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Research Article

The extended irregular domination problem

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Abstract: In this paper we introduce a new domination problem strongly related to the following one recently proposed by Broe, Chartrand and Zhang. One says that a vertex v of a graph Γ labeled with an integer ℓ dominates the vertices of Γ having distance ℓ from v. An irregular dominating set of a given graph Γ is a set S of vertices of Γ, having distinct positive labels, whose elements dominate every vertex of Γ. Since it has been proven that no connected vertex-transitive graph admits an irregular dominating set, here we introduce the concept of an *extended* irregular dominating set, where we admit that precisely one vertex, labeled with 0, dominates itself. Then we present existence or non existence results of an extended irregular dominating set S for several classes of graphs, focusing in particular on the case in which S is as small as possible. We also propose two conjectures.

Keywords: dominating set, vertex-transitive graph, starter.

AMS Subject classification: 05C69, 05C78, 05B99

1. Introduction

The concept of domination plays an important role in graph theory, having a large variety of applications and being closely related to other topics in graphs, such as, just to make an example, independent sets. This area began with the work of Berge [2] and Ore [19], but it became active only 15 years later thanks to the survey by Cockayne and Hedetniemi [8].
In the following, by Γ we denote an undirected simple graph with vertex set V and edge set E. A set $S \subseteq V$ is said to be a *dominating set* if for every $u \in V$ there exists

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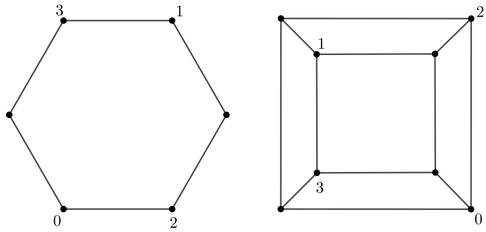
a vertex $v \in S$ such that $\{u, v\} \in E$. Over the course of the years, various kinds of domination have been introduced and investigated, see [12]. In this work, we propose a new variation of a domination problem first considered in [5] and further studied in [1, 4, 6, 16, 17], that we report in what follows.
A vertex v of Γ, labeled with a positive integer ℓ, is said to *dominate*, or *cover*, the vertices of Γ having distance ℓ from v. Similarly, a set $S \subseteq V$ of labeled vertices is said to *dominate* (or *cover*) a vertex $u \in V$ if there exists a vertex $v \in S$ covering u. Then, an *irregular dominating set* of Γ is a set S of vertices of Γ having distinct positive labels that covers every vertex of V. Note that the definition does not require that the labelings are consecutive integers. As it is standard in the topic of domination, it is interesting to determine the minimum value of k such that an irregular dominating set of cardinality k exists, such a value is denoted by $\gamma_i(\Gamma)$ and it is called the *irregular domination number* of Γ.
By a counting argument, in [6] it has been proven that no connected vertex-transitive graph admits an irregular dominating set. However, due to the many symmetries and desirable properties of vertex-transitive graphs, it is natural to wonder if there exists a labeling of the vertices of such a graph, different from the standard one, so that it is possible to obtain something similar to an irregular dominating set. Here, we propose the concept of an extended irregular dominating set of a graph, where we admit that precisely one vertex, labeled with 0, dominates itself. More formally, we give the following new definition.

Definition 1. Let $\Gamma = (V, E)$ be an undirected graph and let $k \geq 0$. A *k-extended irregular dominating set* $S \subseteq V$ is a set of k vertices having distinct non-negative labelings that covers every vertex of V. A labeling λ realizing the covering property for S is said to be a *k-extended irregular dominating labeling.*

Clearly, if $\lambda(v) \neq 0$ for every $v \in S$ then we find again the concept of an irregular dominating set/labeling, hence in the following we always assume that there exists a vertex $v \in S$ labeled by 0 which dominates itself.
We have chosen the term "*extended* irregular dominating labeling", since the connection between these and classical irregular dominating labelings is very similar to the one between extended and classical Skolem sequences, see [20] for further details.
We call the minimum cardinality of an extended irregular dominating set of Γ, denoted by $\gamma_e(\Gamma)$, the *extended irregular domination number* of Γ.

Example 1. Here we show an extended irregular dominating labeling inducing an extended irregular dominating set for the cycle of length 6 and for the cube:



We underline that a graph does not necessarily admit an extended irregular dominating labeling, take for example the cycle of length three. Clearly, if such a labeling exists, the most interesting problem is that of determining a k-extended irregular dominating labeling for a given graph Γ with k as small as possible, or in other words that of determining $\gamma_e(\Gamma)$. We say that a k-extended irregular dominating set (labeling, respectively) is *optimal* if $k = \gamma_e(\Gamma)$, that is if there is no k'-extended irregular dominating set (labeling, respectively) with $k' < k$. One can easily check that the labelings of Example 1 are optimal.
The paper is organized as follows. In Section 2 we determine $\gamma_e(\Gamma)$ for every vertex-transitive graph Γ. This allows us to show that, for this class of graphs, an extended irregular dominating set is necessarily optimal. Then, in Section 3, we prove the existence or non existence of an optimal extended irregular dominating set for several classes of vertex-transitive graphs. In Section 4 we show that the existence of an optimal extended irregular dominating labeling of a cycle of odd length n is equivalent to the existence of a strong starter of $\mathbb{Z}_n$. Then we present some results for an optimal extended irregular dominating labeling of odd cycles obtained as a consequence of known results on strong starters, as well as some new results in the case of cycles of single even length. In Section 5 we focus on a class of non vertex-transitive graphs: the paths. We point out that, in general, it is not easy to establish the value of $\gamma_e(\Gamma)$ if Γ is a non vertex-transitive graph. Here we firstly present a complete answer to the existence problem for an extended irregular dominating set for this class of graphs, then we give a lower bound for $\gamma_e(\Gamma)$, Γ being a path, and then we establish when this bound is reached. To conclude, in the last section, we propose two conjectures: the first one about the existence of an optimal extended irregular dominating labeling of cycles of length divisible by 4, while the second regards the value of $\gamma_e(\Gamma)$ when Γ is a path.

2. Preliminary results

In this section, we show that an extended irregular dominating set of a vertex-transitive graph, if it exists, is necessarily optimal.
Firstly we need to introduce some notation and to recall some basic concepts of graph theory. Given two integers a, b with $a \leq b$, by $[a, b]$ we mean the set $\{a, a+1, \ldots, b\}$.

The *degree* of a vertex v of Γ, denoted by $\deg(v)$, is the number of neighbours of v in Γ. A graph is said to be *regular* if all its vertices have the same degree. We also recall that the *diameter* of a graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is the largest distance between any pair of vertices of Γ.

Remark 1. The labels of an extended irregular dominating set of a graph Γ, if it exists, can assume value in $[0, \mathrm{diam}(\Gamma)]$ and hence $\gamma_e(\Gamma) \leq \mathrm{diam}(\Gamma)+1$.

