Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-14
https://doi.org/10.22049/cco.2024.29897.2214

CCO
Commun. Comb. Optim.

Research Article

Unbalanced complete bipartite signed graphs ${K_{m,n}}^\sigma$ having m and n as Laplacian eigenvalues with maximum multiplicities

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Received: 19 July 2024; Accepted: 11 October 2024
Published Online: 23 October 2024

Abstract: A signed graph $G^\sigma = (G, \sigma)$ consists of an underlying graph $G = (V, E)$ along with a signature function $\sigma : E \to \{-1, 1\}$. A cycle in a signed graph is termed positive if it contains an even number of negative edges, and negative if it contains an odd number of negative edges. A signed graph is considered balanced if it has no negative cycles; otherwise, it is unbalanced. Let $K_{m,n}$ be a complete bipartite graph on $m+n$ vertices. It is well known that for a balanced complete bipartite signed graph ${K_{m,n}}^\sigma$, the parameters m and n are Laplacian eigenvalues with multiplicities $n-1$ and $m-1$, respectively. This raises a natural question about the maximum multiplicities of Laplacian eigenvalues m and n in an unbalanced complete bipartite signed graph ${K_{m,n}}^\sigma$. In this paper, we demonstrate that the multiplicities of the Laplacian eigenvalues m and n in an unbalanced complete bipartite signed graph ${K_{m,n}}^\sigma$, $m < n$, are at most $n-2$ and $m-2$, respectively. Additionally, we characterise all the signed graphs for which m and n are Laplacian eigenvalues with these maximum multiplicities.

Keywords: signed graph, maximum multiplicity, Laplacian matrix, complete bipartite signed graph, minimum multiplicity.

AMS Subject classification: 05C50, 05C09, 05C92

1. Introduction

All the underlying graphs discussed in this paper are assumed to be simple and finite. A *signed graph* $G^\sigma = (G, \sigma)$ consists of an underlying graph $G = (V(G), E(G))$ along with a signature function $\sigma : E(G) \to \{-1, 1\}$. The graph G is referred to as the underlying graph of G^σ, and the function σ is known as the signature of G^σ. An *induced signed subgraph* $H^{\sigma_H} = (H, \sigma_H)$ of a signed graph $G^{\sigma_G} = (G, \sigma_G)$ is a signed graph in which the underlying graph H is an induced subgraph of G, and σ_H is the restriction of σ_G to $E(H)$. Let G^σ be a signed graph with a sign function σ, and $U \subseteq V(G)$ be a subset of vertices in G. The notation $G^\sigma_{(U)}$ represents the signed subgraph induced by U. The edge uv in a signed graph is called positive (negative) if

uv gets the sign $+1$ (respectively, -1). In a signed graph, a cycle is termed positive if it contains an even number of negative edges, and negative if it contains an odd number of negative edges. A signed graph is considered *balanced* if it has no negative cycles; otherwise, it is *unbalanced*. Unsigned graphs (simply graphs) can be viewed as balanced signed graphs where all edges have a positive sign, known as the all-positive signature.

The *adjacency matrix* of G^σ is defined as $A_{G^\sigma} = (\sigma_{ij})$, where $\sigma_{ij} = \sigma(u_i v_j)$ if there is an edge $u_i v_j$ between the vertices u_i and v_j, and $\sigma_{ij} = 0$ otherwise. The *Laplacian matrix* of G^σ is defined as $L_{G^\sigma} = D_G - A_{G^\sigma}$, where D_G is the diagonal matrix of vertex degrees of the underlying graph of G^σ. For any square matrix M of order n, the polynomial, $\det(xI_n - M)$, where I_n is the identity matrix of order n, will be denoted by $\varphi(M; x)$. In particular, the Laplacian characteristic polynomial, $\det(xI_n - L_{G^\sigma})$, of G^σ will be denoted by $\varphi_{G^\sigma}(x)$. The Laplacian eigenvalues of G^σ are the eigenvalues of L_{G^σ}. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct Laplacian eigenvalues of G^σ and $\mathcal{M}_{G^\sigma}(\lambda_k)$ represents the multiplicity of the Laplacian eigenvalue λ_k, then the Laplacian spectrum of G^σ is expressed as

$$\mathrm{Spec}_L(G^\sigma) = \left\{ \lambda_1^{(\mathcal{M}_{G^\sigma}(\lambda_1))}, \lambda_2^{(\mathcal{M}_{G^\sigma}(\lambda_2))}, \ldots, \lambda_k^{(\mathcal{M}_{G^\sigma}(\lambda_k))} \right\}.$$

For a subset $X \subset V(G)$, define $G^{\sigma_X} = (G, \sigma_X)$ as the signed graph obtained from $G^\sigma = (G, \sigma)$ by reversing the signs of the edges in the cut $[X, V(G) \setminus X]$. This means $\sigma_X(e) = -\sigma(e)$ for any edge e between X and $V(G) \setminus X$, and $\sigma_X(e) = \sigma(e)$ for all other edges. Two signed graphs G^σ and G^{σ_X} are called *switching equivalent*, denoted $G^\sigma \sim G^{\sigma_X}$. It is important to note that switching equivalent signed graphs have similar adjacency and Laplacian matrices. Any switching determined by X can be represented by a diagonal matrix $S = \mathrm{diag}(s_1, s_2, \ldots, s_n)$, where $s_i = 1$ for each $i \in X$, and $s_i = -1$ otherwise. Consequently, we have $A_{G^\sigma} = S A_{G^{\sigma_X}} S^{-1}$ and $L_{G^\sigma} = S L_{G^{\sigma_X}} S^{-1}$, showing that the matrices A_{G^σ} and $A_{G^{\sigma_X}}$, and L_{G^σ} and $L_{G^{\sigma_X}}$ are signature similar, respectively. When studying a signed graph G^σ from an adjacency and Laplacian spectral perspective, we always consider its switching equivalent class.

