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Weak signed double Roman domination in graphs

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Abstract: A *weak signed double Roman dominating function* (WSDRDF) of a graph G with vertex set $V(G)$ is defined as a function $f : V(G) \to \{-1, 1, 2, 3\}$ having the property that $\sum_{x \in N[v]} f(x) \geq 1$ for each $v \in V(G)$, where $N[v]$ is the closed neighborhood of v. The weight of a WSDRDF is the sum of its function values over all vertices. The weak signed double Roman domination number of G, denoted by $\gamma_{wsdR}(G)$, is the minimum weight of a WSDRDF in G. We initiate the study of the weak signed double Roman domination number, and we present different sharp bounds on $\gamma_{wsdR}(G)$. In addition, we determine the weak signed double Roman domination number of some classes of graphs.

Keywords: domination, signed double Roman domination, weak signed double Roman domination.

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1. Terminology and introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [9]. Specifically, let G be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the *order* and the *size* of the graph G, respectively. The *open neighborhood* of vertex v is $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $d_G(v) = d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively. A graph G is *regular* or *r-regular* if $\delta(G) = \Delta(G) = r$. The *complement* of a graph G is denoted by $\overline{G}$. Let K_n be the complete graph of order n, C_n the cycle of order n and P_n the path of order n. In addition, let $K_{n_1,n_2,\ldots,n_p}$ be the *complete p-partite graph* with the partite sets $X_1, X_2, \ldots, X_p$ such that $|X_i| = n_i$ for $1 \leq i \leq p$. Also let $S(r,s)$ be the *double star* with exactly two adjacent vertices u and v that are not leaves such that u is adjacent to $r \geq 1$ leaves and v is adjacent to $s \geq 1$ leaves.

A set S of vertices of G is called a *dominating set* if $N[S] = \bigcup_{v\in S} N[v] = V(G)$. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G. Cockayne, Dreyer, S.M. Hedetniemi and S.T. Hedetniemi [7] introduced the concept of *Roman domination* in graphs, and since then a lot of related variations and generalizations has been studied (see, for example, the survey articles [5]-[6]). In this paper we continue the study of signed Roman domination in graphs (see, for example, [3]-[4],[8],[10]-[11]).

A *signed double Roman dominating function* (SDRDF) on a graph G is defined in [1, 2] as a function $f : V(G) \to \{-1, 1, 2, 3\}$ having the property that $f(N[v]) = \sum_{x\in N[v]} f(x) \geq 1$ for each $v \in V(G)$ and if $f(u) = -1$, then the vertex u must have a neighbor w with $f(w) = 3$ or two neighbors assigned 2 under f, and if $f(v) = 1$, then v must have at least one neighbor w with $f(w) \geq 2$. The weight of an SDRDF f is the value $f(V(G)) = \sum_{u\in V(G)} f(u)$. The *signed double Roman domination number* $\gamma_{sdR}(G)$ is the minimum weight of a signed double Roman dominating function on G. A $\gamma_{sdR}(G)$-function is an SDRDF of weight $\gamma_{sdR}(G)$.

A *weak signed double Roman dominating function* (WSDRDF) of a graph G is defined as a function $f : V(G) \to \{-1, 1, 2, 3\}$ having the property that $f(N[v]) \geq 1$ for each $v \in V(G)$. The weight of a WSDRDF is the value $\omega(f) = f(V(G))$. The *weak signed double Roman domination number* of G, denoted by $\gamma_{wsdR}(G)$, is the minimum weight of a WSDRDF in G. A $\gamma_{wsdR}(G)$-function is a WSDRDF of weight $\gamma_{wsdR}(G)$. For a WSDRDF f on G, let $V_i = \{v \in V(G) \mid f(v) = i\}$ for $i = -1, 1, 2, 3$. A WSDRDF f can be represented by the ordered partition $f = (V_{-1}, V_1, V_2, V_3)$.

The definitions lead to $\gamma_{wsdR}(G) \leq \gamma_{sdR}(G)$. Therefore each lower bound of $\gamma_{wsdR}(G)$ is also a lower bound of $\gamma_{sdR}(G)$, and each upper bound of $\gamma_{sdR}(G)$ is an upper bound of $\gamma_{wsdR}(G)$.

Our purpose in this work is to initiate the study of the weak signed double Roman domination number. We present basic properties and sharp bounds for the (weak) signed double Roman domination number of a graph. In particular, we show that many lower bounds on $\gamma_{sdR}(G)$ are also valid for $\gamma_{wsdR}(G)$. In addition, we show that the difference $\gamma_{sdR}(G) - \gamma_{wsdR}(G)$ can be arbitrarily large, and we determine the weak signed double Roman domination number of some classes of graphs.

We make use of the following known results.

Proposition 1. *[2] For $n \geq 5$ or $n = 3$, we have $\gamma_{sdR}(K_n) = 1$ and $\gamma_{sdR}(K_n) = 2$ for $n = 1, 2, 4$.*

Proposition 2. *[1]* If $n \geq 1$, then $\gamma_{sdR}(K_{1,n}) = 1$, unless $n = 1, 3$, in which cases $\gamma_{sdR}(K_{1,1}) = \gamma_{sdR}(K_{1,3}) = 2$.

Proposition 3. *[1]* Let P_n be a path of order $n \geq 2$. Then $\gamma_{sdR}(P_n) = n/3$ when $n \equiv 0 \pmod 3$ and $\gamma_{sdR}(P_n) = \lceil n/3\rceil + 1$ when $n \equiv 1, 2 \pmod 3$.

Proposition 4. *[2]* Let C_n be a cycle of length $n \geq 3$. Then $\gamma_{sdR}(C_n) = n/3$ when $n \equiv 0 \pmod 3$, $\gamma_{sdR}(C_n) = \lceil n/3 \rceil + 2$ when $n \equiv 1 \pmod 3$ and $\gamma_{sdR}(C_n) = \lceil n/3 \rceil + 1$ when $n \equiv 2 \pmod 3$.

Proposition 5. *[2]* If $2 \leq p \leq q$ are integers, then $\gamma_{sdR}(K_{2,q}) = 3$ when $q \geq 3$, $\gamma_{sdR}(K_{3,q}) = 5$ and $\gamma_{sdR}(K_{p,q}) = 4$ for $p \geq 4$ or $p = q = 2$.

2. Preliminary results and first bounds

In this section we present basic properties and some first bounds on the weak signed double Roman domination number. The definitions lead to the first observation immediately.

Observation 1. *If $f = (V_{-1}, V_1, V_2, V_3)$ is a WSDRDF of a graph G of order n, then the following holds.*

(a) $|V_{-1}| + |V_1| + |V_2| + |V_3| = n$.

(b) $\omega(f) = |V_1| + 2|V_2| + 3|V_3| - |V_{-1}|$.

(c) Every vertex of V_{-1} is dominated by one vertex of $V_2 \cup V_3$ or two vertices of V_1.

(d) $V_1 \cup V_2 \cup V_3$ is a dominating set of G.

The proof of the next proposition is identically with the proof of Proposition 2.2 in [2] and is therefore omitted.

Proposition 6. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a WSDRDF of a graph G of order n, $\Delta = \Delta(G)$ and $\delta = \delta(G)$. Then the following holds.

(a) $(3\Delta + 2)|V_3| + (2\Delta + 1)|V_2| + \Delta|V_1| \geq (\delta + 2)|V_{-1}|$.

(b) $(3\Delta + \delta + 4)|V_3| + (2\Delta + \delta + 3)|V_2| + (\Delta + \delta + 2)|V_1| \geq (\delta + 2)n$.

(c) $(\Delta + \delta + 2)\omega(f) \geq (\delta - \Delta + 2)n + (\delta - \Delta)|V_2| + 2(\delta - \Delta)|V_3|$.

(d) $\omega(f) \geq (\delta - 3\Delta)n/(3\Delta + \delta + 4) + |V_2| + 2|V_3|$.

As an immediate consequence of Proposition 6 (c), we obtain a lower bound on the weak signed double Roman domination number of regular graphs.

Corollary 1. If G is an r-regular graph of order n, then $\gamma_{wsdR}(G) \geq \lceil n/(r+1) \rceil$.

Proposition 7. If $n \geq 1$, then $\gamma_{wsdR}(K_n) = 1$.