Proposition 1. *A k-extended irregular dominating set cannot exist for $k = 2, 3$.*

Proof. When $k = 2$, if the label different from 0 is assigned to a vertex v, then it is not possible to dominate v with the remaining label 0.
Suppose now $k = 3$ and let v and w be the vertices with a non-zero label. It is easy to see that necessarily w has to be dominated by v and vice versa, but this is not possible since we use distinct labelings. □

Corollary 1. *The complete graph K_n, with $n > 1$, and the complete bipartite graph $K_{m,n}$ do not admit an extended irregular dominating set.*

Clearly, a graph has a 1-extended irregular dominating set if and only if it is an isolated vertex. Hence, if a non trivial graph Γ admits a 4-extended irregular dominating set S, then S is optimal and $\gamma_e(\Gamma) = 4$.
With the following lemma we show an interesting property of vertex-transitive graphs, that implies the well-known result that these graphs are regular.

Lemma 1. *Let $\Gamma = (V, E)$ be a vertex-transitive graph. For every vertex $v \in V$ and for every $i \in [0, \mathrm{diam}(\Gamma)]$ let $s_i(v)$ be the number of vertices having distance i from v. Then, the sequence $(s_0(v), \ldots, s_{\mathrm{diam}(\Gamma)}(v))$ is invariant on the choice of v, and $\sum s_i(v) = |V|$.*

Theorem 1. *Let $\Gamma = (V, E)$ be a vertex-transitive graph admitting an extended irregular dominating set S. Then, for every vertex $u \in V$ there exists a unique vertex $v \in S$ covering u.*

Proof. From Remark 1 we have that an extended irregular dominating labeling of Γ takes values in $[0, \mathrm{diam}(\Gamma)]$. Let S be an extended irregular dominating set and $W = \{w_1, \ldots, w_a\}$ be the set of vertices that are covered by at least two vertices of S. We have to prove that $W = \emptyset$. For every $j \in [1, a]$, let $m_j \geq 2$ be the number of vertices of S covering w_j. Then, by Lemma 1, the number of vertices covered by S is given by:

$$|V| - \sum_{j=1}^{a} (m_j - 1) \leq |V| - a.$$

Since S is an extended irregular dominating set, we deduce that necessarily $a = 0$, hence every vertex of Γ is covered by precisely one vertex of S. □

As a consequence, we have that if there exists a k-extended irregular dominating set of a vertex-transitive graph Γ, then Γ cannot admit a k'-extended irregular dominating set with $k' \neq k$, otherwise at least one vertex should be dominated more than once. By the above arguments and by Lemma 1 we have the following.

Corollary 2. *A k-extended irregular dominating set S of a vertex-transitive graph Γ, if it exists, is optimal and $k = \gamma_e(\Gamma) = \mathrm{diam}(\Gamma) + 1$. Moreover, the k vertices of S are labeled with all the elements in $[0, \mathrm{diam}(\Gamma)]$.*

3. Some results obtained using the diameter of the graph

In what follows, we show some existence and non-existence results of extended irregular dominating sets for vertex-transitive graphs in which the diameter of the graph plays a crucial role.
First note that as a consequence of Proposition 1, we have the following.

Proposition 2. *Let Γ be a vertex-transitive graph such that $\mathrm{diam}(\Gamma) \in \{1, 2\}$. Then, there does not exist an extended irregular dominating set in Γ.*

We consider now graphs with diameter equal to three. Firstly, we recall the definition of the crown graph. Let $K_{n,n}$ be the complete regular bipartite graph on $2n$ vertices, and denote by $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ the two parts of $K_{n,n}$. Note that $M = \{\{a_i, b_i\} : i \in [1, n]\}$ is a perfect matching of $K_{n,n}$. Given an integer $n \geq 3$, the *crown graph of order* $2n$ is the graph $\Gamma = K_{n,n} \setminus M$. We recall that these graphs are vertex-transitive, and have diameter equal to three. We are going to prove that this is the unique class of bipartite vertex-transitive graphs with diameter 3 admitting an extended irregular dominating set.

Proposition 3. *The crown graph admits an optimal extended irregular dominating set.*

Proof. Let Γ be a crown graph, and pick any two pairs of non-adjacent vertices (a_i, b_i) and (a_j, b_j), and let $\lambda : \{a_i, a_j, b_i, b_j\} \to [0, 3]$ be the following labeling:

$$\lambda(a_i) = 0, \quad \lambda(a_j) = 3, \quad \lambda(b_i) = 1, \quad \lambda(b_j) = 2.$$

We have that b_i dominates $A \setminus \{a_i\}$, while b_j dominates $B \setminus \{b_j\}$. Trivially, a_i and a_j dominate itself and b_j, respectively, thus concluding the proof. □

Theorem 2. *Let Γ be a vertex-transitive bipartite graph with $\mathrm{diam}(\Gamma) = 3$. Then, Γ admits an optimal extended irregular dominating set if and only if it is a crown graph.*

Proof. Let Γ be a vertex-transitive bipartite graph with $\operatorname{diam}(\Gamma) = 3$, and assume that it admits a k-extended irregular dominating labeling. Recall that, by Corollary 2, we have $k = 4$. Let u be the vertex having label 1, and partition the vertex set of Γ into the sets S_0, S_1, S_2, S_3, where $v \in S_i$ if and only if $d(u,v) = i$ (in particular, $S_0 = \{u\}$). Clearly, all the vertices in S_1 are dominated by u. We split the proof into two cases.

Case 1. u is dominated by a vertex v in S_2 having label 2.
By vertex-transitivity, v dominates $|S_2|$ vertices, hence, since Γ is bipartite, by a counting reasoning it dominates $S_0 \cup S_2 \setminus \{v\}$. Clearly v must be dominated by the vertex having label 3, and since every other vertex in $S_0 \cup S_2 \setminus \{v\}$ is already covered, by vertex-transitivity we have $|S_3| = 1$. Call then w the vertex having label 3. Clearly, $w \notin S_3$, otherwise it would dominate u, hence $w \in S_1$. Since $\operatorname{diam}(\Gamma) = 3$, every vertex in S_2 has either distance 1 or 3 from w, and from $|S_3| = 1$ and vertex-transitivity we deduce that v is the unique vertex having distance 3 from w. Thus, w is adjacent to every vertex of $S_2 \setminus \{v\}$. We then have $\deg(w) = |S_0| + |S_2 \setminus \{v\}| = |S_2|$, and since Γ is regular and bipartite $\deg(w) = |S_2| = |S_1| = \deg(u)$. It then follows that Γ is a regular bipartite graph on $2(|S_1| + 1)$ vertices, having degree $|S_1|$, hence Γ is a crown graph.

Case 2. u is dominated by a vertex v in S_3 having label 3.
Since Γ is bipartite, v dominates u and $|S_3| - 1$ vertices of S_2, hence to dominate the vertices in S_3 (in particular v) we use the remaining labels 0 and 2. Let w be the vertex having label 2.

- If $w \in S_1$, then w must be adjacent to every vertex in S_1, otherwise there would be a vertex that is dominated by both u and w. If $|S_1| > 1$, the graph Γ would not be bipartite, while if $|S_1| = 1$, that is if $S_1 = \{w\}$, then Γ would be a regular graph of degree 1, that is the path of length 1, and $\operatorname{diam}(\Gamma) = 1 \neq 3$. In any case, we reach a contradiction.