In a graph G, the neighbourhood of a vertex v, denoted as $N(v)$, comprises the set of vertices adjacent to v. Many well-established graph concepts can be directly applied to signed graphs. For example, a signed graph is *regular* if its underlying graph is regular, and the degree of a vertex v, denoted by $d_{G^\sigma}(v) = |N(v)|$, in G^σ matches its degree in G. However, certain notions are unique to signed graphs. The set of vertices adjacent to a vertex v in G^σ via a negative (positive) edge is called the negative (respectively, positive) neighborhood. The positive degree $d^+_{G^\sigma}(v)$ of a vertex v in G^σ is defined as the number of positive edges incident to v. Conversely, the negative degree $d^-_{G^\sigma}(v)$ is the number of negative edges incident to v. The net-degree of a vertex v in G^σ is defined as the difference between its positive degree and its negative degree, denoted as $d^\pm_{G^\sigma}(v) = d^+_{G^\sigma}(v) - d^-_{G^\sigma}(v)$. A signed graph is classified as net-regular if the net-degree is consistent across all vertices. The rank of an $n \times n$ symmetric matrix M, denoted as rank(M), is $n - m(0)$, where $m(0)$ is the multiplicity of 0 as an eigenvalue of M. As usual, let K_n represents the complete graph on n vertices. $K_{m,n}$ denotes a complete bipartite graph on $m+n$ vertices with the bipartition (U_m, V_n), where $U_m = \{u_1, u_2, \ldots, u_m\}$ and $V_n = \{v_1, v_2, \ldots, v_n\}$. A signed graph G^σ whose negative edges (positive edges) induce a graph H will be denoted by (G, H^-) (respectively, (G, H^+)).

The *index* of a signed graph is the largest eigenvalue of its adjacency matrix. Kafai and Heydari [10] characterised all the signed graphs achieving the maximum index in the class of complete signed graphs ${K_n}^\sigma$, whose negative edges induce a unicyclic graph (a graph having the same number of vertices and edges) with a cycle length of at least 4. Koledin and Stanić [12] conjectured that if ${K_n}^\sigma$ is a complete signed graph with k negative edges, $k < n-1$, and has the maximum index, then the negative edges induce the signed star $K_{1,k}$. Akbari, Dalvandi, Heydari, and Maghasedi [3] proved that the conjecture holds for complete signed graphs whose negative edges form a tree. Very recently, Ghorbani and Majidi [9] completely confirmed the conjecture. Dalvandi, Heydari, and Maghasedi [8] characterized complete signed graphs with exactly m non-negative adjacency eigenvalues. Akbari, Dalvandi, Heydari, and Maghasedi [1] studied the multiplicities of the adjacency eigenvalues -1 and 1 for the complete signed graphs. More results on the spectral theory of complete signed graphs can be found in [2, 5, 7, 11, 15]. The spectral theory of complete bipartite signed graphs is much less explored, see [4, 13]. Motivated by these, in this study, we explore the Laplacian spectral properties of complete bipartite signed graphs.

In the context of the *complete bipartite signed graphs*, it is well-known that [6] for a balanced complete bipartite signed graph ${K_{m,n}}^\sigma$, the parameters m and n are Laplacian eigenvalues with multiplicities $n-1$ and $m-1$, respectively. This raises a natural question about the unbalanced signed graphs. To explore this, we consider the following:

Problem 1.1. For an unbalanced complete bipartite signed graph ${K_{m,n}}^\sigma$, are m and n still Laplacian eigenvalues? What are the maximum multiplicities of the Laplacian eigenvalues m and n?

Problem 1.2. Which unbalanced complete bipartite signed graphs ${K_{m,n}}^\sigma$ have m and n as Laplacian eigenvalues with the maximum multiplicities?

The main objective of this paper is to give the answer to these questions and identify the configurations that yield maximum multiplicities for the Laplacian eigenvalues m and n. Complete bipartite signed graphs have a wide range of applications, particularly in data structures, as discussed in the notable paper [16] and the references therein.

The remainder of the paper is organised as follows: In Section 2, we state some preliminary results that will be used in the sequel. In Section 3, we demonstrate that the multiplicities of the Laplacian eigenvalues m and n in an unbalanced complete bipartite signed graph ${K_{m,n}}^\sigma$ are at most $n-2$ and $m-2$, respectively. Additionally, we characterise all the signed graphs for which m and n are Laplacian eigenvalues with these maximum multiplicities.

2. Preliminaries

Let G^σ be a signed graph consisting of a bipartite underlying graph with the bipartition (U_m, V_n), where $U_m = \{u_1, u_2, \ldots, u_m\}$ and $V_n = \{v_1, v_2, \ldots, v_n\}$. By applying suitable vertex labelling, the Laplacian matrix takes the form

$$L_{G^\sigma} = \begin{pmatrix} D(U_m)_{m\times m} & M_{m\times n} \\ M^\top_{n\times m} & D(V_n)_{n\times n} \end{pmatrix},$$

where $D(U_m)_{m\times m}$ represents the diagonal matrix of vertex degrees for vertices in U_m, $D(V_n)_{n\times n}$ represents the diagonal matrix of vertex degrees for vertices in V_n, and $M_{m\times n} = (m_{ij})$ is an $m \times n$ matrix defined as

$$m_{ij} = \begin{cases} -\sigma(u_i v_j) & \text{if there is an edge } u_i v_j \text{ between vertices } u_i \in U_m \text{ and } v_j \in V_n \text{ of } G^\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\sigma(u_i v_j)$ denotes the sign associated with the edge between vertices u_i and v_j. The matrix $M_{m\times n}$ is known as the Laplacian block incidence matrix of the signed graph G^σ.
The subsequent result provides an easy method for determining the characteristic polynomial of the Laplacian matrix of a complete bipartite signed graph.