Proof. According to Corollary 1, we have $\gamma_{wsdR}(K_n) \geq 1$. If n is even, then assign to one vertex the weight 2, to $n/2$ vertices the weight -1 and to the remaining $(n-2)/2$ vertices the weight 1. On the other hand, if n is odd, then assign to $(n+1)/2$ vertices the weight 1 and to the remaining $(n-1)/2$ vertices the weight -1. In both cases, we produce a WSDRDF of weight 1, and thus $\gamma_{wsdR}(K_n) \leq 1$ and so $\gamma_{wsdR}(K_n) = 1$. □

Proposition 8. If G is an $(n-2)$-regular graph of order $n \geq 4$, then $\gamma_{wsdR}(G) = 2$.

Proof. Since G is $(n-2)$-regular, the graph is isomorphic to the complete r-partite graph $K_{n_1,n_2,\ldots,n_r}$ with $r \geq 2$ and $n_1 = n_2 = \ldots = n_r = 2$. Corollary 1 implies $\gamma_{wsdR}(G) \geq \lceil n/(n-1) \rceil = 2$.
Now let $X_i = \{x_i, y_i\}$ be the partite sets of G for $1 \leq i \leq r$. Define $f(x_i) = f(y_1) = 1$ for $1 \leq i \leq r$ and $f(y_i) = -1$ for $2 \leq i \leq r$. Then f is a WSDRDF on G of weight 2 and thus $\gamma_{wsdR}(G) \leq 2$. Therefore $\gamma_{wsdR}(G) = 2$. □

Example 1. *Let H be the complete r-partite graph with $r \geq 2$ and the partite sets $X_1, X_2, \ldots, X_r$ such that $|X_1| = |\{a, b, u, v\}| = 4$ and $|X_i| = 3$ for $2 \leq i \leq r$. Now let G consisting of H with the additional edges ab and uv. Then G is an $(n-3)$-regular graph of order $n = 3r + 1$. Corollary 1 implies $\gamma_{sdR}(G) \geq \gamma_{wsdR}(G) \geq \lceil n/(n-2) \rceil = 2$.*
Now let $X_i = \{x_i, y_i, z_i\}$ be the partite sets of G for $2 \leq i \leq r$. Define $f(x_i) = f(a) = f(u) = 2$ for $2 \leq i \leq r$ and $f(b) = f(v) = f(y_i) = f(z_i) = -1$ for $2 \leq i \leq r$. Then f is a WSDRDF (even an SDRDF) on G of weight 2 and thus $\gamma_{wsdR}(G) \leq \gamma_{sdR}(G) \leq 2$. Therefore $\gamma_{sdR}(G) = \gamma_{wsdR}(G) = 2$.

Propositions 7, 8 and Example 1 show that Corollary 1 is sharp. If G is an r-regular graph of order n, then Corollay 1 implies the known bound $\gamma_{sdR}(G) \geq \lceil n/(r+1) \rceil$ (see [2]). Example 1 demonstrates that this bound is sharp too.

In the case that G is not regular, Proposition 6 (c) and (d) lead to the following lower bound.

Corollary 2. Let G be a graph of order n, maximum degree Δ and minimum degree δ. If $\delta < \Delta$, then

$$\gamma_{wsdR}(G) \geq \frac{-3\Delta + 3\delta + 4}{3\Delta + \delta + 4} n.$$

Proof. Multiplying both sides of the inequality in Proposition 6 (d) by $\Delta - \delta$ and adding the resulting inequality to the inequality in Proposition 6 (c), we yield the desired lower bound. □

Example 2. *Let $p \geq 2$ be an integer, and let $v_1, v_2, \ldots, v_p$ be the vertex set of the complete graph K_p. Let H_1 be the graph consisting of K_p such that each vertex v_i is adjacent to $3p-1$ leaves for $1 \leq i \leq p$. Then H_1 has $(3p-1)p$ leaves $b_1, b_2, \ldots, b_{(3p-1)p}$. Now let H be the graph consisting of H_1 together with the edges $b_1b_2, b_3b_4, \ldots, b_{(3p-1)p-1}b_{(3p-1)p}$. Then $n(H) = p + p(3p-1) = 3p^2$, $\Delta(H) = 3p - 1 + p - 1 = 4p - 2$ and $\delta(H) = 2$. Define the* *function* $f : V(H) \to \{-1, 1, 2, 3\}$ *by* $f(v_i) = 3$ *for* $1 \le i \le p$ *and* $f(x) = -1$ *otherwise. Then* f *is a WSDRDF on* H *of weight*

$$3p - (3p - 1)p = 4p - 3p^2 = \frac{-3\Delta(H) + 3\delta(H) + 4}{3\Delta(H) + \delta(H) + 4} n(H).$$

Therefore Corollary 2 shows that $\gamma_{wsdR}(H) = 4p - 3p^2$ *and thus Corollary 2 is sharp.*

Since f is also a signed double Roman dominating function on H, Example 2 also shows that the inequality

$$\gamma_{sdR}(G) \ge \frac{-3\Delta + 3\delta + 4}{3\Delta + \delta + 4} n,$$

which can be found in [2], and which follows from Corollary 2, is sharp too.
The next example will demonstrate that the difference $\gamma_{sdR}(G) - \gamma_{wsdR}(G)$ can be arbitrarily large.

Example 3. *Let* $p \ge 2$ *be an integer, and let* $v_1, v_2, \ldots, v_p$ *be the vertex set of the complete graph* K_p. *Let* H *be the graph consisting of* K_p *such that each vertex* v_i *is adjacent to* $2p - 1$ *leaves for* $1 \le i \le p$. *Then* $n(H) = p + p(2p - 1) = 2p^2$. *Define the function* $f : V(H) \to \{-1, 1, 2, 3\}$ *by* $f(v_i) = 2$ *for* $1 \le i \le p$ *and* $f(x) = -1$ *otherwise. Then* f *is a WSDRDF on* H *of weight* $2p-(2p-1)p = 3p-2p^2$ *and thus* $\gamma_{wsdR}(H) \le 3p-2p^2$. *In fact we have* $\gamma_{wsdR}(H) = 3p-2p^2$. *On the other hand, let* $a_i^1, a_i^2, \ldots, a_i^{2p-1}$ *be the leaves of* v_i, *and let* g *be a* $\gamma_{sdR}(H)$*-function. If* $g(v_i) = -1$, *then* $g(a_i^j) \ge 2$ *for* $1 \le j \le 2p-1$, *if* $g(v_i) = 1$, *then* $g(a_i^j) \ge 1$ *for* $1 \le j \le 2p-1$, *if* $g(v_i) = 2$, *then* $g(a_i^j) \ge 1$ *for* $1 \le j \le 2p-1$, *and if* $g(v_i) = 3$, *then* $g(a_i^j) \ge -1$ *for* $1 \le j \le 2p - 1$. *This leads to* $\gamma_{sdR}(H) = \omega(g) \ge p(4 - 2p) = 4p - 2p^2$, *even* $\gamma_{sdR}(H) = 4p - 2p^2$.
Consequently, we deduce that $\gamma_{sdR}(H) - \gamma_{wsdR}(H) \ge p$.

Proposition 9. If G is a graph of order n, then $\gamma_{wsdR}(G) \le n$, with equality if and only if $G = \overline{K_n}$.

Proof. Define the function $f : V(G) \to \{-1, 1, 2, 3\}$ by $f(v) = 1$ for each $v \in V(G)$. Then f is a WSDRDF on G of weight n and thus $\gamma_{wsdR}(G) \le n$. If $G = \overline{K_n}$, then $\gamma_{wsdR}(G) = n$ is obviously.
Conversely, assume that $\Delta(G) \ge 1$. Then G contains a component H with $\delta(H) \ge 1$. If $\delta(H) = 1$, then there exists a vertex $v \in V(H)$ with $d(v) = 1$. If u is a neighbor of v, then define the function $f : V(G) \to \{-1, 1, 2, 3\}$ by $f(v) = -1$, $f(u) = 2$ and $f(x) = 1$ for $x \in V(G) \setminus \{u, v\}$. It is straightforward to verify that f is a WSDRDF on G of weight $n-1$ and thus $\gamma_{wsdR}(G) \le n-1$. If $\delta(H) \ge 2$, then define $f(w) = -1$ for an arbitrary vertex $w \in V(H)$ and $f(x) = 1$ for $x \in V(G) \setminus \{w\}$. Then f is a WSDRDF on G of weight $n - 2$ and thus $\gamma_{wsdR}(G) \le n - 2$. □

Theorem 2. If G is a connected graph of order $n \ge 2$, then $\gamma_{wsdR}(G) = n - 1$ if and only if $G = K_2$.