- If $w \in S_3$, then there exists a path P realizing the minimum distance between u and w, namely $P = [u, s_1, s_2, w]$, with $s_1 \in S_1$ and $s_2 \in S_2$. We have then that u and w dominate s_1, hence by Theorem 1 it is not an extended irregular dominating labeling.

Hence, Case 2 cannot occur. □

In the remaining part of this section we present a complete solution for the existence problem of an optimal extended irregular dominating set in the case of hypercubes and Möbius ladders.
We recall that given a positive integer n, the *hypercube* of dimension n, that we denote here by $\mathcal{Q}_n$, is the graph whose vertex set is identified by the sequences in $\{0,1\}^n$, and whose edges connect vertices having Hamming distance equal to 1. Clearly, $\mathcal{Q}_n$ is a bipartite graph with diameter n.

Theorem 3. *The n-dimensional hypercube admits an optimal extended irregular dominating set if and only if $n = 0, 3$.*

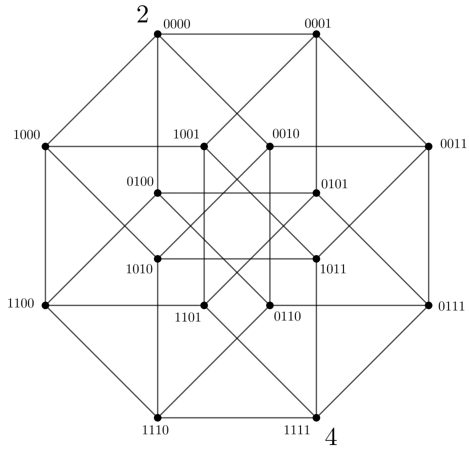


Figure 1. **The hypercube $\mathcal{Q}_4$.**

Proof. The existence of an extended irregular dominating set is trivial for $\mathcal{Q}_0$, which consists of a single vertex, and Example 1 shows the case $\mathcal{Q}_3$. Also, the non existence for $\mathcal{Q}_1$ and $\mathcal{Q}_2$ follows from Corollary 1.
Consider now $\mathcal{Q}_4$ shown in Figure 1. It is not restrictive to assume that the vertex $v = (0, 0, 0, 0)$ has label 2. A direct check shows that v cannot be covered by a vertex having labels 1 or 3, otherwise there should be some vertex covered twice. Hence it must be covered by a vertex having label 4, that has to be $w = (1, 1, 1, 1)$. It is then easy to see that this cannot be completed to an extended irregular dominating set.
Assume now that there exists an irregular dominating set in $\mathcal{Q}_n$ for some integer $n > 4$. For any fixed vertex v of $\mathcal{Q}_n$, and for every $d \in [0, n]$ let S_d denote the set of vertices in $\mathcal{Q}_n$ having distance d from v, where it is understood that $S_0 = \{v\}$. It is easy to see that $|S_d| = \binom{n}{d}$, and that if $u \in S_d$ for some $d \in [1, n]$, then $N_{\mathcal{Q}_n}(u) \subseteq S_{d-1} \cup S_{d+1}$ where $N_{\mathcal{Q}_n}(u)$ denotes the set of vertices of $\mathcal{Q}_n$ adjacent to u.
Without loss of generality assume that $v = (0, 0, \ldots, 0)$ is the vertex having label 4. We then have that $\mathcal{Q}_n \setminus S_4$ has two connected components, and in particular the one containing v, say C, has $1 + n + \binom{n}{2} + \binom{n}{3}$ vertices. If A is the part of the bipartition of $\mathcal{Q}_n$ containing v, then $|A \cap C| = 1 + \binom{n}{2}$.
Assume that the vertex v is dominated by a vertex $w = (w_1, \ldots, w_n)$ having label d: this implies that $w \in S_d$, thus there is a d-set $I = \{i_1, \ldots, i_d\}$ such that $w_i = 1$ if and only if $i \in I$. Now, if $2 \leq d \leq n - 2$, let i_1, i_2, j_1, j_2 be four distinct indexes, with $i_1, i_2 \in I$ and $j_1, j_2 \notin I$; let $z = (z_1, \ldots, z_n)$ be the vertex having coordinates:

$$z_i = \begin{cases} 1 & \text{if } i \in \{i_1, i_2, j_1, j_2\}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $z \in S_4$, and that the Hamming distance between z and w is

d, hence z is dominated by v and w, and the labeling does not induce an extended dominating set.

We now show that it is not possible to dominate v using the labels $d = 1$ or $d = n-1$. Let then w in S_1 be the vertex that dominates v, having label 1 (it is easy to see that the case where the label is $n-1$ is analogous to this one). If, without loss of generality, $w = (w_1, w_2, \ldots, w_n) = (1, 0, \ldots, 0)$, then w covers the $n-1$ vertices of S_2 having the first coordinate equal to 1. We show that it is not possible to cover the remaining vertices of S_2, with a vertex u having label ℓ. As a first remark, the label $\ell = 2$ cannot be used to cover the remaining vertices of S_2; indeed, it is easy to see that this implies $u = (u_1, \ldots, u_n) \in S_4$, hence, since $n > 4$, then there exists at least one component $u_i = 0$. If $u_j = 1$ for some $j \in [1, n]$, then the vertex $z = (z_1, \ldots, z_n)$ having $z_m = u_m$ for each $m \in [1, n] \setminus \{i, j\}$, $z_i = 1$ and $z_j = 0$ has distance 2 from u, but belongs to S_4, thus it is dominated twice.

Let then u be labeled with $\ell > 2$. Clearly, $u \notin S_\ell$, otherwise it would dominate v another time, and if $u \in S_m$, with $m \leq \ell - 4$ or $m \geq \ell + 4$, then u does not dominate any vertex of S_2. Then:

- if $u \in S_{\ell-2}$, then there are precisely $\ell - 2 \geq 1$ indexes $i_1, \ldots, i_{\ell-2}$ such that $u_{i_j} = 1$ for $j \in [1, \ell - 2]$, and 0 otherwise. In particular, if $n - 1 > \ell$, then there are at least three zero entries u_a, u_b, u_c (indeed, two of them are required to dominate vertices of S_2, while the third one is ensured by $n - 1 > \ell$). Let then $z = (z_1, \ldots, z_n)$ be the vertex such that $z_m = 1$ if and only if $m \in \{a, b, c, i_1\}$: z belongs to S_4 and has distance ℓ from u, thus it is dominated twice. Now, if $\ell = n$, then u would dominate precisely one vertex of S_2, but the vertices of S_2 that are not dominated by w are:

  $$\binom{n}{2} - (n-1) = \frac{n(n-3)}{2} + 1.$$

  It would be then necessary to use other labels to cover the remaining vertices of S_2, thus returning in one of the other cases.

- if $u \in S_{\ell+2}$, then there are precisely $\ell + 2 \geq 3$ indexes $i_1, \ldots, i_{\ell+2}$ such that $u_{i_j} = 1$ for $j \in [1, \ell - 2]$, and 0 otherwise. If $\ell \leq n - 3$, then there is at least one zero entry u_a: if $z = (z_1, \ldots, z_n)$ is the vertex such that $z_m = 1$ if and only if $m \in \{a, i_1, i_2, i_3\}$, then $z \in S_4$ and is dominated twice. If $\ell = n - 2$, necessarily $u \in S_n$, but then u would dominate the whole set S_2, that is already partially covered by w.