Lemma 1. (Shamsher [14]) *Let ${K_{m,n}}^\sigma$ be a complete bipartite signed graph on $m+n$ vertices, and let $B_{p\times q}$ be the Laplacian block incidence matrix of the induced signed subgraph ${K_{m,n}}^\sigma_{(U_p\cup V_q)}$, which contains the minimum number of vertices, including all the negative edges of ${K_{m,n}}^\sigma$. Then, we have the following:*

(i) *If $p \leq q$, then the Laplacian characteristic polynomial of ${K_{m,n}}^\sigma$ is given by*

$$\varphi_{{K_{m,n}}^\sigma}(x) = (x-m)^{n-p-1}(x-n)^{m-p-1}\varphi\left(\begin{pmatrix} B_{p\times q}B^\top_{q\times p} + (n-q)J_{p\times p} & C \\ C^\top & n(m-p) \end{pmatrix}; (x-m)(x-n)\right), \tag{2.1}$$

where $C = \sqrt{m-p}(D^\pm(U_p) + (n-q)J_{p\times 1})$, $J_{r\times s}$ is a matrix of order $r \times s$ with each entry equal to 1, and $D^\pm(U_p)$ is the column vector of the net vertex degrees for vertices in U_p.

(ii) *If $q \leq p$, then the Laplacian characteristic polynomial of ${K_{m,n}}^\sigma$ is given by*

$$\varphi_{{K_{m,n}}^\sigma}(x) = (x-m)^{n-q-1}(x-n)^{m-q-1}\varphi\left(\begin{pmatrix} B^\top_{q\times p}B_{p\times q} + (m-p)J_{q\times q} & E \\ E^\top & m(n-q) \end{pmatrix}; (x-m)(x-n)\right), \tag{2.2}$$

where $E = \sqrt{n-q}(D^\pm(V_q) + (m-p)J_{q\times 1})$ and $D^\pm(V_q)$ is the column vector of the net vertex degrees for vertices in V_p.

Let M be a square matrix of order n, written in block form as:

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,s-1} & M_{1,s} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,s-1} & M_{2,s} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{s-1,1} & M_{s-1,2} & \cdots & M_{s-1,s-1} & M_{s-1,s} \\ M_{s,1} & M_{s,2} & \cdots & M_{s,s-1} & M_{s,s} \end{pmatrix}. \quad (2.3)$$

The quotient matrix $Q = (q_{ij})_{s\times s}$ is a square matrix of order s, where the (i,j)-th entry of Q is the average row sum of the block M_{ij} in M. If each block M_{ij} of M has a constant row sum, then Q is called the equitable quotient matrix.
The following theorem describes the relationship between the eigenvalues of M and those of Q.

Lemma 2. (Brouwer and Haemers [6]) *Let M be a real symmetric matrix of order n in the form given in (2.3), and let Q be its quotient matrix of order s (with $n > s$). Then the eigenvalues of Q are included in the spectrum of M.*

3. Unbalanced complete bipartite signed graphs ${K_{m,n}}^{\sigma}$ having m and n as Laplacian eigenvalues with maximum multiplicities

In this section, we provide an upper bound for the multiplicities of Laplacian eigenvalues m and n in an unbalanced complete bipartite signed graph $K^{\sigma}_{m,n}$ and characterise all the signed graphs containing m and n as Laplacian eigenvalues with these maximum multiplicities. Before that, we construct two families of graphs that will be utilized in the main theorem:
Let K_{m_1,n_1} be a complete bipartite graph with the bipartition $U_{m_1} \cup V_{n_1}$. From this complete bipartite graph, we construct the new bipartite graphs of the class $\mathcal{G}_{K_{m_1,n_1}}(m_2, m_3; n_2)$; with the bipartition $U \cup V$, where $U = U_{m_1} \cup U_{m_2} \cup U_{m_3}$ and $V = V_{n_1} \cup V_{n_2}$. That is by adding two vertex sets U_{m_2} and U_{m_3} to the vertex set U_{m_1}, and a vertex set V_{n_2} to the vertex set V_{n_1}. In the resulting graph, we have the following properties:
i) $d(u) = n_1$ for each vertex $u \in U_{m_1}$,
ii) $N(u) = N(u')$ for each pair of vertices $u, u' \in U_{m_2}$,
iii) $N(u) = N(u')$ for each pair of vertices $u, u' \in U_{m_3}$,
iv) $N(u) \cap N(u') = \emptyset$ for each pair of vertices $u \in U_{m_2}$ and $u' \in U_{m_3}$,
v) $N(u) \cup N(u') = V_{n_1} \cup V_{n_2}$ for each pair of vertices $u \in U_{m_2}$ and $u' \in U_{m_3}$.
Note that m_2, m_3, and n_2 are non-negative integers and $m_2 + m_3 > 0$. Moreover, a graph of type $\mathcal{G}_{K_{m_1,n_1}}(m_2, m_3; n_2)$ is not uniquely determined by the parameters only. If the neighbours of any vertex u, $u \in U_{m_2}$, or v, $v \in U_{m_3}$, are known, then it is uniquely determined by the parameters. A graph of type $\mathcal{G}_{K_{2,3}}(2, 2; 2)$ is shown in Figure 1.

