distinct elements t_i and t_j in T_Γ which are adjacent in $IG(\Gamma)$, the rainbow path that connects t_i and t_j is the edge $t_i t_j$.
5. For any two distinct elements s_i and s_j in S_Γ which are adjacent in $IG(\Gamma)$, the rainbow path between s_i and s_j is the edge $s_i s_j$.

Therefore, we find that every pair of distinct vertices in $IG(\Gamma)$ is linked by a rainbow path, and hence $rc(IG(\Gamma)) \leq 2$. Note that $IG(\Gamma)$ is not a complete graph. Therefore, $rc(IG(\Gamma)) \geq 2$. Thus, we deduce that $rc(IG(\Gamma)) = 2$. □

Theorem 8 is a generalization of Theorem 5. In theorem 5, for a finite group Γ of even order, if $s * t = t * s$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$, then $rc(IG(\Gamma)) = 2$. If $s * t = t * s$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$, then $s * t \neq t^{-1} * s$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$. However, the converse is not true. Therefore, the condition of Theorem 8 is more general than the condition of Theorem 5.

The alternating group A_4 is an example of a group that satisfies Theorem 8. This group is a group of all even permutations on four elements, with $S_{A_4} = \{(1), (14)(23), (13)(24), (12)(34)\}$ and $T_{A_4} = \{(123), (132), (124), (142), (134), (143), (234), (243)\}$. The binary operation of this group is the composition of permutations. For the elements of T_{A_4}, the inverse of (123) is (132), the inverse of (124) is (142), the inverse of (134) is (143), and the inverse of (234) is (243). It is easy to check that the group satisfies $s * t \neq t^{-1} * s$ for every $s \in S_{A_4}$ and every $t \in T_{A_4}$. Therefore, according to Theorem 8, $rc(IG(A_4)) = 2$. Figure 1 shows a minimum rainbow coloring of $IG(A_4)$ with two colors.

Corollary 1. *Let n be a natural number and for every $i \in \{1, 2, \ldots, n\}$, Γ_i be a finite group of even order with $T_{\Gamma_i} \neq \emptyset$ that satisfies $s_i * t_i \neq t_i^{-1} * s_i$ for every $s_i \in S_{\Gamma_i}$ and every $t_i \in T_{\Gamma_i}$. If $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$, then $rc(IG(\Gamma)) = rc(IG(\Gamma_1)) = \cdots = rc(IG(\Gamma_n)) = 2$.*

Proof. According to Theorem 8, $rc(IG(\Gamma_i)) = 2$ for every $i \in \{1, 2, \ldots, n\}$. Let $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$. Since for every $i \in \{1, 2, \ldots, n\}$, $T_{\Gamma_i} \neq \emptyset$ and $s_i * t_i \neq t_i^{-1} * s_i$ for every $s_i \in S_{\Gamma_i}$ and every $t_i \in T_{\Gamma_i}$, we have $T_\Gamma \neq \emptyset$ and $s * t \neq t^{-1} * s$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$. Therefore, $rc(IG(\Gamma)) = 2$ according to Theorem 8. □

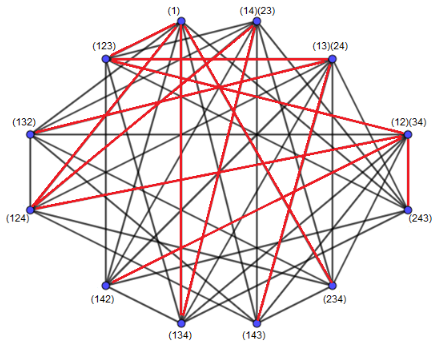

Figure 1. **A minimum rainbow coloring of $IG(A_4)$ with two colors**

The rainbow connection number of the inverse graph of a finite group Γ can be used to determine a specific property of Γ. Theorem 9 shows that if $rc(IG(\Gamma))$ is known, then every self-invertible element of Γ is a product of r non-self-invertible elements of Γ, where $r \leq rc(IG(\Gamma))$.

Theorem 9. *Let $k \geq 2$ be a natural number, Γ be a finite group, and $T_\Gamma \neq \emptyset$ be a set of generators of Γ. If $rc(IG(\Gamma)) = k$, then every self-invertible element of Γ is a product of r non-self-invertible elements of Γ for some $r \leq k$.*