Proof. If $G = K_2$, then Proposition 7 implies $\gamma_{wsdR}(G) = 1 = n - 1$.
Conversely, assume that $\gamma_{wsdR}(G) = n - 1$. If $n = 2$, then $G = K_2$ and we are done. Assume next that $n \geq 3$. If $\delta(G) \geq 2$, then, as in the proof of Proposition 9, we have $\gamma_{wsdR}(G) \leq n - 2$, a contradiction with $\gamma_{wsdR}(G) = n - 1$. Let now $\delta(G) = 1$. If G has a strong support vertex v with leaf neighbors u_1, u_2, then the function g defined on G by $g(v) = 3$, $g(u_1) = g(u_2) = -1$ and $g(x) = 1$ for the remaining vertices, is a WSDRDF on G of weight $n - 2$, a contradiction. Thus G has no strong support vertex. Let u be a vertex of degree 1 in G and u' be the neighbor of u in G. Assume that T is a spanning tree of G. Then T has at least two leaves. Let v be a leaf of T different from u and v' the support vertex of v in T. If $u' = v'$, then the function g defined on G by $g(u') = 3$, $g(u) = g(v) = -1$ and $g(x) = 1$ for the remaining vertices, is a WSDRDF on G and so $\gamma_{wsdR}(G) \leq n - 2$, a contradiction with $\gamma_{wsdR}(G) = n - 1$. Hence $u' \neq v'$. In this case define the function g on G by $g(u') = g(v') = 2$, $g(u) = g(v) = -1$ and $g(x) = 1$ for the remaining vertices. It is not hard to see that g is a WSDRDF on G and so $\gamma_{wsdR}(G) \leq n-2$, a contradiction. This completes the proof. □

Corollary 3. Let G be a graph of order $n \geq 2$. Then $\gamma_{wsdR}(G) = n - 1$ if and only if $G = K_2 \cup (n - 2)K_1$.

Proof. If $G = K_2 \cup (n - 2)K_1$, then we deduce from Propositions 7 and 9 that

$$\gamma_{wsdR}(G) = \gamma_{wsdR}(K_2) + \sum_{v \in V(G)-V(K_2)} \gamma_{wsdR}(K_1) = n - 1,$$

as desired.
Conversely, assume that $\gamma_{wsdR}(G) = n-1$. If $n = 2$, then $G = K_2$ and $\gamma_{wsdR}(G) = 1$, as desired. Let $n \geq 3$. Theorem 2 implies that G is disconnected. If G has two components of order at least two, say G_1, G_2, then Proposition 9 and Theorem 2 lead to

$$\begin{aligned}\gamma_{wsdR}(G) &= \gamma_{wsdR}(G_1) + \gamma_{wsdR}(G_2) + \gamma_{wsdR}(G - (G_1 \cup G_2)) \\ &\leq (n(G_1) - 1) + (n(G_2) - 1) + n(G - (G_1 \cup G_2)) \\ &\leq n - 2,\end{aligned}$$

a contradiction with $\gamma_{wsdR}(G) = n - 1$. If G has a component of order at least three, say G_1, then it follows from Proposition 9 and Theorem 2 that

$$\begin{aligned}\gamma_{wsdR}(G) &= \gamma_{wsdR}(G_1) + \gamma_{wsdR}(G - G_1) \\ &\leq (n(G_1) - 2) + n(G - G_1) \\ &\leq n - 2,\end{aligned}$$

a contradiction with $\gamma_{wsdR}(G) = n - 1$. This completes the proof. □

Theorem 3. If G is a graph of order n, then $\gamma_{wsdR}(G) \geq 2\gamma(G) - n$, with equality if and only if $G = \overline{K_n}$.

Proof. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma_{wsdR}(G)$-function. Then it follows from Observation 1 that

$$\begin{aligned}\gamma_{wsdR}(G) &= |V_1| + 2|V_2| + 3|V_3| - |V_{-1}| = 2|V_1| + 3|V_2| + 4|V_3| - n \\ &\geq 2|V_1 \cup V_2 \cup V_3| - n \geq 2\gamma(G) - n, \end{aligned} \tag{2.1}$$

and the desired inequality is proved. Clearly, if $G = \overline{K_n}$, then $\gamma_{wsdR}(G) = n = 2\gamma(G) - n$. Now assume that G contains at least one edge. Using Proposition 9, we observe that $\gamma_{wsdR}(G) \leq n - 1$ and therefore $|V_{-1}| \geq 1$. If $|V_2 \cup V_3| \geq 1$, then it follows from (2.1) that

$$\gamma_{wsdR}(G) = 2|V_1| + 3|V_2| + 4|V_3| - n > 2|V_1 \cup V_2 \cup V_3| - n \geq 2\gamma(G) - n.$$

Therefore assume now that $|V_2 \cup V_3| = 0$. Let $u \in V_{-1}$, and let $x, y \in V_1$ be two neighbors of u. The condition $f(N[x]) \geq 1$ shows that x has a neighbor in $V_1 \setminus \{x\}$. Furthermore, since every vertex of V_{-1} has at least two neighbors in V_1, we conclude that $V_1 \setminus \{x\}$ is a dominating set of G. Hence we deduce from (2.1) that

$$\gamma_{wsdR}(G) = 2|V_1| - n > 2\gamma(G) - n.$$

□

Proposition 10. If G is a graph of order n with minimum degree $\delta \geq 2$, then $\gamma_{wsdR}(G) \leq n - 2\lfloor \delta/2 \rfloor$.

Proof. Let $t = \lfloor \delta/2 \rfloor$, and let $A = \{v_1, v_2, \ldots, v_t\}$ be a set of t vertices of G. Define the function $f : V(G) \to \{-1, 1, 2, 3\}$ by $f(x) = -1$ for $x \in A$ and $f(x) = 1$ for $x \in V(G) \setminus A$. Then

$$f(N[w]) \geq -t + (\delta + 1 - t) = \delta + 1 - 2t = \delta + 1 - 2\lfloor \delta/2 \rfloor \geq 1$$

for each $w \in V(G)$. Therefore f is a WSDRDF on G of weight $n - 2t$ and thus $\gamma_{wsdR}(G) \leq n - 2t$. □

For odd $n \geq 3$, Proposition 7 shows that Proposition 10 is sharp, and for even $n \geq 4$, Proposition 8 shows that Proposition 10 is sharp.

Proposition 11. If G is a graph of order n, then $\gamma_{wsdR}(G) \geq \Delta(G) + 2 - n$.

Proof. Let w be a vertex of maximum degree, and let f be a $\gamma_{wsdR}(G)$-function. Then the definitions imply the desired bound as follows:

$$\begin{aligned}\gamma_{wsdR}(G) &= \sum_{x\in V(G)} f(x) = \sum_{x\in N[w]} f(x) + \sum_{x\in V(G)\setminus N[w]} f(x) \\ &\geq 1 + \sum_{x\in V(G)\setminus N[w]} f(x) \geq 1-(n-(\Delta(G)+1)) = \Delta(G)+2-n.\end{aligned}$$

□

Corollary 4. *[2] If G is a graph of order n, then $\gamma_{sdR}(G) \geq \Delta(G)+2-n$.*

Example 2 shows that Proposition 11 and Corollary 4 are sharp.

3. Bounds on $\gamma_{sdR}(G)$

In this section we present more bounds on the signed double Roman domination number.

Theorem 4. If $G \neq K_4$ is a connected graph of order n and minimum degree $\delta \geq 3$, then

$$\gamma_{sdR}(G) \leq 2n-2\delta-1.$$

Proof. We proceed with three cases.