Let $K_{m_1,n_1} \cup K_{m_2,n_2}$ be a disjoint union of two complete bipartite graphs with the

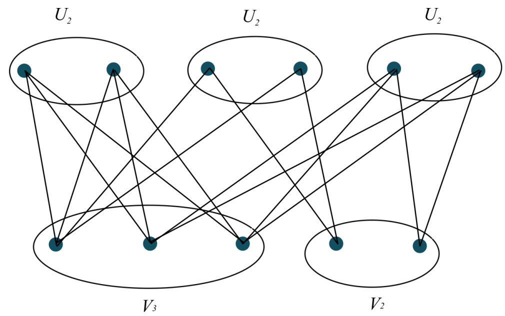

Figure 1. **A graph of type** $\mathcal{G}_{K_{2,3}}(2,2;2)$

partition $U \cup V$, where $U = U_{m_1} \cup U_{m_2}$ and $V = V_{n_1} \cup V_{n_2}$. From this bipartite graph, we construct the new bipartite graphs of the class $\mathcal{G}_{K_{m_1,n_1} \cup K_{m_2,n_2}}(m_3, m_4)$; with the bipartition $U' \cup V'$, where $U' = U_{m_1} \cup U_{m_2} \cup U_{m_3} \cup U_{m_4}$ and $V' = V_{n_1} \cup V_{n_2}$. That is, by adding two vertex sets U_{m_3} and U_{m_4} to the vertex set $U_{m_1} \cup U_{m_2}$. In the resulting graph, we have the following properties:
i) $d(u) = n_1$ for each vertex $u \in U_{m_1}$,
ii) $d(u) = n_2$ for each vertex $u \in U_{m_2}$,
iii) $N(u) = N(u')$ for each pair of vertices $u, u' \in U_{m_3}$,
iv) $N(u) = N(u')$ for each pair of vertices $u, u' \in U_{m_4}$,
v) $N(u) \cap N(u') = \emptyset$ for each pair of vertices $u \in U_{m_3}$ and $u' \in U_{m_4}$,
vi) $N(u) \cup N(u') = V_{n_1} \cup V_{n_2}$ for each pair of vertices $u \in U_{m_3}$ and $u' \in U_{m_4}$.
Note that m_3 and m_4 are non-negative integers and $m_3 + m_4 > 0$. Moreover, a graph in $\mathcal{G}_{K_{m_1,n_1} \cup K_{m_2,n_2}}(m_3, m_4)$ is not uniquely determined by the parameters only. If the neighbours of any vertex u, $u \in U_{m_3}$, or v, $v \in U_{m_4}$, are known then it is uniquely determined by the parameters. An example of a graph in $\mathcal{G}_{K_{2,3} \cup K_{2,2}}(2,2)$ is shown in Figure 2.
The following two results are essential in establishing the main conclusion.

Lemma 3. *For positive integers m_1, m_2, n_1, and n_2, the following holds:*

i) *If $m = m_1 + m_2$, then the signed graph $\left(K_{m,n}, (K_{m_1,n_1} \cup K_{m_2,n_2})^-\right)$ is switching equivalent to $\left(K_{m,n}, K^-_{m_1,n_1+n_2}\right)$.*

ii) *If $n = n_1 + n_2$, then the signed graph $\left(K_{m,n}, (K_{m_1,n_1} \cup K_{m_2,n_2})^-\right)$ is switching equivalent to $\left(K_{m,n}, K^-_{m_1+m_2,n_1}\right)$.*

Proof. Let $U_{m_1} \cup V_{n_1}$ and $U_{m_2} \cup V_{n_2}$ be the bipartitions of K_{m_1,n_1} and K_{m_2,n_2} respectively.
i) The signed graph $\left(K_{m,n}, K^-_{m_1,n_1+n_2}\right)$ can be obtained from $\Big(K_{m,n}, (K_{m_1,n_1} \cup K_{m_2,n_2})^-\Big)$ by performing a switching operation determined by the vertex set V_{n_2}.

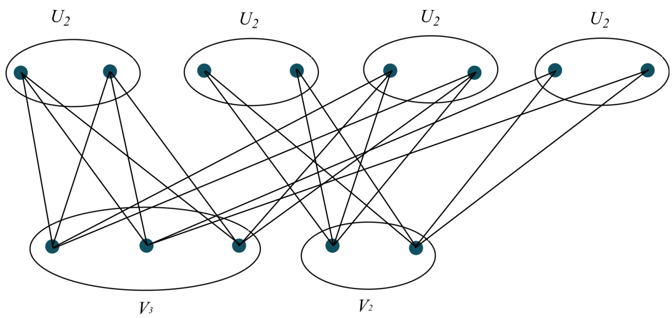

Figure 2. **A graph of type** $\mathcal{G}_{K_{2,3}\cup K_{2,2}}(2,2)$

ii) Similarly, the signed graph $\left(K_{m,n}, K^{-}_{m_1+m_2,n_1}\right)$ can be derived from $\left(K_{m,n}, (K_{m_1,n_1} \cup K_{m_2,n_2})^{-}\right)$ by applying a switching operation determined by the vertex set U_{m_2}. □

Lemma 4. *For positive integers m_1, m_2, m_3, m_4, n_1, and n_2, the following holds:*

i) *Let $\Gamma = (K_{m,n}, \Lambda^{-})$ be a signed graph whose negative edges form the bipartite graph $\Lambda \in \mathcal{G}_{K_{m_1,n_1}}(m_2, m_3; n_2)$ with $m_1 + m_2 + m_3 = m$ and $n_1 + n_2 = n$. Then, Γ is switching equivalent to $\left(K_{m,n}, K^{-}_{m_1,k_1+k_2}\right)$, where*

$$k_1 := \left|N^{-}(U_{m_3}) \cap V_{n_1}\right|, \quad k_2 := \left|N^{-}(U_{m_2}) \cap V_{n_2}\right|,$$

and $N^{-}(U_a)$ is the set of vertices connected to elements in U_a with a negative edge.