To conclude, assume that v is dominated by the vertex $w = (1, \ldots, 1) \in S_n$ having label n. Assume now that some of the vertices of S_2 are dominated by a vertex $u = (u_1, \ldots, u_n)$ having label ℓ. Clearly, $\ell \neq 1$, otherwise u would cover either v or vertices of S_4. If $\ell = 2$, since $n > 4$ there are vertices in S_4 that are dominated by u (see above). If $\ell > 2$, then the reasoning explained above can be applied for almost all the cases: the only exceptions are for $\ell = n - 2$, that in this case would imply $u = w$, and for $\ell = n$, that it is not possible as that label is already used.

It then follows that it is not possible to cover the vertices in $S_0 \cup S_2$ with a labeling that induces an extended irregular dominating set, hence the statement follows. $\square$

We recall now the definition of a Mobius ladder. For every positive integer $n \geq 2$, the *Mobius ladder* on $2n$ vertices, denoted by M_{2n}, is the graph having $\{x_1, x_2, \dots, x_{2n}\}$ as vertex set, and such that $E(M_{2n}) = \{\{x_i, x_{i+1}\} \mid 1 \leq i \leq 2n\} \cup \{\{x_i, x_{i+n}\} \mid 1 \leq i \leq n\}$, where the subscripts have to be considered modulo $2n$. Note that M_4 is nothing but the complete graph on 4 vertices.

Lemma 2. *Let $n \geq 3$ and let x be a vertex of M_{2n}. Assign a label $\ell \in [1, \lceil \frac{n}{2} \rceil]$ to x. Then, for every vertex v covered by x there exists a vertex u, dominated by x, such that $d(u, v) = 2$.*

Proof. Clearly, by vertex-transitivity, we can assume that $x = x_1$. The vertices dominated by x_1 are $x_{1+\ell}$, $x_{2n-\ell+1}$, $x_{n+\ell}$ and $x_{n-\ell+2}$ (note that for n odd and $\ell = \frac{n+1}{2}$ we have $x_{1+\ell} = x_{n-\ell+2} = x_{\frac{n+3}{2}}$ and $x_{2n-\ell+1} = x_{n+\ell} = x_{\frac{3n+1}{2}}$). Since $\{x_i, x_{i+n}\}$ is an edge of M_{2n} for every $i \in [1, n]$, we have $d(x_{1+\ell}, x_{n+\ell}) = d(x_{n-\ell+2}, x_{2n-\ell+1}) = 2$. $\square$

Theorem 4. *For every $n \geq 2$, the Mobius ladder M_{2n} does not admit an extended irregular dominating set.*

Proof. Since M_4 is the complete graph of order 4 and M_6 is the complete bipartite graph $K_{3,3}$, for $n = 2, 3$, the result follows from Corollary 1.
Suppose now $n \geq 4$. Assume by contradiction that there exists an extended irregular dominating set of M_{2n}. By vertex-transitivity, we can suppose, without loss of generality, that x_1 is the vertex that receives label 2, hence dominating $X = \{x_3, x_n, x_{n+2}, x_{2n-1}\}$.
If n is even, from Lemma 2, it can be seen that it is not possible to cover x_1 with a vertex having label in $[1, \frac{n}{2} - 1]$: indeed, there would be a vertex between x_3, x_n, x_{n+2}, x_{n-1} covered twice. Hence, x_1 has to be covered with a vertex having label $\frac{n}{2}$. It can be seen that the graph induced by the vertices dominated by a vertex having label $n/2$ is a path P on 4 vertices. However, $M_{2n} \setminus X$ is a disconnected graph, where the connected component containing x_1 is isomorphic to the complete bipartite graph $K_{1,3}$: since P is not a subgraph of $K_{1,3}$, it follows that x_1 cannot be covered. Hence an extended irregular dominating set of M_{2n} does not exist.
Suppose now n odd. By Lemma 2 it can be seen that if a label in $\ell \in [1, \frac{n+1}{2}]$ is assigned to a vertex w, and $d(w, x_1) = \ell$, then there exists a vertex $x \in X$ such that $d(w, x) = \ell$, that is then covered twice. Hence, there does not exist an extended irregular dominating set. $\square$

4. Results for cycles via strong starters

In this section, we show that optimal extended irregular dominating labelings of cycles are equivalent to a combinatorial structure that has been thoroughly studied over the course of the years, that is strong starters in a cyclic group, see [9].

Definition 2. Let G be an additive abelian group of odd order g, where the neutral element is denoted by 0. A *starter* L in G is a set of unordered pairs $\{\{x_i, y_i\} \mid 1 \leq i \leq (g-1)/2\}$ such that:

(1) $\{x_i, y_i \mid 1 \leq i \leq (g-1)/2\} = G \setminus \{0\}$;

(2) $\{\pm(x_i - y_i) \mid 1 \leq i \leq (g-1)/2\} = G \setminus \{0\}$.

A starter $L = \{\{x_i, y_i\}\}$ in G is called a *strong starter* if the additional property:

(3) $x_i + y_i = x_j + y_j$ implies $i = j$ and for any i, $x_i + y_i \neq 0$

is satisfied. In other words a starter is called strong if $\{(x_i + y_i) \mid 1 \leq i \leq (g-1)/2\}$ comprises of distinct elements in $G \setminus \{0\}$.
A starter $L = \{\{x_i, y_i\}\}$ in G is said to be *skew* if the following additional property holds:

(4) $x_i + y_i = \pm(x_j + y_j)$ implies $i = j$ and for any i, $x_i + y_i \neq 0$.

or equivalently if:

(4) $\{\pm(x_i + y_i) \mid 1 \leq i \leq (g-1)/2\} = G \setminus \{0\}$.

It is clear that a skew starter is also a strong starter. Both strong and skew starters have been studied in many groups, achieving various existence results. Here, we are interested in the case of the cyclic group $\mathbb{Z}_n$ for some positive integer n. An hill-climbing algorithm to find strong starters in cyclic groups has been developed in [10]. In [22], Stinson presented several results for strong starters in view of which he proposed the following conjecture.

Conjecture 4. Let $n \geq 5$ be an odd integer. There exists a strong starter in $\mathbb{Z}_n$ if and only if $n \neq 5, 9$.

It is easy to see that a starter of $\mathbb{Z}_5$ does not exist. Also, the non existence of a strong starter of $\mathbb{Z}_9$ is well-known. In the same paper Stinson proved the existence of a strong starter in $\mathbb{Z}_n$ for every odd $n \neq 9$, with $7 \leq n \leq 99$. We point out that the conjecture proposed in [22] has a more general statement that Conjecture 4, but for the purpose of this paper it is sufficient to focus on this special case. We also underline that this special case is contained in the following conjecture [13] proposed by Horton in 1990, which is not restricted to cyclic groups and which is still far from being solved.

Conjecture 5. Suppose that G is an abelian group of odd order $g \geq 3$. Then there is a strong starter in G if and only if $G \neq \mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_9$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Now we summarize the main results for skew starters in a cyclic group, obtained in [7, 14, 15, 18].