Case 1. Assume that $\delta = 3p$ with an integer $p \geq 1$. Let $q = 2p$, and let $A = \{v_1, v_2 \ldots, v_q\}$ be a set of q vertices of G. Define the function $f : V(G) \to \{-1, 1, 2, 3\}$ by $f(x) = -1$ for $x \in A$, $f(u) = 1$ for a vertex $u \in V(G) \setminus A$ and $f(x) = 2$ for $x \in V(G) \setminus (A \cup \{u\})$. Then

$$f(N[w]) \geq -q+1+2(\delta-q) = 2\delta-3q+1 = 6p-6p+1 = 1$$

for each $w \in V(G)$. In addition, the vertex u has at least $\delta - q = 3p - 2p = p \geq 1$ neighbors of weight 2, and each v_i has at least $\delta - q = p$ neighbors of weight 2. Therefore f is an SDRDF on G of weight $2(n-q-1)+1-q = 2n-3q-1 = 2n-2\delta-1$ when $p \geq 2$.
Let now $p = 1$, that means $\delta = 3$. Since $G \neq K_4$, we observe that we can choose $A = \{v_1, v_2\}$ as an independent set. If we define f as above, then we observe that each v_i has at least two neighbors of weight 2. Thus f is also an SDRDF on G of weight $2n-7 = 2n-2\delta-1$. Altogether we see that $\gamma_{sdR}(G) \leq 2n-2\delta-1$ in the first case.

Case 2. Assume that $\delta = 3p + 1$ with an integer $p \geq 1$. Let $q = 2p + 1$, and let $A = \{v_1, v_2, \ldots, v_q\}$ be a set of q vertices of G. Define the function $f : V(G) \to \{-1, 1, 2, 3\}$ by $f(x) = -1$ for $x \in A$ and $f(x) = 2$ for $x \in V(G) \setminus A$. Then

$$f(N[w]) \geq -q + 2(\delta + 1 - q) = 2\delta - 3q + 2 = 6p + 4 - 6p - 3 = 1$$

for each $w \in V(G)$, and each v_i has at least $\delta-(q-1) = 3p+1-2p = p+1 \geq 2$ neighbors of weight 2. Thus f is an SDRDF on G of weight $2(n-q)-q = 2n-3q = 2n-2\delta-1$, and so $\gamma_{sdR}(G) \leq 2n - 2\delta - 1$ in the second case too.

Case 3. Assume that $\delta = 3p + 2$ with an integer $p \geq 1$. Let $q = 2p + 1$, and let $A = \{v_1, v_2, \ldots, v_q\}$ be a set of q vertices of G. Define the function $f : V(G) \to \{-1, 1, 2, 3\}$ by $f(x) = -1$ for $x \in A$, $f(u) = f(z) = 1$ for two different vertices $u, z \in V(G) \setminus A$ and $f(x) = 2$ for $x \in V(G) \setminus (A \cup \{u, z\})$. Then

$$f(N[w]) \geq -q + 2 + 2(\delta - q - 1) = 2\delta - 3q = 6p + 4 - 6p - 3 = 1$$

for each $w \in V(G)$. In addition, the vertices u and z have at least $\delta - q - 1 = 3p+2-2p-1-1 = p \geq 1$ neighbors of weight 2, and each v_i has at least $\delta - q - 1 = p$ neighbors of weight 2. Therefore f is an SDRDF on G of weight $2(n-q-2)+2-q = 2n - 3q - 2 = 2n - 2\delta - 1$ when $p \geq 2$.

Let now $p = 1$, that means $\delta = 5$. If G is complete, then Proposition 1 implies $\gamma_{sdR}(K_6) = 1 = 2n - 2\delta - 1$. If G is not complete, then let v_1 and v_2 be two non-adjacent vertices. Now let $A = \{v_1, v_2, v_3\}$ be a set of 3 vertices with an arbitary vertex $v_3 \in V(G) \setminus \{v_1, v_2\}$. Since $\delta = 5$ and v_1 and v_2 are not adjacent, the vertex v_3 has at least three neighbors $a_1, a_2, a_3 \in V(G) \setminus A$, and v_1 has a neighbor $b \notin \{a_1, a_2, a_3, v_2, v_3\}$. Define the function $f : V(G) \to \{-1, 1, 2, 3\}$ by $f(x) = -1$ for $x \in A$, $f(b) = 1$ and $f(c) = 1$ for a further vertex $c \in V(G) \setminus (A \cup \{b\})$ and $f(x) = 2$ for $x \in V(G) \setminus (A \cup \{b, c\})$. Then each vertex v_i has at least two neighbors of weight 2, and as above, we observe that f is an SDRDF on G of weight $2n - 2\delta - 1$. This completes the proof. □

If $n \geq 5$, then Proposition 1 shows that Theorem 4 is sharp. If $n \geq 6$ and $\delta \geq \frac{n-1}{2}$, then Theorem 4 shows that $\gamma_{sdR}(G) \leq n$. I think that this is valid for all connected graphs.

Conjecture 1. If G is a connected graph of order $n \geq 2$, then $\gamma_{sdR}(G) \leq n$.

Proposition 12. If G is an $(n-2)$-regular graph of order $n \geq 6$, then $\gamma_{sdR}(G) = 3$.

Proof. Since G is $(n-2)$-regular, the graph is isomorphic to the complete r-partite graph $K_{n_1,n_2,\ldots,n_r}$ with $r \geq 3$ and $n_1 = n_2 = \ldots = n_r = 2$. Let $X_i = \{u_i, v_i\}$ be the partite sets of G for $1 \leq i \leq r$. Define the function $f : V(G) \to \{-1, 1, 2, 3\}$ by

$f(u_i) = -1$ for $1 \leq i \leq r$, $f(v_1) = f(v_2) = f(v_3) = 2$ and $f(v_i) = 1$ for $4 \leq i \leq r$. Then f is an SDRDF on G of weight 3 and thus $\gamma_{sdR}(G) \leq 3$.
Suppose on the contrary that $\gamma_{sdR}(G) \leq 2$. Let f be a $\gamma_{sdR}(G)$-function. Assume that there exists a vertex, say u_1, with $f(u_1) = 3$. If $f(v_1) = a$ for $a \in \{-1, 1, 2, 3\}$, then the condition $f(N[v_1]) \geq 1$ implies $f(V(G) \setminus \{u_1, v_1\}) \geq 1 - a$. This leads to the contradiction $2 \geq \gamma_{sdR}(G) = f(V(G)) \geq 1 - a + f(u_1) + f(v_1) = 4$. Next assume that there exists a vertex, say u_1, with $f(u_1) = 2$. If $f(v_1) = a$ for $a \in \{-1, 1, 2, 3\}$, then $f(V(G) \setminus \{u_1, v_1\}) \geq 1 - a$. Thus $2 \geq \gamma_{sdR}(G) = f(V(G)) \geq 1 - a + f(u_1) + f(v_1) = 3$, a contradiction. Consequently, $f(u_i), f(v_i) \leq 1$ for all $1 \leq i \leq r$, but this is impossible. Hence $\gamma_{sdR}(G) \geq 3$ and so $\gamma_{sdR}(G) = 3$. □

For $n = 2p$ with $p \geq 3$, the $(n-2)$-regular graphs in Proposition 12 are further examples which demonstrate the sharpness of Theorem 4. Next we improve Theorem 4 for small δ, more precisely for $\delta < \frac{n+2}{2}$.

Theorem 5. If G is a connected graph of order $n \geq 3$, then $\gamma_{sdR}(G) \leq \lfloor \frac{3n}{2} \rfloor - 2$.

Proof. Assume that T is a spanning tree of G. Let X, Y be the bipartite sets of T with $|Y| \leq |X|$. We proceed with two cases.
Case 1. Assume that $|Y| = 1$ with $Y = \{y\}$. If $X = \{x_1, x_2, \ldots, x_{n-1}\}$, then define the function f by $f(y) = 3$, $f(x_1) = f(x_2) = -1$ and $f(x_i) = 1$ for $3 \leq i \leq n-1$. Then f is an SDRDF on T of weight $n-2$. Since f is also an SDRDF on G, we deduce that

$$\gamma_{sdR}(G) \leq n - 2 \leq \left\lfloor \frac{3n}{2} \right\rfloor - 2.$$

Case 2. Assume that $|Y| \geq 2$ with $Y = \{y_1, y_2, \ldots, y_t\}$ and $X = \{x_1, x_2, \ldots, x_{n-t}\}$. Since T is connected, we observe that there exists a vertex, say x_1, with $d_T(x_1) \geq 2$. Now define the function f by $f(x_1) = -1$, $f(x_i) = 1$ for $2 \leq i \leq n-t$ and $f(y_j) = 2$ for $1 \leq j \leq t$. Then f is an SDRDF on T of weight $2t + n - t - 2 = n + t - 2$. Since f is also an SDRDF on G, we deduce that

$$\gamma_{sdR}(G) \leq n + t - 2 \leq \frac{3n}{2} - 2,$$

and since $\gamma_{sdR}(G)$ is an integer, we obtain the desired bound. □

A set $S \subseteq V(G)$ is a 2-*packing* of the graph G if $N[u] \cap N[v] = \emptyset$ for any two distinct vertices $u, v \in S$. The maximum cardinality of a 2-packing in G is the 2-*packing number*, denoted by $\rho(G) = \rho$.