ii) *Let $\Gamma = (K_{m,n}, \Lambda^{-})$ be a signed graph whose negative edges form the bipartite graph $\Lambda \in (K_{m,n}, \mathcal{G}_{K_{m_1,n_1}\cup K_{m_2,n_2}}(m_3, m_4)^{-})$ with $m_1 + m_2 + m_3 + m_4 = m$ and $n_1 + n_2 = n$. Then, Γ is switching equivalent to $\left(K_{m,n}, (K_{m_1,k_1+k_2} \cup K_{m_2,k_3+k_4})^{-}\right)$, where*

$$k_1 := \left|N^{-}(U_{m_4}) \cap V_{n_1}\right|, \quad k_2 := \left|N^{-}(U_{m_3}) \cap V_{n_2}\right|, k_3 := \left|N^{-}(U_{m_3}) \cap V_{n_1}\right|, \quad k_4 := \left|N^{-}(U_{m_4}) \cap V_{n_2}\right|,$$

and $N^{-}(U_a)$ is the set of vertices connected to elements in U_a with a negative edge.

Proof. i) Let $\Gamma = (K_{m,n}, \Lambda^{-})$, $\Lambda \in \mathcal{G}_{K_{m_1,n_1}}(m_2, m_3; n_2)$, be a signed graph with the bipartition $U \cup V$, where $U = U_{m_1} \cup U_{m_2} \cup U_{m_3}$ and $V = V_{n_1} \cup V_{n_2}$. Let $k_1 := |N^{-}(U_{m_3}) \cap V_{n_1}|$ and $k_2 := |N^{-}(U_{m_2}) \cap V_{n_2}|$, where $N^{-}(U_a)$ is the set of vertices connected to elements in U_a with a negative edge. The signed graph $\left(K_{m,n}, K^{-}_{m_1,k_1+k_2}\right)$ can be obtained from $(K_{m,n}, \Lambda^{-})$ by performing a switching operation determined by the vertex set $U_{m_3} \cup N^{-}(U_{m_2})$.
ii) Let $\Gamma = (K_{m,n}, \Lambda^{-})$, $\Lambda \in \mathcal{G}_{K_{m_1,n_1}\cup K_{m_2,n_2}}(m_3, m_4)$, be a signed graph with the bipartition $U \cup V$, where $U = U_{m_1} \cup U_{m_2} \cup U_{m_3} \cup U_{m_4}$ and $V = V_{n_1} \cup V_{n_2}$.

Let $k_1 := |N^-(U_{m_4}) \cap V_{n_1}|, k_2 := |N^-(U_{m_3}) \cap V_{n_2}|, k_3 := |N^-(U_{m_3}) \cap V_{n_1}|$, and $k_4 := |N^-(U_{m_4}) \cap V_{n_2}|$. The signed graph $(K_{m,n}, (K_{m_1,k_1+k_2} \cup K_{m_2,k_3+k_4})^-)$ can be obtained from $(K_{m,n}, \Lambda^-)$ by performing a switching operation determined by the vertex set $U_{m_4} \cup N^-(U_{m_3})$. □

The following constitutes the main result of the paper, establishing upper bounds for the multiplicities of the Laplacian eigenvalues m and n in an unbalanced complete bipartite signed graph ${K_{m,n}}^\sigma$. Additionally, it characterises all the signed graphs for which m and n are Laplacian eigenvalues with maximum multiplicities.

Theorem 1. *Let ${K_{m,n}}^\sigma$, $m < n$, be an unbalanced complete bipartite signed graph on $m+n$ vertices. Then*

i) $\mathcal{M}_{{K_{m,n}}^\sigma}(m) \leq n-2$. *The equality holds if and only if ${K_{m,n}}^\sigma$ is switching equivalent to $(K_{m,n}, K^-_{m_1,n_1})$, where $m_1+n_1 < m+n$.*

ii) $\mathcal{M}_{{K_{m,n}}^\sigma}(n) \leq m-2$. *The equality holds if and only if ${K_{m,n}}^\sigma$ is switching equivalent to $(K_{m,n}, K^-_{m_1,n_1})$ where $m_1+n_1 < m+n$.*

Proof. i) Let ${K_{m,n}}^\sigma$ be an unbalanced complete bipartite signed graph on $m+n$ vertices. Let $B_{p\times q}$ be the Laplacian block incidence matrix of the induced signed subgraph ${K_{m,n}}^\sigma_{(U_p\cup V_q)}$, which includes all the negative edges of ${K_{m,n}}^\sigma$ with the minimum number of vertices. We show that $\mathcal{M}_{{K_{m,n}}^\sigma}(m) \leq n-2$ by analysing two cases:

Case 1. If $p \leq q$, then by Lemma 1 (i), the Laplacian characteristic polynomial of ${K_{m,n}}^\sigma$ is given by

$$\varphi_{{K_{m,n}}^\sigma}(x) = (x-m)^{n-p-1}(x-n)^{m-p-1}\varphi\left(\begin{pmatrix} B_{p\times q}B^\top_{q\times p} + (n-q)J_{p\times p} & C \\ C^\top & n(m-p)\end{pmatrix}; (x-m)(x-n)\right), \tag{3.1}$$

where $C = \sqrt{m-p}(D^\pm(U_p) + (n-q)J_{p\times 1})$ and $D^\pm(U_p)$ represents the column vector of net vertex degrees for vertices in U_p. We know that $x-m$ is a factor of the polynomial $\varphi(M; (x-m)(x-n))$ if and only if 0 is an eigenvalue of the matrix M. Therefore, by the Rank-Nullity Theorem and Equation (3.1), $\mathcal{M}_{{K_{m,n}}^\sigma}(m) \leq n-2$ if and only if

$$\operatorname{rank}(Q) \geq 2, \tag{3.2}$$

where

$$Q = \begin{pmatrix} B_{p\times q}B^\top_{q\times p} + (n-q)J_{p\times p} & C \\ C^\top & n(m-p)\end{pmatrix}.$$