Theorem 5. *Let n be a positive integer. Then, there exists a skew starter in $\mathbb{Z}_n$ in the following cases:*

- $n = 2^k t + 1$ *is a prime power, where $t > 1$ is an odd integer (Mullin-Nemeth starters);*
- $n = 16t^2 + 1$ *(Chong-Chang-Dinitz starters);*
- $\gcd(n, 6) = 1$ *and either $n \neq 0 \pmod 5$ or $n = 0 \pmod{125}$.*

As remarked by Stinson in [21], the parameters for the Mullin-Nemeth and Chong-Chang-Dinitz starters allow to construct skew starters in the notable class of cyclic groups whose order is a prime number larger than 5.
We now establish the following equivalence, where by C_n we denote the cycle of length n.

Proposition 6. *Let n be an odd integer. Then, an optimal extended irregular dominating set over C_n is equivalent to a strong starter in $\mathbb{Z}_n$.*

Proof. Let S be an extended irregular dominating set over $C_n = (v_0, v_1, \ldots, v_{n-1})$, with labeling function $\lambda : S \to [0, \frac{n-1}{2}]$. Recall that, by Corollary 2, $|S| = \frac{n+1}{2}$, that is, λ is a bijection. Observe that, since n is odd, 2 admits a multiplicative inverse in $\mathbb{Z}_n$, that we denote by 2^{-1} for the sake of brevity. Assume without loss of generality that $v_0 \in S$ and $\lambda(v_0) = 0$, and construct the following set $L \subset \mathbb{Z}_n \times \mathbb{Z}_n$:

$$L = \{\{i, j\} \mid \ v_i, v_j \text{ are dominated by } v \in S \setminus \{v_0\} \ \}.$$

We prove that L is a strong starter in $\mathbb{Z}_n$. Since every vertex of C_n is dominated exactly once, it is immediate to verify that property (1) of Definition 2 holds. Moreover, let $\{i, j\} \in L$, with $i > j$, and let v_k be the vertex that dominates v_i and v_j, receiving label ℓ. It can be immediately seen that precisely one between $i - j$ and $n + j - i$ is even. In the first case, we necessarily have $v_k = v_{(i+j)/2}$ and $\ell = \frac{i-j}{2}$, while in the second case $v_k = v_{(n+i+j)/2}$ and $\ell = \frac{n+j-i}{2}$. Suppose now that property (2) of Definition 2 does not hold, and let $\{x, y\}$ and $\{r, s\}$ be two pairs such that $\{\pm(x - y)\} = \{\pm(r - s)\}$, with $x > y$ and $r > s$. Note that it is not restrictive to assume that $x - y = r - s$ is an even number (otherwise, consider $y + n$ and $s + n$). Then, the labels assigned to the vertices $v_{\frac{x+y}{2}}$ and $v_{\frac{r+s}{2}}$ are not distinct, hence λ is not a bijection and S is not an extended irregular dominating set, that is a contradiction. Thus property (2) of Definition 2 holds. Hence we have proved that L is a starter. Finally, it is easy to see that if property (3) of Definition 2 does not hold, there exist $\{x, y\}$ and $\{r, s\}$ in L such that $x + y = r + s$, then the vertices dominating v_x, v_y and v_r, v_s must coincide. It would follow that S is not an extended irregular dominating set, hence also property (3) of Definition 2 holds, and L is a strong starter in $\mathbb{Z}_n$.
Let now $L = \{\{x_i, y_i\} \mid 1 \le i \le (n-1)/2\}$ be a strong starter in $\mathbb{Z}_n$ with $x_i > y_i$ for each $1 \le i \le (n-1)/2$. Let $V = \{v_0, v_1, \ldots, v_{n-1}\}$ be the vertex set of C_n and $E(C_n) = \{\{v_i, v_{i+1}\} \mid i \in [0, n-1]\}$ where the indexes are understood modulo n.

Consider the following set $\left\{v_{\frac{x_i+y_i}{2}} \mid \{x_i, y_i\} \in S\right\} \subseteq V$, where if $x_i + y_i < n$ is odd, by $v_{\frac{x_i+y_i}{2}}$ we mean $v_{\frac{n+x_i+y_i}{2}}$. Take now the labeling so defined:

$$\lambda\left(v_{\frac{x_i+y_i}{2}}\right) = \begin{cases} \dfrac{x_i - y_i}{2} & \text{if } x_i - y_i \pmod n \text{ is even,} \\ \dfrac{n-(x_i-y_i)}{2} & \text{otherwise.} \end{cases}$$

One can easily check that λ is an optimal extended irregular dominating labeling of C_n. □

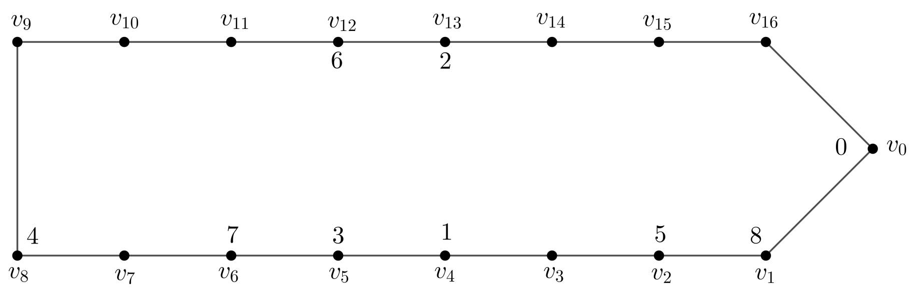
Since every skew starter is a strong starter, the existence of an optimal extended irregular dominating set is granted for cycles of length n, for every n belonging to one of the cases of Theorem 5 and for every n for which Conjecture 4 holds. In particular, we have.

Corollary 3. *The cycle C_n admits an optimal extended irregular dominating set for every odd integer $n \neq 9$, such that $7 \leq n \leq 99$ and for every prime $n > 5$.*

Example 2. Starting from the following strong starter of $\mathbb{Z}_{17}$

$$\{\{9,10\},\{3,5\},\{13,16\},\{11,15\},\{1,6\},\{2,8\},\{7,14\},\{4,12\}\}$$

one can construct the optimal extended irregular dominating labeling of C_{17} below.



In what follows, we show that there exists an optimal extended irregular dominating set for many cycles, having singly even length.