Proof. Let Γ be a finite group and $T_\Gamma \neq \emptyset$ be a set of generators of Γ. Then the inverse graph $IG(\Gamma)$ is connected. Let $k \geq 2$ be a natural number such that $rc(IG(\Gamma)) = k$. The identity e can be written as $e = t * t^{-1}$ for any $t \in T_\Gamma$. Since every non-identity element $s \in S_\Gamma$ is not adjacent to e and $IG(\Gamma)$ is connected, for every non-identity element $s \in S_\Gamma$, there exists a shortest path between s and e in $IG(\Gamma)$ with length $r \geq 2$. Let the path be $sh_1 \ldots h_{r-1}e$. Since h_{r-1} is adjacent to e, we get $h_{r-1} * e = e * h_{r-1} = h_{r-1} \in T_\Gamma$. For every $i \in \{1, \ldots, r-2\}$, since h_i is adjacent to h_{i+1}, we get $h_i * h_{i+1} = t_{i+1} \in T_\Gamma$ or $h_{i+1} * h_i = t'_{i+1} \in T_\Gamma$. Hence, $h_i = t_{i+1} * h_{i+1}^{-1}$ or $h_i = h_{i+1}^{-1} * t'_{i+1}$ for every $i \in \{1, \ldots, r-2\}$. Since s and h_1 are adjacent, we get $s * h_1 = t_1 \in T_\Gamma$ or $h_1 * s = t'_1 \in T_\Gamma$. Hence, $s = t_1 * h_1^{-1}$ or $s = h_1^{-1} * t'_1$. Recall that if t is in T_Γ, then t^{-1} is also in T_Γ. Thus, s is a product of r elements of T_Γ. Since the length of the shortest path of any pair of vertices in $IG(\Gamma)$ is less than or equal to $diam(IG(\Gamma))$ and $diam(IG(\Gamma)) \leq rc(IG(\Gamma)) = k$, then $r \leq k$. □

If the rainbow connection number of the inverse graph of a finite group is 2, then every self-invertible element of the group can be expressed as a product of two non-self-invertible elements of the group, as shown in the following corollary.

Corollary 2. *For a finite group Γ with a connected $IG(\Gamma)$, if $rc(IG(\Gamma)) = 2$, then every self-invertible element of Γ is a product of two non-self-invertible elements of Γ.*

Proof. Let Γ be a finite group with a connected $IG(\Gamma)$ and $rc(IG(\Gamma)) = 2$. According to Theorem 2, since $IG(\Gamma)$ is connected, T_Γ is a set of generators of Γ. Therefore, every $s \in S_\Gamma$ is a product of at least two elements of T_Γ. Since $rc(IG(\Gamma)) = 2$, according to Theorem 9, every $s \in S_\Gamma$ is a product of at most two elements of T_Γ. Thus, we conclude that every $s \in S_\Gamma$ is a product of two elements of T_Γ. □

For an example of Corollary 2, consider the inverse graph of the alternating group A_4. We have observed that $rc(IG(A_4)) = 2$. It is easy to check that $(1) = (123) * (132)$, $(14)(23) = (243) * (124)$, $(13)(24) = (234) * (123)$, $(12)(34) = (243) * (132)$, where '$*$' is composition of permutations. Therefore, every self-invertible element of A_4 is a product of two non-self-invertible elements of A_4. This result is in accordance with Corollary 2.

4. Conclusions and future work

In this paper, we obtain some conclusions. For a finite group Γ of even order, we get $2 \leq rc(IG(\Gamma)) \leq 4 + m$, with $m = |S''_\Gamma|$, where $S''_\Gamma = \{s \in S_\Gamma | s * t = t^{-1} * s$ for all $t \in T_\Gamma\}$. Moreover, the lower bound is tight. For a direct product of finite groups $\Gamma = \Gamma_1 \times \ldots \times \Gamma_n$, where $n \geq 2$, $|\Gamma_i|$ is even for some $i \in \{1, \ldots, n\}$, and T_{Γ_i} is nonempty and generates Γ_i for every $i \in \{1, \ldots, n\}$, we get $2 \leq rc(IG(\Gamma)) \leq 4$ and the lower bound is tight. If Γ is a finite group of even order and $s * t$ is not self-invertible for all $s \in S_\Gamma$ and all $t \in T_\Gamma$, then $rc(IG(\Gamma)) = 2$. If $rc(IG(\Gamma)) = k$ for a finite group Γ, then every self-invertible element of Γ is a product of r non-self-invertible elements of Γ for some $r \leq k$. If $rc(IG(\Gamma)) = 2$ for a finite group Γ, then every self-invertible element of Γ is a product of two non-self-invertible elements of Γ.

A finite group Γ of even order that satisfies $rc(IG(\Gamma)) = 4 + m$ and a direct product of finite groups $\Gamma_1 \times \cdots \times \Gamma_n$ that satisfies $rc(IG(\Gamma_1 \times \cdots \times \Gamma_n)) = 4$ are still unknown. Therefore, proving that these upper bounds are tight, or finding better upper bounds, remains an unresolved issue.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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