Now, the following two subcases arise:

Subcase 1.1. Let $p \neq m$. If $D^\pm(U_p) = -nJ_{p\times 1}$, then it is evident that ${K_{m,n}}^\sigma_{(U_p\cup V_q)} = (K_{p,q}, -)$ and $q = n$. Clearly, the signed graph ${K_{m,n}}^\sigma$ is switching equivalent to a signed graph with all positive signature, which contradicts the fact that ${K_{m,n}}^\sigma$ is an unbalanced signed graph. Hence, there must exist at least one vertex of U_p in ${K_{m,n}}^\sigma_{(U_p\cup V_q)}$ where at least one positive and one negative edge are incident. Without loss of generality, assume that vertex to be u_p. The submatrix obtained by deleting the first $p-1$ rows and columns of Q, resulting in

$$R = \begin{pmatrix} n & \sqrt{m-p}(r_p + (n-q)) \\ \sqrt{m-p}(r_p + (n-q)) & n(m-p) \end{pmatrix},$$

where r_p is the net vertex degree of the vertex u_p in U_p. Given that at least one negative and one positive edge are incident on the vertex $u_p \in U_p$ in ${K_{m,n}}^\sigma_{(U_p\cup V_q)}$, it follows that $r_p < q$. Thus, the determinant $(m-p)(n^2 - (r_p + n - q)^2)$ of the matrix R is nonzero. Consequently,

$$\text{rank}(Q) \geq \text{rank}(R) = 2.$$

This establishes that the rank of Q is at least 2, thereby proving the result for this subcase.

Subcase 1.2. If $p = m$, then it is sufficient to show the rank of the matrix $B_{p\times q}B^\top_{q\times p} + (n-q)J_{p\times p}$ is at least 2. Suppose to the contrary that $\text{rank}(B_{p\times q}B^\top_{q\times p} + (n-q)J_{p\times p}) = 1$. If $q = n$, then it is clear that $\text{rank}(B_{p\times q}B^\top_{q\times p} + (n-q)J_{p\times p}) = \text{rank}(B_{p\times q}B^\top_{q\times p}) = \text{rank}(B_{p\times q})$. As the entries of the matrix $B_{p\times q}$ are either $+1$ or -1 and ${K_{m,n}}^\sigma$ is an unbalanced signed graph. Therefore, $\text{rank}(B_{p\times q}) = 1$ if and only if the matrix $B_{p\times q}$ is permutationally similar to one of the matrices, $J_{p\times q}$ or

$$\begin{pmatrix} J_{m_1\times n_1} & -J_{m_1\times n_2} \\ -J_{m_2\times n_1} & J_{m_2\times n_2} \end{pmatrix},$$

where $m_1 + m_2 = p$ and $n_1 + n_2 = q$. The signed graphs corresponding to these matrices are ${K_{m,n}}^\sigma_{(U_p\cup V_q)} \sim (K_{p,q}, -)$ or ${K_{m,n}}^\sigma_{(U_p\cup V_q)} \sim (K_{p,q}, (K_{m_1,n_1} \cup K_{m_2,n_2})^-)$. In these cases, the signed graph ${K_{m,n}}^\sigma$ is switching equivalent to a balanced signed graph. Which is a contradiction. If $q \neq n$ and ${K_{m,n}}^\sigma_{(U_p\cup V_q)}$ is not switching equivalent to $(K_{p,q}, -)$, then one can easily see that $\text{rank}(B_{p\times q}B^\top_{q\times p} + (n-q)J_{p\times p}) = \text{rank}(B_{p\times q}B^\top_{q\times p}) + 1 = \text{rank}(B_{p\times q}) + 1$. As the entries of the matrix $B_{p\times q}$ are either $+1$ or -1, therefore, $\text{rank}(B_{p\times q}) + 1 = 1$, leads to a contradiction. Hence, the result is proved in this subcase.

Case 2. If $q \leq p$, then the Laplacian characteristic polynomial of ${K_{m,n}}^\sigma$ is given by

$$\varphi_{{K_{m,n}}^\sigma}(x) = (x-m)^{n-q-1}(x-n)^{m-q-1}\varphi\left(\begin{pmatrix} B^\top_{q\times p}B_{p\times q} + (m-p)J_{q\times q} & E \\ E^\top & m(n-q) \end{pmatrix}; (x-m)(x-n)\right), \tag{3.3}$$

where $E = \sqrt{n-q}(D^\pm(V_q) + (m-p)J_{q\times 1})$ and $D^\pm(V_q)$ is the column vector

of the net vertex degrees for vertices in V_q. Following a similar approach as in Case 1, we arrive at the desired conclusion for this case.
This proves that $\mathcal{M}_{K_{m,n}{}^{\sigma}}(m) \leq n-2$.