Proposition 13. *[2]* If G is a graph of order n, minimum degree δ and packing number ρ, then $\gamma_{sdR}(G) \geq \rho(\delta + 2) - n$.

The proof of the next result is identically with the proof of Proposition 3.8 in [2] and is therefore omitted.

Proposition 14. If G is a graph of order n, minimum degree δ and packing number ρ, then $\gamma_{wsdR}(G) \geq \rho(\delta + 2) - n$.

The next example will demonstrate that the bounds in Propositions 13 and 14 are sharp.

Example 4. *Let F be an arbitrary graph of order $t \geq 1$, and for each vertex $v \in V(F)$ add a vertex-disjoint copy of a complete graph K_s ($s \geq 5$) and identify the vertex v with one vertex of the added complete graph. Let H denote the resulting graph. Furthermore, let $H_1, H_2, \ldots, H_t$ be the added copies of K_s. For $i = 1, 2, \ldots, t$, let v_i be the vertex of H_i that is identified with a vertex of F. We now construct an SDRDF on H as follows. For each $i = 1, 2, \ldots, t$, let $f_i : V(H_i) \to \{-1, 1, 2, 3\}$ be the SDRDF on the complete graph defined as in the proof of Proposition 4.1 in [2] such that $f_i(v_i) \geq 1$. As shown in Proposition 4.1 in [2], we have $\omega(f_i) = 1$. Now let $f : V(H) \to \{-1, 1, 2, 3\}$ be the function defined by $f(v) = f_i(v)$ for each $v \in V(H_i)$. Then f is an SDRDF of H of weight t and hence $\gamma_{wsdR}(H) \leq \gamma_{sdR}(H) \leq t$. Since $n(H) = ts$, $\delta(H) = s - 1$ and $\rho(H) = t$, Proposition 14 implies that $\gamma_{sdR}(H) \geq \gamma_{wsRd}(H) \geq \rho(H)(\delta(H)+2) - n(H) = t$. Consequently, $\gamma_{wsdR}(H) = \gamma_{sdR}(H) = \rho(H)(\delta(H) + 2) - n(H) = t$.*

4. Special classes of graphs

In this section, we determine the weak signed double Roman domination number for special classes of graphs.

Proposition 15. If $n \geq 1$, then $\gamma_{wsdR}(K_{1,n}) = 1$.

Proof. Let w be the central vertex of $G = K_{1,n}$, and let f be a $\gamma_{wsdR}(G)$-function. The definitions imply $\gamma_{wsdR}(G) = \sum_{x \in N[w]} f(x) \geq 1$. For $n \neq 1, 3$ it follows from Proposition 2 that $\gamma_{wsdR}(G) \leq \gamma_{sdR}(G) = 1$. Therefore $\gamma_{wsdR}(G) = 1$ for $n \neq 1, 3$. Since it is straightforward to verify that $\gamma_{wsdR}(K_{1,1}) = \gamma_{wsdR}(K_{1,3}) = 1$, the proof is complete. □

Proposition 16. If C_n is a cycle of length $n \geq 3$, then $\gamma_{wsdR}(C_n) = \lceil n/3 \rceil$ when $n \equiv 0, 1 \pmod 3$ and $\gamma_{wsdR}(C_n) = \lceil n/3 \rceil + 1$ when $n \equiv 2 \pmod 3$.

Proof. Let $C_n = v_1 v_2 \ldots v_n v_1$. Applying Corollary 1, we observe that $\gamma_{wsdR}(C_n) \geq \lceil n/3 \rceil$.
Let first $n = 3p$ for an integer $p \geq 1$. Define $f(v_{3i}) = -1$ and $f(v_{3i-2}) = f(v_{3i-1}) = 1$ for $1 \leq i \leq p$. Then f is a WSDRDF on C_n of weight $p = n/3$ and thus $\gamma_{wsdR}(C_n) \leq p$. Therefore $\gamma_{wsdR}(C_n) = \lceil n/3 \rceil$ in this case.
Let second $n = 3p+1$ for an integer $p \geq 1$. Define $f(v_{3i}) = -1$, $f(v_{3i-2}) = f(v_{3i-1}) = 1$ for $1 \leq i \leq p$ and $f(v_{3p+1}) = 1$. Then f is a WSDRDF on C_n of weight $p+1 = \lceil n/3 \rceil$ and thus $\gamma_{wsdR}(C_n) \leq \lceil n/3 \rceil$. Therefore $\gamma_{wsdR}(C_n) = \lceil n/3 \rceil$ in this case.

Finally, let $n = 3p + 2$ for an integer $p \geq 1$, and let f be a $\gamma_{wsdR}(C_n)$-function. If $f(x) \geq 1$ for each $x \in V(C_n)$, then $\gamma_{wsdR}(C_n) \geq n \geq \lceil n/3 \rceil + 1$. Let next, without loss of generality, $f(v_1) = -1$. Then v_1 has a positive neighbor, say v_2. If $f(v_2) = 3$, then $f(v_1) + f(v_2) = 2$. If $f(v_2) = 2$ or $f(v_2) = 1$, then $f(v_3) \geq 1$ and thus $f(v_2) + f(v_3) \geq 2$. Hence we have found two adjacent vertices v_j and v_{j+1} such that $f(v_j) + f(v_{j+1}) \geq 2$. If we assume, without loss of gerality, that $f(v_{3p+1}) + f(v_{3p+2}) \geq 2$, then we obtain

$$\gamma_{wsdR}(C_n) = f(V(C_n)) = f(v_{3p+1}) + f(v_{3p+2}) + \sum_{i=1}^{p} f(N[v_{3i-1}]) \geq p + 2 = \left\lceil \frac{n}{3} \right\rceil + 1.$$

Otherwise, define $g(v_{3i}) = -1$, $g(v_{3i-2}) = g(v_{3i-1}) = 1$ for $1 \leq i \leq p$ and $g(v_{3p+1}) = g(v_{3p+2}) = 1$. Then g is a WSDRDF on C_n of weight $p + 2 = \lceil n/3 \rceil + 1$ and thus $\gamma_{wsdR}(C_n) \leq \lceil n/3 \rceil + 1$. Therefore, $\gamma_{wsdR}(C_n) = \lceil n/3 \rceil + 1$ in this case. □

Proposition 17. Let P_n be a path of order $n \geq 1$. Then $\gamma_{wsdR}(P_2) = 1$, $\gamma_{wsdR}(P_n) = \lceil n/3 \rceil$ when $n \equiv 0, 1 \pmod 3$ and $\gamma_{wsdR}(P_n) = \lceil n/3 \rceil + 1$ when $n \equiv 2 \pmod 3$ and $n \geq 3$.

Proof. Let $P_n = v_1 v_2 \ldots v_n$, and let f be a $\gamma_{wsdR}(P_n)$-function.
First assume that $n = 3p + 1$ for an integer $p \geq 0$. If $p = 0$, then the result is trivial. If $p \geq 1$, then we observe that $f(v_1) + f(v_2) \geq 1$ and $f(v_{3p}) + f(v_{3p+1}) \geq 1$. This leads to

$$\begin{aligned}\gamma_{wsdR}(P_n) &= f(V(P_n)) = f(v_1) + f(v_2) + f(v_{3p}) + f(v_{3p+1}) + \sum_{i=1}^{p-1} f(N[v_{3i+1}]) \\ &\geq 2 + p - 1 = p + 1 = \left\lceil \frac{n}{3} \right\rceil.\end{aligned}$$