Proposition 7. *Let n be an odd integer. If C_n admits an optimal extended irregular dominating set, then C_{2n} admits an optimal extended irregular dominating set too.*

Proof. Since n is an odd integer, for every $x \in \mathbb{Z}_n^*$ precisely one element between x and $n-x$ is odd. Let $C_{2n} = (v_0, v_1, \ldots, v_{2n-1})$, and for any vertex v_i by $-v_i$ we mean

v_{i+n}, where indexes are read modulo $2n$. Let now λ and λ' be two labelings on C_{2n} such that $\lambda(v) = x$ and $\lambda'(-v) = n - x$ for some vertex v, where $x, n - x \in \mathbb{Z}_n^*$. Then v through λ and $-v$ through λ' dominate the same set of vertices. Set $A = \{v_{2i} : i \in [0, n-1]\}$ and $B = \{v_{2i+1} : i \in [0, n-1]\}$.
Let $V(C_n) = (x_0, \ldots, x_{n-1})$, and set $\lambda_1 : X \subset V(C_n) \to [0, \frac{n-1}{2}]$ be an optimal extended irregular dominating labeling of C_n which exists by hypothesis. Let ψ be the natural bijection $\psi : V(C_n) \to A \subset V(C_{2n})$, where $x_i \mapsto \psi(x_i) = v_{2i}$. Construct the labeling $\lambda_2 : \psi(X) \subset V(C_{2n}) \to [0, n]$, where $\lambda_2(v_{2i}) = \lambda_2\psi(x_i) = 2\lambda_1(x_i)$. Since the vertices in $\psi(X)$ are contained in A and have even labels, from the fact that λ_1 induces an optimal extended irregular dominating set it follows that the vertices of $\psi(X)$ dominate A.
Let now $v_{2i} \in A$ be any vertex that is not labeled by λ_2, that is $v_{2i} \in A \setminus \psi(X)$. Let $Y = \{v_{2i}, -v_{2i}\}$ be a cut of C_{2n}, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the vertex set of the two connected components of $C_{2n} \setminus Y$. For the sake of brevity, denote by Z the set $(\psi(X) \cap \mathcal{C}_1) \cup (-(\psi(X) \cap \mathcal{C}_2))$, and let $\lambda_3 : Z \subset \mathcal{C}_1 \to [0, n]$ be the following labeling:

$$\lambda_3(v) = \begin{cases} \lambda_2(v) & \text{if } v \in \psi(X) \cap \mathcal{C}_1, \\ -\lambda_2(v) \pmod{n} & \text{if } \lambda_2(v) \neq 0 \text{ and } v \in -(\psi(X) \cap \mathcal{C}_2), \\ n & \text{if } \lambda_2(v) = 0 \text{ (and } v \in -(\psi(X) \cap \mathcal{C}_2)). \end{cases}$$

As previously remarked, it follows that λ_3 induces a set of labeled vertices dominating A. Let now $\eta : \mathcal{C}_1 \to \mathcal{C}_2$, $v \mapsto \eta(v) = w$, where w is the vertex of $\mathcal{C}_2$ such that the distance between w and v_{2i} is equal to the distance between v and v_{2i}. We conclude the proof by constructing the labeling $\lambda : Z \cup \eta(Z) \to [0, n]$, where:

$$\lambda(v) = \begin{cases} \lambda_3(v) & \text{if } v \in Z, \\ -\lambda_3(v) & \text{if } \lambda_3(v) \neq n \text{ and } v \in -Z, \\ 0 & \text{if } \lambda_3(v) = n \text{ (and } v \in -Z). \end{cases}$$

Since $Z \subset \mathcal{C}_1$, it follows that λ is well-defined, and as the dominated set induced by λ_3 is A, λ is an optimal extended irregular dominating labeling inducing an optimal extended irregular dominating set on C_{2n}. □

It is easy to see that the converse of the previous result does not hold. For example, even if C_3 and C_5 do not admit an extended irregular dominating set, C_6 and C_{10} have such a set as shown below:

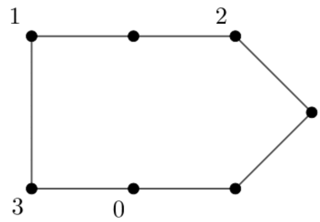
$$C_6 : \quad (3, 1, \square, 2, 0, \square), \qquad C_{10} : \quad (1, 4, 2, \square, \square, 3, 0, \square, 5, \square),$$

where □ denotes a vertex with no label.

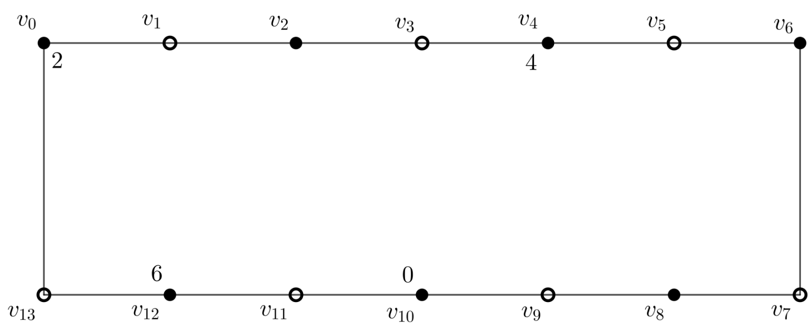
Corollary 4. *Let p be a prime with $p \geq 7$. Then, there exists an optimal extended irregular dominating set of C_{2p}.*

Corollary 5. *Let n be an odd integer with $n \neq 9$ and $7 \leq n \leq 99$. Then, there exists an optimal extended irregular dominating set of C_{2n}.*

Example 3. We show the construction provided in Proposition 7 for $n = 7$. Consider the following optimal extended irregular dominating labeling of C_7:



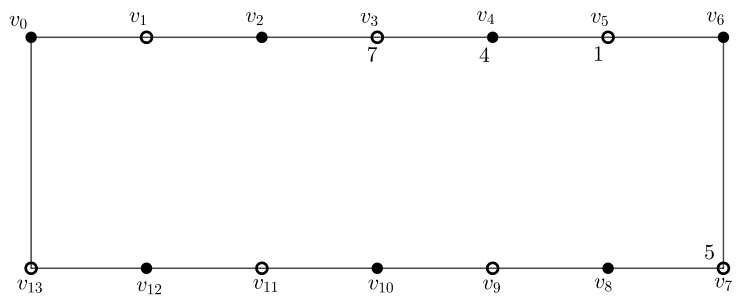
We now consider the cycle $C_{14} = (v_0, v_1, \dots, v_{13})$, and a possible choice for the labeling obtained by doubling the labels of C_7:



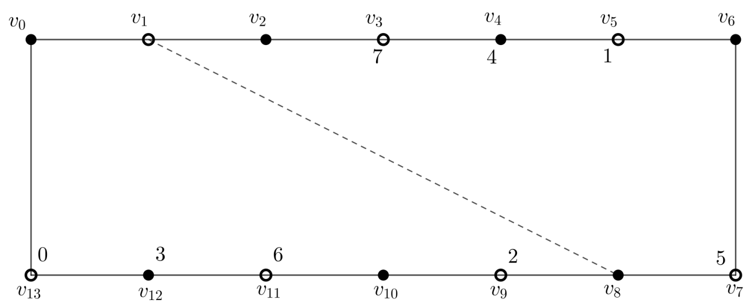
It can be seen that the labeled vertices cover all the vertices of the form $\{v_{2i} : i \in [0, 6]\}$. Consider now the cut $Y = \{v_1, v_8\}$, and the vertex sets $\mathcal{C}_1 = \{v_i : i \in [2, 7]\}$ and $\mathcal{C}_2 = \{v_i : i \in \{0\} \cup [9, 13]\}$ of the connected components of $C_{14} \setminus Y$. We then have $Z = \{v_4\} \cup \{v_3, v_5, v_7\} \subset \mathcal{C}_1$, and we construct the labeling $\lambda_3 : Z \to [0, 7]$:

$$\lambda_3(v_4) = 4, \quad \lambda_3(v_3) = 7,$$
$$\lambda_3(v_5) = 1, \quad \lambda_3(v_7) = 5,$$

that we show below:



We then conclude the example by showing the labeling λ. Since λ can also be obtained by reflecting the labeled vertices along the line joining the vertices of the cut Y, we have added the dashed line between v_1 and v_8:



It can then be seen that the labeling λ so defined induces an optimal extended irregular dominating set.