For equality, consider that $\mathcal{M}_{K_{m,n}{}^{\sigma}}(m) = n-2$. Now, we have the following two cases:

Case 3. Let $p \neq m$ or $q \neq n$. Assume, without loss of generality, that $p \neq m$ and $p \leq q$. As $\mathcal{M}_{K_{m,n}{}^{\sigma}}(m) = n-2$, therefore, the Equation (3.2) turns into equality. That is

$$\operatorname{rank}\left(\begin{pmatrix} B_{p\times q}B^{\top}_{q\times p} + (n-q)J_{p\times p} & C \\ C^{\top} & n(m-p) \end{pmatrix}\right) = 2$$

If $n = q$, then it is easy to see that

$$\begin{aligned}\operatorname{rank}\left(\begin{pmatrix} B_{p\times q}B^{\top}_{q\times p} + (n-q)J_{p\times p} & C \\ C^{\top} & n(m-p) \end{pmatrix}\right) &= \operatorname{rank}(B_{p\times q}B^{\top}_{q\times p} + (n-q)J_{p\times p}) + 1 \\ &= \operatorname{rank}(B_{p\times q}B^{\top}_{q\times p}) + 1 \\ &= \operatorname{rank}(B_{p\times q}) + 1.\end{aligned}$$

Thus, we get $\operatorname{rank}(B_{p\times q}) = 1$. Since, the entries of the matrix $B_{p\times q}$ are either $+1$ or -1, and the signed graph $K_{m,n}{}^{\sigma}$ is unbalanced, therefore, $K_{m,n}{}^{\sigma}$ is switching equivalent to $(K_{m,n}, K^{-}_{m_1,n_1})$, $m_1+n_1 < m+n$, or $(K_{m,n}, (K_{m_1,n_1} \cup K_{m_2,n_2})^{-})$, $m_1 + m_2 = m$ or $n_1 + n_2 = n$, and $m_1 + m_2 + n_1 + n_2 < m + n$. By lemma 3, $(K_{m,n}, (K_{m_1,n_1} \cup K_{m_2,n_2})^{-})$, $m_1 + m_2 = m$ or $n_1 + n_2 = n$, and $m_1+m_2+n_1+n_2 < m+n$, is switching equivalent to $\big(K_{m,n}, K^{-}_{m_1,n_1+n_2}\big)$ or $\big(K_{m,n}, K^{-}_{m_1+m_2,n_1}\big)$. Thus, $K_{m,n}{}^{\sigma}$ is switching equivalent to $(K_{m,n}, K^{-}_{m'_1,n'_1})$, $m'_1+n'_1 < m+n$, for suitable m'_1 and n'_1 in this case. If $n \neq q$, then it can again be seen that $K_{m,n}{}^{\sigma}$ is switching equivalent to $(K_{m,n}, K^{-}_{m'_1,n'_1})$, $m'_1+n'_1 < m+n$.

Case 4. Let $p = m$ and $q = n$. By Equation (3.2) and Lemma 1, we have

$$\operatorname{rank}(B_{p\times q}B^{\top}_{q\times p} + (n-q)J_{p\times p}) = 2.$$

Which further implies that

$$\operatorname{rank}(B_{p\times q}) = 2.$$

Let $X = (\underbrace{1, 1, \ldots, 1}_{n_1}, \underbrace{-1, -1, \ldots, -1}_{n_2})$ be a vector in $\mathbb{R}^q$ and Y, $Y \in \mathbb{R}^q$, be a vector with entries $+1$ and -1 which is linearly independent to X. Then, the matrix $B_{p\times q}$ is permutationally similar to one of the matrices $[\underbrace{X\ X\ \cdots\ X}_{m_1}\ \ \mathbf{1}^{\top}\ \ \mathbf{1}^{\top}\ \cdots\ \mathbf{1}^{\top}]^{\top}$

$(Y = \mathbf{1}^\top)$, $[\underbrace{X\ X\ \cdots\ X}_{m_1}\ \underbrace{-X\ -X\ \cdots\ -X}_{m_2}\ \mathbf{1}^\top\ \mathbf{1}^\top\ \cdots\ \mathbf{1}^\top]^\top$, $[\underbrace{X\ X\ \cdots\ X}_{m_1}\ \underbrace{Y\ Y\ \cdots\ Y}_{m_2}\ \underbrace{-Y\ -Y\ \cdots\ -Y}_{m_3}]^\top$, or $[\underbrace{X\ X\ \cdots\ X}_{m_1}\ \underbrace{-X\ -X\ \cdots\ -X}_{m_2}\ \underbrace{Y\ Y\ \cdots\ Y}_{m_3}\ \underbrace{-Y\ -Y\ \cdots\ -Y}_{m_4}]^\top$, where $\mathbf{1}$ is a column vector of appropriate size with each entry equal to 1. The signed graphs corresponding to these matrices are $(K_{m,n}, K^+_{m_1,n_2})$, $(K_{m,n}, (K_{m_1,n_2} \cup K_{m_2,n_1})^+)$, $(K_{m,n}, \Lambda^-)$ $(\Lambda \in G_{K_{m_1,n_2}}(m_2, m_3; n_1))$, or $(K_{m,n}, \Lambda^-)$ $(\Lambda \in G_{K_{m_1,n_1} \cup K_{m_2,n_2}}(m_3, m_4))$, respectively. It is easy to see that the signed graphs $(K_{m,n}, K^+_{m_1,n_2})$ and $(K_{m,n}, (K_{m_1,n_2} \cup K_{m_2,n_1})^+)$ are switching equivalent to $(K_{m,n}, K^-_{m_1,n_1})$ and $(K_{m,n}, (K_{m_1,n_1} \cup K_{m_2,n_2})^-)$, respectively. Hence, considering Lemmas 3 and 4, we obtain ${K_{m,n}}^\sigma$ is switching equivalent to $(K_{m,n}, K^-_{m_1',n_1'})$ for suitable m_1' and n_1' with $m_1' + n_1' < m + n$.

This demonstrates that if $\mathcal{M}_{{K_{m,n}}^\sigma}(m) = n - 2$, then ${K_{m,n}}^\sigma$ is switching equivalent to $(K_{m,n}, K^-_{m_1,n_1})$ with $m_1 + n_1 < m + n$.