Next define $g(v_{3i+1}) = -1$ for $0 \leq i \leq p$, $g(v_2) = g(v_{3p}) = 2$ and $g(v_{3i}) = g(v_{3i+2}) = 1$ for $1 \leq i \leq p - 1$. Then g is a WSDRDF on P_n of weight $p + 1 = \lceil n/3 \rceil$ and thus $\gamma_{wsdR}(P_n) \leq \lceil n/3 \rceil$. Therefore $\gamma_{wsdR}(P_n) = \lceil n/3 \rceil$ in this case.
Next let $n = 3p$ for an integer $p \geq 1$. According to Propostion 3, we note that $\gamma_{wsdR}(P_n) \leq \gamma_{sdR}(P_n) = n/3$. Furthermore, we observe that

$$\gamma_{wsdR}(P_n) = f(V(P_n)) = \sum_{i=0}^{p-1} f(N[v_{3i+2}]) \geq p = n/3$$

and thus $\gamma_{wsdR}(P_n) = n/3$ in this case.
Finally, asume that $n = 3p + 2$ for an integer $p \geq 0$. Clearly, $\gamma_{wsdR}(P_2) = 1$. Let now $p \geq 1$. If $f(v_1) + f(v_2) \geq 2$, then we have

$$\gamma_{wsdR}(P_n) = f(V(P_n)) = f(v_1) + f(v_2) + \sum_{i=1}^{p} f(N[v_{3i+1}]) \geq 2 + p = p + 2 = \left\lceil \frac{n}{3} \right\rceil + 1.$$

Next assume that $f(v_1)+f(v_2)=1$. Then $f(v_1)=-1$ and $f(v_2)=2$ or $f(v_1)=2$ and $f(v_2)=-1$. In both cases we observe that $f(v_3)\geq 1$ and so $f(v_1)+f(v_2)+f(v_3)\geq 2$. This leads to

$$\begin{aligned}\gamma_{wsdR}(P_n) &= f(V(P_n))\\ &= f(v_1)+f(v_2)+f(v_3)+\sum_{i=1}^{p-1} f(N[v_{3i+2}])+f(v_{3p+1})+f(v_{3p+2})\\ &\geq 2+p-1+1=p+2=\left\lceil\frac{n}{3}\right\rceil+1.\end{aligned}$$

Conversely, Proposition 3 yields $\gamma_{wsdR}(P_n)\leq\gamma_{sdR}(P_n)=\lceil n/3\rceil+1$ and thus $\gamma_{wsdR}(P_n)=\lceil n/3\rceil+1$ in the last case. □

Proposition 18. Let $S(r,s)$ be a double star with $1\leq r\leq s$.

(a) If $7\leq r$, then $\gamma_{wsdR}(S(r,s))=-4$.

(b) $\gamma_{wsdR}(S(6,s))=-3$.

(c) $\gamma_{wsdR}(S(5,s))=-4$ for $s\neq 6$ and $\gamma_{wsdR}(S(5,6))=-3$.

(d) $\gamma_{wsdR}(S(4,s))=-3$.

(e) $\gamma_{wsdR}(S(3,s))=-2$.

(f) $\gamma_{wsdR}(S(2,s))=-1$ for $s\geq 3$ and $\gamma_{wsdR}(S(2,2))=0$.

(g) $\gamma_{wsdR}(S(1,s))=0$ for $s\geq 3$, $\gamma_{wsdR}(S(1,1))=2$ and $\gamma_{wsdR}(S(1,2))=1$.

Proof. Let u and v be two adjacent vertices of $S(r,s)$ such that u is adjacent to $r\geq 1$ leaves and v is adjacent to $s\geq 1$ leaves. If f is a $\gamma_{wsdR}(S(r,s))$-function, then the definition implies

$$\omega(f)=f(N[u])+f(N[v])-f(u)-f(v)\geq 1+1-3-3=-4$$

and so $\gamma_{wsdR}(S(r,s))\geq -4$.

Let $u_1,u_2,\ldots,u_r$ be the leaves adjacent to u and $v_1,v_2,\ldots,u_s$ be the leaves adjacent to v.

(a) Let $r\geq 7$. Define the function $g:V(G)\to\{-1,1,2,3\}$ by $g(u)=g(v)=3$. Furthermore, if $r=2t$ even for an integer $t\geq 4$, then let $g(u_1)=2$, $g(u_2)=g(u_3)=\ldots=g(u_{t-3})=1$ and $g(u_{t-2})=g(u_{t-1})=\ldots=g(u_{2t})=-1$ and if $r=2t+1\geq 7$ is odd for an integer $t\geq 3$, then let $g(u_1)=g(u_2)=\ldots=g(u_{t-2})=1$ and $g(u_{t-1})=g(u_t)=\ldots=g(u_{2t+1})=-1$. In addition, if $s=2t$ is even for an integer $t\geq 4$, then let $g(v_1)=2$, $g(v_2)=g(v_3)=\ldots=g(v_{t-3})=1$ and $g(v_{t-2})=g(v_{t-1})=\ldots=g(v_{2t})=-1$ and if $s=2t{+}1$ is odd for an integer $t\geq 3$, then let $g(v_1)=g(v_2)=\ldots=g(v_{t-2})=1$ and $g(v_{t-1})=g(v_t)=\ldots=g(v_{2t+1})=-1$.

Then g is a WSDRDF on $S(r, s)$ of weight -4 and therefore $\gamma_{wsdR}(S(r, s)) = -4$ in this case.

(b) Let $r = 6$. Let f be a $\gamma_{wsdR}(S(6, s))$-function. If $f(u) + f(v) \leq 5$, then $\omega(f) = f(N[u]) + f(N[v]) - f(u) - f(v) \geq 1 + 1 - 5 = -3$. However, if $f(u) + f(v) = 6$, then $f(N[u]) \geq 7 - 5 = 2$ and so $\omega(f) = f(N[u]) + f(N[v]) - f(u) - f(v) \geq 2 + 1 - 6 = -3$. Conversely, define the function $g : V(G) \rightarrow \{-1, 1, 2, 3\}$ by $g(u_1) = 1$ and $g(u_i) = -1$ for $2 \leq i \leq 6$.
Futhermore, if $s = 2t + 1$ is odd for an integer $t \geq 3$, then let $g(u) = g(v) = 3$, $g(v_1) = g(v_2) = \ldots = g(v_{t-2}) = 1$ and $g(v_{t-1}) = g(v_t) = \ldots = g(v_{2t+1}) = -1$. Then g is a WSDRDF on $S(r, s)$ of weight -3 and therefore $\gamma_{wsdR}(S(r, s)) = -3$ in this case.
If $s = 2t$ is even for an integer $t \geq 3$, then let $g(u) = 2$, $g(v) = 3$, $g(v_1) = g(v_2) = \ldots = g(v_{t-2}) = 1$ and $g(v_{t-1}) = g(v_t) = \ldots = g(v_{2t}) = -1$. Again g is a WSDRDF on $S(r, s)$ of weight -3 and therefore $\gamma_{wsdR}(S(r, s)) = -3$ also in this case.

(c) $r = 5$. If $s \neq 6$, then define the function $g : V(G) \rightarrow \{-1, 1, 2, 3\}$ by $g(u) = g(v) = 3$ and $g(u_i) = -1$ for $1 \leq i \leq 5$. In addition, if $s = 5$, then define $g(v_i) = -1$ for $1 \leq i \leq 5$. If $s = 2t + 1$ odd for an integer $t \geq 3$, then define $g(v_1) = g(v_2) = \ldots = g(v_{t-2}) = 1$ and $g(v_{t-1}) = g(v_t) = \ldots = g(v_{2t+1}) = -1$. If $s = 2t$ is even for an integer $t \geq 4$, then define $g(v_1) = 2$, $g(v_2) = \ldots = g(v_{t-3}) = 1$ and $g(v_{t-2}) = g(v_{t-1}) = \ldots = g(v_{2t}) = -1$. In all these cases, we observe that g is a WSDRDF on $S(5, s)$ of weight -4 and therefore $\gamma_{wsdR}(S(r, s)) = -4$ in these cases.
It is straightforward to verify that $\gamma_{wsdR}(S(5, 6)) = -3$.

(d) Let $r = 4$. If f is a $\gamma_{wsdR}(S(4, s))$-function, then we observe as in Case (b) that $\gamma_{wsdR}(S(4, s)) \geq -3$.
Conversely, define the function $g : V(G) \rightarrow \{-1, 1, 2, 3\}$ by $g(u_i) = -1$ for $1 \leq i \leq 4$. If $s = 2t$ is even for an integer $t \geq 2$, then define $g(u) = 2$, $g(v) = 3$, $g(v_1) = g(v_2) = \ldots = g(v_{t+2}) = -1$ and $g(v_{t+3}) = g(v_{t+4}) = \ldots = g(v_{2t}) = 1$. If $s = 2t + 1$ is odd for an integer $t \geq 2$, then define $g(u) = g(v) = 3$, $g(v_1) = g(v_2) = \ldots = g(v_{t+3}) = -1$ and $g(v_{t+4}) = g(v_{t+5}) = \ldots = g(v_{2t+1}) = 1$. In both cases, g is a WSDRDF on $S(4, s)$ of weight -3 and therefore $\gamma_{wsdR}(S(4, s)) = -3$.