We then summarize these results:

Corollary 6. *Let n be a positive integer, where either*

- $n \geq 7$ *is a prime,*
- n *is an odd integer with* $n \neq 9$ *and* $7 \leq n \leq 99$*,*
- $n = 2^k t + 1$ *is a prime power, where* $t > 1$ *is an odd integer,*
- $n = 16t^2 + 1$*, or*
- $\gcd(n, 6) = 1$ *and either* $n \neq 0 \pmod 5$ *or* $n = 0 \pmod{125}$*.*

Then, there exists an optimal extended irregular dominating set of C_n and C_{2n}.

5. A class of non vertex-transitive graphs: the paths

Obviously it makes sense to consider the irregular extended domination problem also for non vertex-transitive graphs. In this section we focus on paths. By P_n we denote the path on n vertices, that we also write as its list of vertices $[x_1, x_2, \ldots, x_n]$, where the edges are $\{x_i, x_{i+1}\}$ for every $i \in [1, n-1]$. First of all we completely solve the existence problem of an extended irregular dominating set for a path, then we establish $\gamma_e(P_n)$ for several values of n. To do this, let us see some connection between irregular dominating sets and extended irregular dominating sets for paths.

Remark 2. Let S be a k-irregular dominating set of P_n. If there exists a vertex v dominating a unique vertex u of P_n and $u \notin S$, then there also exists a k-extended irregular dominating set of P_n. In fact it is sufficient to remove the label from v and to label u by 0.

Remark 3. If there exists a k-irregular dominating labeling, say λ, of the path $P_n = [x_1, x_2, \ldots, x_n]$, then there exists a $(k+1)$-extended irregular dominating labeling, say λ', of $P_{n+1} = [x_1, x_2, \ldots, x_n, x_{n+1}]$. In fact it is sufficient to extend λ by labeling the vertex x_{n+1} by 0. Unfortunately, if λ is optimal, this does not necessarily imply that λ' is optimal too.

Clearly, the same reasoning can be applied also to other classes of graphs.

Example 4. In [4] the following 6-irregular dominating labeling of P_8 is presented

$$[\square, 5, 3, 1, 4, 2, \square, 6],$$

where $\square$ denotes a vertex with no label, and it is proved that it is optimal, that is $\gamma_i(P_8) = 6$. Such a labeling can be extended to the following 7-extended irregular dominating labeling of P_9:

$$[\square, 5, 3, 1, 4, 2, \square, 6, 0]$$

which is not optimal; in fact there exists a 6-extended irregular dominating labeling of P_9:

$$[\square, \square, 2, \square, 3, 0, 4, 1, 5].$$

The existence problem for an irregular dominating set of a path has been completely solved in [4], where the authors proved the following.

Proposition 8. *The path P_n has an irregular dominating labeling if and only if $n \geq 4$ except for $n = 6$.*

We point out that the proof is constructive, but the resulting labeling is not necessarily optimal. Actually, the problem of establishing the exact value of $\gamma_i(P_n)$ is still open. The previous proposition allows us to prove the following.

Proposition 9. *Given $n \geq 1$, the path P_n admits an extended irregular dominating labeling if and only if $n \neq 2, 3$.*

Proof. Since P_2 and P_3 do not have an extended irregular dominating labeling by Proposition 1, it remains to verify the converse. For $n = 1$ the existence is trivial. An extended irregular dominating labeling for the paths P_4 and P_7 is given by:

$$\begin{aligned} P_4 &: [3, 1, 0, 2], \\ P_7 &: [\square, 0, 2, 3, 1, \square, 4], \end{aligned}$$

where $\square$ denotes a vertex with no label. The other cases follow from the Remark 3 and Proposition 8. $\square$

By Remark 3 the extended irregular dominating labelings of previous proposition are, in general, not optimal. In particular, in [4], a k-irregular dominating labeling of P_n is constructed with $k = n-2$ for $n = 7, 8$, with $k = n-3$ for $n = 9$, and with $k = n-4$ for every $n \geq 10$. Starting for these labelings, we immediately have a k'-extended irregular dominating labeling of P_{n+1} with $k' = k+1$. On the other hand, we believe that this result can be improved, that is that there exists a k''-extended irregular dominating labeling of P_{n+1} for $k'' < k'$.
In the remaining part of this section, we make some consideration on $\gamma_e(P_n)$, we start with a lemma whose proof is trivial.

Lemma 3. *For every $n \geq 1$, $\gamma_e(P_n) \geq \lceil \frac{n+1}{2} \rceil$.*

In the following result we establish when the equality holds in Lemma 3.

Theorem 6. *Let $n \geq 1$, $\gamma_e(P_n) = \lceil \frac{n+1}{2} \rceil$ if and only if $n = 1, 6, 10$.*

Proof. Suppose firstly n odd, and set $n = 2m+1$ for some integer m. If $m = 0$ the result is trivial, so assume that $m \geq 1$. Note that $\gamma_e(P_{2m+1}) = m+1$ implies that we have to use the labels from the set $[0, m]$, where every vertex labeled with an integer different from 0 has to cover 2 vertices. For this reason, the label m has to be given to the vertex x_{m+1}, covering x_1 and x_{2m+1}. However, it is not possible to assign the label $m-1$ and cover two vertices, hence an extended irregular dominating set of size $m+1$ cannot exist. In other words, $\gamma_e(P_n) > \lceil \frac{n+1}{2} \rceil$.
Let now $n = 2m$ for some integer m. By Corollary 1, P_2 and P_4 do not admit an extended irregular dominating set, while for P_6 and P_{10} we have the following labelings:

$$\begin{aligned} P_6 &: [3, 1, \square, 2, 0, \square], \\ P_{10} &: [\square, 6, 0, 2, 4, 1, 3, \square, \square, \square]. \end{aligned}$$

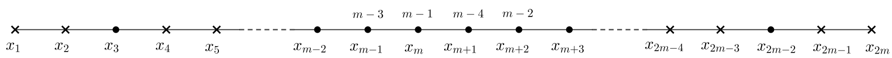
Hence $\gamma_e(P_6) = 4$ and $\gamma_e(P_{10}) = 6$ hold.
Assume now that $m \geq 8$. Also here, $\gamma_e(P_{2m}) = m+1$ implies we have to use labels in the set $[0, m-1] \cup \bar{\lambda}$, where $\bar{\lambda} \in [m, 2m-1]$ in such a way that the vertices receiving labels 0 and $\bar{\lambda}$ cover exactly one vertex, and the vertices receiving labels from $[1, m-1]$ have to cover two vertices. To this aim, it is easy to see that the label $m-1$ has to be assigned either to x_m or to x_{m+1}. By symmetry we can assume without loss of generality that this label is assigned to x_m, thus covering x_1 and x_{2m-1}. This assignment forces the following labeling:

(1) the label $m-2$ has to be assigned to x_{m+2}, covering x_4 and x_{2m};

(2) the label $m-3$ has to be assigned to x_{m-1}, dominating x_2 and x_{2m-4};

(3) the label $m-4$ has to be assigned to x_{m+1}, covering x_5 and x_{2m-3}.