Conversely, if ${K_{m,n}}^\sigma$ is switching equivalent to $(K_{m,n}, K^-_{m_1,n_1})$ with $m_1 + n_1 < m + n$. Then, we show that m is a Laplacian eigenvalue of ${K_{m,n}}^\sigma$ with multiplicity $n - 2$. Clearly, ${K_{m,n}}^\sigma_{(U_p \cup V_q)} \sim (K_{m_1,n_1}, -)$. The Laplacian block incidence matrix of ${K_{m,n}}^\sigma_{(U_p \cup V_q)}$ is given by $B_{p\times q} = J_{m_1 \times n_1}$. By Lemma 1 (i), the Laplacian characteristic polynomial of ${K_{m,n}}^\sigma$ is given by

$$\varphi_{{K_{m,n}}^\sigma}(x) = (x-m)^{n-m_1-1}(x-n)^{m-m_1-1}\varphi\left(\begin{pmatrix} nJ_{m_1\times m_1} & C \\ C^\top & n(m-m_1) \end{pmatrix}; (x-m)(x-n)\right),$$

where $C = \sqrt{m - m_1}(n - 2n_1)J_{m_1 \times 1}$. Nothing that $(x - m)(x - n) - \gamma$ is a factor of $\varphi(M; (x - m)(x - n))$ whenever γ is an eigenvalue of M, it is sufficient to show that 0 is an eigenvalue of the matrix

$$\begin{pmatrix} nJ_{m_1\times m_1} & \sqrt{m-m_1}(n-2n_1)J_{m_1\times 1} \\ \sqrt{m-m_1}(n-2n_1)J_{1\times m_1} & n(m-m_1) \end{pmatrix}$$

with multiplicity $m_1 - 1$. Let $\{X_1, X_2, \ldots, X_p\}$ is an orthogonal basis of $\mathbb{R}^p$ with $X_1 = \mathbf{1}_p$. Then, we have

$$\begin{pmatrix} nJ_{m_1\times m_1} & \sqrt{m-m_1}(n-2n_1)J_{m_1\times 1} \\ \sqrt{m-m_1}(n-2n_1)J_{1\times m_1} & n(m-m_1) \end{pmatrix}\begin{pmatrix} X_i \\ 0 \end{pmatrix} = 0\begin{pmatrix} X_i \\ 0 \end{pmatrix}, \quad i = 2, \ldots, p.$$

Therefore, 0 for $i = 2, \ldots, m_1$, is an eigenvalue of the given matrix with multiplicity $m_1 - 1$. The remaining two eigenvalues are given by the equitable quotient matrix (by Lemma 2)

$$Q = \begin{pmatrix} nm_1 & \sqrt{m-m_1}(n-2n_1) \\ \sqrt{m-m_1}(n-2n_1)m_1 & n(m-m_1) \end{pmatrix}.$$

As $m_1+n_1<m+n$, the determinant of the matrix Q is non-zero. Thereby, proving the result.
ii) The proof is similar to i). □

The proof of the following result is similar to Theorem 1.

Theorem 2. *Let ${K_{m,m}}^{\sigma}$ be an unbalanced complete bipartite signed graph on $2m$ vertices. Then*

$$\mathcal{M}_{{K_{m,m}}^{\sigma}}(m) \leq 2(m-2).$$

The equality holds if and only if ${K_{m,n}}^{\sigma}$ is switching equivalent to $(K_{m,n}, K^{-}_{m_1,n_1})$, where $m_1+n_1<2m$.

Conclusion. In this paper, we establish an upper bound for the multiplicities of Laplacian eigenvalues m and n in an unbalanced complete bipartite signed graph $K^{\sigma}_{m,n}$. Furthermore, we characterize all the signed graphs for which m and n are Laplacian eigenvalues with their respective maximum multiplicities. As demonstrated in Lemma 1, the lower bound for the multiplicities of Laplacian eigenvalues m and n in $K^{\sigma}_{m,n}$ are at least $n-\min(p,q)-1$ and $m-\min(p,q)-1$, respectively. Consider the complete bipartite signed graph $K^{\sigma}_{4,6}$ depicted in Figure 3. Here, the dashed lines show the negative edges, and bold lines show the positive edges. The Laplacian spectrum of $K^{\cdot}_{4,6}$ is given by $\left\{2^{(1)}, 4^{(3)}, 6^{(1)}, 8^{(1)}, 5\pm\sqrt{7+2\sqrt{3}}^{(1)}, 5\pm\sqrt{7-2\sqrt{3}}^{(1)}\right\}$. Thus, it indicates that the provided lower bounds are optimal. Consequently, the following problem remains open.

Problem. To characterise all complete bipartite signed graphs for which m and n are Laplacian eigenvalues with the minimum multiplicities.

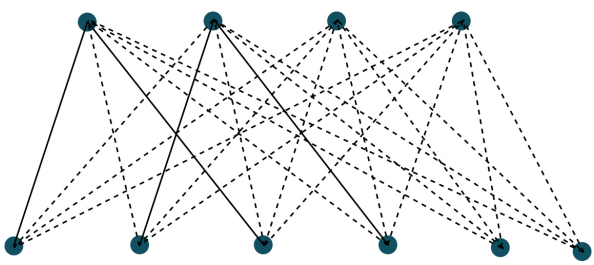

Figure 3. **Signed graph** $K^{\cdot}_{4,6}$

Acknowledgements: The author is thankful to the anonymous referee for a careful reading of the article and the encouraging comments made in the report. The corresponding author is thankful to IIT Bhubaneswar, India for providing the post doctoral fellowship (Grant No: F.15-12/2021-Acad/SBS-PDF-01).

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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