(e) Let $r = 3$. Let f be a $\gamma_{wsdR}(S(3, s))$-function. If $f(u) + f(v) \leq 4$, then $\omega(f) = f(N[u]) + f(N[v]) - f(u) - f(v) \geq 1 + 1 - 4 = -2$. If $f(u) + f(v) = 5$, then $f(N[u]) \geq 5 - 3 = 2$ and so $\omega(f) = f(N[u]) + f(N[v]) - f(u) - f(v) \geq 2 + 1 - 5 = -2$. If $f(u) + f(v) = 6$, then $f(N[u]) \geq 6 - 3 = 3$ and so $\omega(f) = f(N[u]) + f(N[v]) - f(u) - f(v) \geq 3 + 1 - 6 = -2$.
Conversely, define the function $g : V(G) \rightarrow \{-1, 1, 2, 3\}$ by $g(u_i) = -1$ for $1 \leq i \leq 3$. If $s = 3$, then define $g(u) = g(v) = 2$ and $g(v_i) = -1$ for $1 \leq i \leq 3$. Then g is a WSDRDF on $S(3, 3)$ of weight -2 and thus $\gamma_{wsdR}(S(3, 3)) = -2$.
If $s = 2t$ is even for an integer $t \geq 2$, then define $g(u) = 2$, $g(v) = 3$,

$g(v_1) = g(v_2) = \ldots = g(v_{t+2}) = -1$ and $g(v_{t+3}) = g(v_{t+4}) = \ldots = g(v_{2t}) = 1$. If $s = 2t + 1$ is odd for an integer $t \geq 2$, then define $g(u) = g(v) = 3$, $g(v_1) = g(v_2) = \ldots = g(v_{t+3}) = -1$ and $g(v_{t+4}) = g(v_{t+5}) = \ldots = g(v_{2t+1}) = 1$. In both cases, g is a WSDRDF on $S(3, s)$ of weight -2 and therefore $\gamma_{wsdR}(S(3, s)) = -2$.

(f) Let $r = 2$. Let f be a $\gamma_{wsdR}(S(2, s))$-function. If $f(u) + f(v) \leq 3$, then $\omega(f) = f(N[u]) + f(N[v]) - f(u) - f(v) \geq 1 + 1 - 3 = -1$. If $f(u) + f(v) = 4$, then $f(N[u]) \geq 2$ and so $\omega(f) = f(N[u]) + f(N[v]) - f(u) - f(v) \geq 2 + 1 - 4 = -1$. If $f(u) + f(v) = 5$, then $f(N[u]) \geq 5 - 2 = 3$ and so $\omega(f) = f(N[u]) + f(N[v]) - f(u) - f(v) \geq 3 + 1 - 5 = -1$. If $f(u) + f(v) = 6$, then $f(N[u]) \geq 6 - 2 = 4$ and so $\omega(f) = f(N[u]) + f(N[v]) - f(u) - f(v) \geq 4 + 1 - 6 = -1$.
Conversely, define the function $g : V(G) \to \{-1, 1, 2, 3\}$ by $g(u_i) = -1$ for $1 \leq i \leq 2$. It is straightforward to verify that $\gamma_{wsdR}(S(2, 2)) = 0$ and $\gamma_{wsdR}(S(2, 3)) = -1$ Let next $s \geq 4$.
If $s = 2t$ is even for an integer $t \geq 2$, then define $g(u) = 2$, $g(v) = 3$, $g(v_1) = g(v_2) = \ldots = g(v_{t+2}) = -1$ and $g(v_{t+3}) = g(v_{t+4}) = \ldots = g(v_{2t}) = 1$. If $s = 2t + 1$ is odd for an integer $t \geq 2$, then define $g(u) = g(v) = 3$, $g(v_1) = g(v_2) = \ldots = g(v_{t+3}) = -1$ and $g(v_{t+4}) = g(v_{t+5}) = \ldots = g(v_{2t+1}) = 1$. As above, g is a WSDRDF on $S(2, s)$ of weight -1 and therefore $\gamma_{wsdR}(S(2, s)) = -1$ also in these cases.

(g) Let $r = 1$. Let f be a $\gamma_{wsdR}(S(1, s))$-function. If $f(u) + f(v) \leq 2$, then $\omega(f) = f(N[u]) + f(N[v]) - f(u) - f(v) \geq 1 + 1 - 2 = 0$. If $f(u) + f(v) = 3$, then $f(N[u]) \geq 2$ and so $\omega(f) = f(N[u]) + f(N[v]) - f(u) - f(v) \geq 2 + 1 - 3 = 0$. If $f(u) + f(v) = 4$, then $f(N[u]) \geq 3$ and so $\omega(f) = f(N[u]) + f(N[v]) - f(u) - f(v) \geq 3 + 1 - 4 = 0$. If $f(u) + f(v) = 5$, then $f(N[u]) \geq 4$ and so $\omega(f) = f(N[u]) + f(N[v]) - f(u) - f(v) \geq 4 + 1 - 5 = 0$. If $f(u) + f(v) = 6$, then $f(N[u]) \geq 5$ and so $\omega(f) = f(N[u]) + f(N[v]) - f(u) - f(v) \geq 5 + 1 - 6 = 0$.
Furthermore, it is a simple matter to verify that $\gamma_{wsdR}(S(1, 1)) = 2$, $\gamma_{wsdR}(S(1, 2)) = 1$ and $\gamma_{wsdR}(S(1, 3)) = 0$. Let now $s \geq 4$.
Define the function $g : V(G) \to \{-1, 1, 2, 3\}$ by $g(u_1) = -1$. If $s = 2t$ is even for an integer $t \geq 2$, then define $g(u) = 2$, $g(v) = 3$, $g(v_1) = g(v_2) = \ldots = g(v_{t+2}) = -1$ and $g(v_{t+3}) = g(v_{t+4}) = \ldots = g(v_{2t}) = 1$. If $s = 2t + 1$ is odd for an integer $t \geq 2$, then define $g(u) = g(v) = 3$, $g(v_1) = g(v_2) = \ldots = g(v_{t+3}) = -1$ and $g(v_{t+4}) = g(v_{t+5}) = \ldots = g(v_{2t+1}) = 1$. Again, g is a WSDRDF on $S(1, s)$ of weight 0 and therefore $\gamma_{wsdR}(S(1, s)) = 0$ for $s \geq 4$. □

Proposition 19. If $2 \leq p \leq q$ are integers, then $\gamma_{wsdR}(K_{2,q}) = 2$, $\gamma_{wsdR}(K_{3,q}) = 3$ and $\gamma_{wsdR}(K_{p,q}) = 4$ for $p \geq 4$.