To summarize, up to this point, the vertices labeled are $\{x_{m-1}, x_m, x_{m+1}, x_{m+2}\}$, and they dominate the set of vertices $\{x_1, x_2, x_4, x_5\} \cup \{x_{2m-4}, x_{2m-3}, x_{2m-1}, x_{2m}\}$, as shown in the following:



It can then be seen that the label $m-5$ has to be assigned either to x_{m-2} or x_{m+3}; by symmetry, it is not again restrictive to assume that this label is given to x_{m-2}, thus dominating x_3 and x_{2m-7}. Then, the label $m-6$ has to be given to x_{m+4}, covering x_{10} and x_{2m-2}. However, it can be seen that now there is no possible assignment of the label $m-7$ to cover two of the remaining vertices, hence proving that for $m \geq 8$ there is no k-extended irregular dominating labeling of the path of even order with $k = \lceil \frac{n+1}{2} \rceil$.
To conclude, for $m = 4, 6, 7$ repeat then the procedure shown before until the label 2 is assigned. It can then be seen that it is not possible to assign the label 1 to any vertex of the path. □

Corollary 7. *Let $n \geq 4$ with $n \neq 6, 10$, then $\gamma_e(P_n) \geq \lceil \frac{n+3}{2} \rceil$.*

The next natural question is the following: when does the equality hold in Corollary 7?
Clearly it holds for $n = 4$, see the labeling of P_4 given in the proof of Proposition 9. Below we show that the equality holds for every $n \in [5, 26] \setminus \{6, 10\}$. Some of these optimal extended irregular dominating labeling of P_n have been obtained thanks to Remarks 2 and 3 starting from an irregular dominating labeling of P_n constructed in [3].

$$
\begin{aligned}
P_5 &: [3, 1, \square, 2, 0],\\
P_7 &: [6, 0, 2, \square, 1, 3, \square]\\
P_8 &: [\square, \square, 3, 1, 4, 2, 0, 6]\\
P_9 &: [3, 6, 0, 2, 4, 1, \square, \square, \square]\\
P_{11} &: [\square, \square, 5, 3, 1, 4, 2, \square, \square, 7, 0],\\
P_{12} &: [0, 10, \square, 5, 3, 1, 4, 2, \square, \square, 7, \square],\\
P_{13} &: [\square, 5, \square, \square, 1, 4, 6, 3, 0, 2, \square, 9, \square],\\
P_{14} &: [7, \square, \square, \square, 2, 4, 6, 3, 5, 1, \square, 0, \square, 8],\\
P_{15} &: [\square, \square, \square, \square, 1, 3, 5, 7, 4, \square, 0, 2, \square, 6, 8],\\
P_{16} &: [\square, 9, \square, \square, 1, 3, 5, 7, 4, \square, \square, 2, \square, 6, 8, 0],\\
P_{17} &: [8, \square, \square, \square, 2, 5, 3, \square, 7, 4, 6, 0, \square, 1, \square, \square, 9],\\
P_{18} &: [8, 10, \square, \square, 2, 5, 3, \square, 7, 4, 6, \square, \square, 1, \square, \square, 9, 0],\\
P_{19} &: [\square, \square, \square, 13, \square, 3, 6, 4, 1, 8, 5, 7, 2, 0, \square, \square, 10, \square, \square],\\
P_{20} &: [\square, \square, \square, 13, 9, 3, 6, 4, 1, 8, 5, 7, 2, \square, \square, \square, 10, \square, \square, 0],\\
P_{21} &: [\square, 10, \square, \square, \square, 1, \square, 5, 7, 9, 3, 8, 2, 4, 6, \square, 0, 12, \square, \square, \square]\\
P_{22} &: [\square, \square, \square, \square, \square, \square, 4, 7, 5, 3, 9, 6, 8, 2, \square, \square, \square, 1, 11, 10, 12, 0],\\
P_{23} &: [12, \square, \square, \square, \square, 3, 0, 4, 8, 10, 5, 7, 9, 1, 6, 2, \square, \square, \square, \square, 11, \square],\\
P_{24} &: [12, \square, \square, \square, \square, 3, \square, 4, 8, 10, 5, 7, 9, 1, 6, 2, \square, \square, \square, 13, 11, \square, 0],\\
P_{25} &: [\square, 12, \square, \square, \square, \square, 3, \square, 4, 8, 10, 5, 7, 9, 1, 6, 2, \square, \square, \square, 13, 11, \square, 0],\\
P_{26} &: [14, 12, \square, \square, \square, \square, 4, 1, \square, 6, 9, 11, 5, 8, 10, 3, 7, \square, 2, \square, \square, \square, \square, \square, 13, 0],
\end{aligned}
$$

where $\square$ denotes a vertex with no label.

6. Conclusions

In this paper we have introduced the concept of an extended irregular dominating set focusing, in particular, on the optimal case. In some cases we presented complete solution to existence problem of an optimal extended irregular dominating set for some classes of graphs, while for other ones we have only some partial results.
For example note that in Section 4 about cycles, we have not considered the class of cycles C_n, with $n \equiv 0 \pmod 4$. A direct check shows that there exists no extended irregular dominating set of C_n for $n = 4, 8$, while here we report an optimal extended irregular dominating labeling for the cycles C_{4n} with $n \in [3, 6]$:

$$
\begin{aligned}
C_{12} &: (6, 3, \square, \square, 1, 4, \square, 5, 0, 2, \square, \square),\\
C_{16} &: (0, \square, \square, \square, \square, 7, 4, 1, \square, \square, 3, 6, 8, 2, 5, \square),\\
C_{20} &: (10, 8, \square, 3, \square, 9, 1, \square, 0, 6, \square, 7, \square, \square, \square, 4, \square, 5, \square, 2),\\
C_{24} &: (0, 11, 8, 2, 12, 1, \square, \square, 9, 6, \square, \square, \square, \square, 5, 7, \square, 4, \square, \square, \square, 10, \square, 3),
\end{aligned}
$$

where the examples for C_{16} and C_{24} have been found with the aid of a computer by Falcón [11]. At the moment we have no further results for this class of graphs, on the other hand for $n \in [3,6]$ we obtain really many labelings of C_{4n} satisfying the required properties, hence we believe that $n = 1, 2$ are the only exception and we propose the following.

Conjecture 10. There exists an optimal extended irregular dominating labeling for C_{4n} for every $n \geq 3$.

About paths we established that $\gamma_e(P_n) = \lceil \frac{n+1}{2} \rceil$ if and only if $n = 1, 6, 10$, and hence that $\gamma_e(P_n) \geq \lceil \frac{n+3}{2} \rceil$ for $n \geq 4$ with $n \neq 6, 10$. Also we proved that the equality holds for every $n \in [4, 26] \setminus \{6, 10\}$. This leads us to propose another conjecture.

Conjecture 11. Let $n \geq 4$.

$$\gamma_e(P_n) = \begin{cases} \left\lceil \dfrac{n+1}{2} \right\rceil & \text{if } n = 6, 10, \\ \left\lceil \dfrac{n+3}{2} \right\rceil & \text{otherwise.} \end{cases}$$

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