Proof. Let $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$ be the bipartite sets of $K_{p,q}$, and let f be a $\gamma_{wsdR}(K_{p,q})$-function. If there exists a vertex $x \in X$ and $y \in Y$ with $f(x) = f(y) = -1$, then the property $f(N[x]) \geq 1$ implies $f(Y) \geq 2$ and the

property $f(N[y]) \geq 1$ implies $f(X) \geq 2$ and thus $\omega(f) \geq 4$.
First let $p \geq 4$. Assume that $f(x) \geq 1$ for each $x \in X$. Then

$$\omega(f) = f(N[x_1]) + \sum_{i=2}^{p} f(x_i) \geq 1 + (p-1) = p \geq 4.$$

Analogously we see that $\omega(f) \geq 4$ when $f(y) \geq 1$ for each $y \in Y$. Therefore we obtain $\omega(f) \geq 4$ in each case. It follows from Proposition 5 that $\gamma_{wsdR}(K_{p,q}) \leq \gamma_{sdR}(K_{p,q}) = 4$ and thus $\gamma_{wsdR}(K_{p,q}) = 4$ in this case.
Next let $p = 3$. As above we observe that $\omega(f) \geq 3$ when $f(x) \geq 1$ for each $x \in X$ or $f(y) \geq 1$ for each $y \in Y$. Therefore $\omega(f) \geq 3$ in each case. Now define the function $g : V(G) \to \{-1, 1, 2, 3\}$ as follows. Let $g(x_1) = g(x_2) = g(x_3) = 1$ and if $Y = \{y_1, y_2, ..., y_{2t}\}$ for an integer $t \geq 2$, then let $g(y_1) = g(y_2) = \ldots = g(y_t) = 1$ and $g(y_{t+1}) = g(y_{t+2}) = \ldots = g(y_{2t}) = -1$ and if $Y = \{y_1, y_2, ..., y_{2t+1}\}$ for an integer $t \geq 1$, then let $g(y_1) = 2$, $g(y_2) = g(y_3) = \ldots = g(y_t) = 1$ and $g(y_{t+1}) = g(y_{t+2}) = \ldots = g(y_{2t+1}) = -1$. Then g is a WSDRDF on $K_{3,q}$ of weight 3 and thus $\gamma_{wsdR}(K_{3,q}) \leq 3$. This yields to $\gamma_{wsdR}(K_{3,q}) = 3$.
Finally, let $p = 2$. As above, we see that $\omega(f) \geq 2$ when $f(x) \geq 1$ for each $x \in X$ or $f(y) \geq 1$ for each $y \in Y$. Therefore $\omega(f) \geq 2$ in each case. Now define the function $g : V(G) \to \{-1, 1, 2, 3\}$ as follows. Let $g(x_1) = g(x_2) = 1$ and if $Y = \{y_1, y_2, ..., y_{2t}\}$ for an integer $t \geq 1$, then let $g(y_1) = g(y_2) = \ldots = g(y_t) = 1$ and $g(y_{t+1}) = g(y_{t+2}) = \ldots = g(y_{2t}) = -1$ and if $Y = \{y_1, y_2, ..., y_{2t+1}\}$ for an integer $t \geq 1$, then let $g(y_1) = 2$, $g(y_2) = g(y_3) = \ldots = g(y_t) = 1$ and $g(y_{t+1}) = g(y_{t+2}) = \ldots = g(y_{2t+1}) = -1$. Then g is a WSDRDF on $K_{2,q}$ of weight 2 and thus $\gamma_{wsdR}(K_{2,q}) \leq 2$. This leads to $\gamma_{wsdR}(K_{2,q}) = 2$. □

Proposition 20. If $G = K_{n_1,n_2,\ldots,n_r}$ is a complete r-partite graph with $3 \leq n_1 \leq n_2 \leq \ldots \leq n_r$ and $r \geq 3$, then $\gamma_{wsdR}(G) = \gamma_{sdR}(G) = 3$.

Proof. Let $X_i = \{v_1^i, v_2^i, \ldots, v_{n_i}^i\}$ be the partite sets of G for $1 \leq i \leq r$, and let f be a $\gamma_{wsdR}(G)$-function. Assume that there exists a partite set X_k such $f(v_i^k) \geq 1$ for each $1 \leq i \leq n_k$. Then

$$\omega(f) = f(N[v_1^k]) + \sum_{i=2}^{n_k} f(v_i^k) \geq 1 + (n_k - 1) = n_k \geq 3.$$

Next assume that each partite set X_i contains a vertex x_i with $f(x_i) = -1$. Then $f(N[x_i]) \geq 1$ yields $f(V(G) \setminus X_i) \geq 2$ for each $1 \leq i \leq r$. Therefore

$$(r-1)\omega(f) = (r-1)f(V(G)) = \sum_{i=1}^{r} f(V(G) \setminus X_i) \geq 2r$$

and thus

$$\omega(f) = f(V(G)) \geq \left\lceil \frac{2r}{r-1} \right\rceil = 3.$$

Consequently, $\gamma_{sdR}(G) \geq \gamma_{wsdR}(G) = \omega(f) \geq 3$ in every case.
Conversely, we shall define a function $f : V(G) \to \{-1, 1, 2, 3\}$ such that $f(X_1) = f(X_2) = f(X_3) = 1$ so that $f(x_i) \geq 2$ for at least one vertex $x_i \in X_i$ for $i = 1, 2, 3$ and $f(X_i) = 0$ for $4 \leq i \leq r$.
First let $i \in \{1, 2, 3\}$. If $X_i = \{v_i^1, v_i^2, v_i^3\}$, then define $f(v_i^1) = 3$ and $f(v_i^2) = f(v_i^3) = -1$. If $X_i = \{v_i^1, v_i^2, ..., v_i^{2p}\}$ for an integer $p \geq 2$, then define $f(v_i^1) = 2$, $f(v_i^2) = f(v_i^3) = \ldots = f(v_i^p) = 1$ and $f(v_i^{p+1}) = f(v_i^{p+2}) = \ldots = f(v_i^{2p}) = -1$. If $X_i = \{v_i^1, v_i^2, ..., v_i^{2p+1}\}$ for an integer $p \geq 2$, then define $f(v_i^1) = f(v_i^2) = 2$, $f(v_i^3) = f(v_i^4) = \ldots = f(v_i^p) = 1$ and $f(v_i^{p+1}) = f(v_i^{p+2}) = \ldots = f(v_i^{2p+1}) = -1$.
Now let $i \geq 4$. If $X_i = \{v_i^1, v_i^2, ..., v_i^{2p}\}$ for an integer $p \geq 2$, then define $f(v_i^1) = f(v_i^2) = \ldots = f(v_i^p) = 1$ and $f(v_i^{p+1}) = f(v_i^{p+2}) = \ldots = f(v_i^{2p}) = -1$. If $X_i = \{v_i^1, v_i^2, ..., v_i^{2p+1}\}$ for an integer $p \geq 1$, then define $f(v_i^1) = 2$, $f(v_i^2) = f(v_i^3) = \ldots = f(v_i^p) = 1$ and $f(v_i^{p+1}) = f(v_i^{p+2}) = \ldots = f(v_i^{2p+1}) = -1$.
We observe that f is an SDRDF on G of weight 3 and thus $\gamma_{wsdR}(G) \leq \gamma_{sdR}(G) \leq 3$.
Altogether, we have $\gamma_{wsdR}(G) = \gamma_{sdR}(G) = 3$. □

Proposition 21. If $G = K_{n_1,n_2,\ldots,n_r}$ is an r-partite graph with $2 = n_1 \leq n_2 \leq \ldots \leq n_r$ and $r \geq 3$, then $\gamma_{wsdR}(G) = 2$.

Proof. Let $X_i = \{v_1^i, v_2^i, \ldots, v_{n_i}^i\}$ be the partite sets of G for $1 \leq i \leq r$, and let f be a $\gamma_{wsdR}(G)$-function. Assume that there exists a partite set X_k such $f(v_i^k) \geq 1$ for each $1 \leq i \leq n_k$. Then

$$\omega(f) = f(N[v_1^k]) + \sum_{i=2}^{n_k} f(v_i^k) \geq 1 + (n_k - 1) = n_k \geq 2.$$

Next assume that each partite set X_i contains a vertex x_i with $f(x_i) = -1$. As in the proof of Proposition 20, we obtain $\omega(f) \geq 3$ in this case. Consequently, $\gamma_{wsdR}(G) = \omega(f) \geq 2$ in every case.
Conversely, we define the function $g : V(G) \to \{-1, 1, 2, 3\}$ as follows. Let $g(v_1^1) = g(v_2^1) = 1$. Let now $i \geq 2$. If $X_i = \{v_i^1, v_i^2, ..., v_i^{2p}\}$ for an integer $p \geq 1$, then define $g(v_i^1) = g(v_i^2) = \ldots = g(v_i^p) = 1$ and $g(v_i^{p+1}) = g(v_i^{p+2}) = \ldots = g(v_i^{2p}) = -1$. If $X_i = \{v_i^1, v_i^2, ..., v_i^{2p+1}\}$ for an integer $p \geq 1$, then define $g(v_i^1) = 2$, $g(v_i^2) = g(v_i^3) = \ldots = g(v_i^p) = 1$ and $g(v_i^{p+1}) = g(v_i^{p+2}) = \ldots = g(v_i^{2p+1}) = -1$. Then g is a WSDRDF on G of weight 2 and thus $\gamma_{wsdR}(G) \leq 2$. Consequently, $\gamma_{wsdR}(G) = 2$. □

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