Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-18
https://doi.org/10.22049/cco.2024.30037.2287

CCO
Commun. Comb. Optim.

Research Article

On norms, spread, characteristic polynomial and determinant of Hankel and Toeplitz matrices with Mersenne sequence

Kalika Prasad[1,†], Munesh Kumari[1,*], Jagmohan Tanti[2]

[1]Department of Applied Science and Humanities (Mathematics),
Government Engineering College Bhojpur, Bihar, India, 802301
[†]klkaprsd@gmail.com
[*]muneshnasir94@gmail.com

[2]Department of Mathematics, Babasaheb Bhimrao Ambedkar University, Lucknow, India, 226025
jagmohan.t@gmail.com

Received: 2 October 2024; Accepted: 16 December 2024
Published Online: 28 December 2024

Abstract: In this work, some new properties of the Hankel and Toeplitz matrices are obtained by considering the Mersenne numbers as entries. We developed efficient formulas to compute matrix norms like $\|.\|_1$, $\|.\|_\infty$, Euclidean norm, spread, and the lower and upper bound for the spectral norm of these matrices. Also, the study shows that these matrices are non-singular for $n = 2$ and singular for $n \geq 3$. Furthermore, we presented rank, eigenvalues, principal minors, and the characteristic polynomial of them explicitly.

Keywords: Mersenne and Fermat numbers, Hankel matrices, Toeplitz matrices, matrix norms, spread, rank, characteristic polynomial, determinant.

AMS Subject classification: 11B39, 11B37

1. Introduction

Let $\{t_n\}_{n\in\mathbb{Z}}$ and $\{h_n\}_{n\in\mathbb{Z}}$ be infinite sequences, then Toeplitz and Hankel matrices of order n with the entries $t_{ij} = t_{i-j}$ and $h_{ij} = h_{i+j-2}$, respectively, are defined as

$$T_n = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{2-n} & t_{1-n} \\ t_1 & t_0 & t_{-1} & \cdots & t_{3-n} & t_{2-n} \\ t_2 & t_1 & t_0 & \cdots & t_{4-n} & t_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{n-2} & t_{n-3} & t_{n-4} & \cdots & t_0 & t_{-1} \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_1 & t_0 \end{bmatrix} \quad \text{and} \quad H_n = \begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_{n-2} & h_{n-1} \\ h_1 & h_2 & h_3 & \cdots & h_{n-1} & h_n \\ h_2 & h_3 & h_4 & \cdots & h_n & h_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{n-2} & h_{n-1} & h_n & \cdots & h_{2n-4} & h_{2n-3} \\ h_{n-1} & h_n & h_{n+1} & \cdots & h_{2n-3} & h_{2n-2} \end{bmatrix}.$$

* *Corresponding Author*

Many researchers have worked on these special matrices involving a recursive sequence like Fibonacci, Lucas, Pell, balancing numbers, etc. in the last decades and still, it is of great interest among researchers. For instance, Akbulak and Bozkurt [1] have obtained the norms for the Toeplitz matrices with entries from Fibonacci and Lucas numbers. Then S. Shen [19] and A. Daşdemir [6] extended this study to the k-Fibonacci and k-Lucas numbers and Pell and Pell-Lucas numbers, respectively. Also, Solak and Bahsi [20] obtained the norms and bounds for the spectral norm of the Hankel matrices involving the Fibonacci and Lucas numbers. This study has been extended for other number sequences, one can see [3, 9, 10, 15, 21, 22, 24]. These types of special matrices have wide applications in various areas like image processing, vibration analysis, cryptography etc. [14, 16, 23].

In this study, we consider the Mersenne numbers $(2^n - 1)$ and Fermat numbers $(2^n + 1)$ as our sequence of entries. The Mersenne and Fermat numbers [17] are special numbers that intrigued mathematicians for centuries. A recurrence relation for these numbers was provided by A.F. Horadam in 1979 [7]. Recently, the recurrence relation and some curious properties of these numbers were revisited by Catarino et al. [4].

Definition 1. For $n \geq 0$, the Mersenne numbers $\{M_n\}$ and Fermat numbers $\{R_n\}$ are defined by the same relation

$$Z_n = 3Z_{n-1} - 2Z_{n-2} \tag{1.1}$$

with initial assumptions $M_0 = 0,\ M_1 = 1$ and $R_0 = 2,\ \ R_1 = 3$.

The Binet's formulae for these numbers is given as

$$M_n = 2^n - 1 \tag{1.2}$$

$$\text{and} \quad R_n = 2^n + 1. \tag{1.3}$$

Note that the sequence (1.1) can be extended in the negative direction too. So, the Binet's formulae in negative subscript is given by $M_{-n} = (1 - 2^n)/2^n$ and $R_{-n} = (1 + 2^n)/2^n$.

After the work of Saba et al. [18], the name 'Mersenne-Lucas numbers' is also used for the Fermat numbers. A study on 'k-Mersenne-Lucas numbers' is reported by Chelgham and Boussayoud [5] which generalizes Mersenne-Lucas numbers.

Now, we give some preliminaries for different norms of any rectangular matrix $A = [a_{ij}] \in \mathbb{R}^{m \times n}$. Maximum absolute column sum (1-norm) and row sum (∞-norm) norms [25] for the matrix A are given as

$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}| \quad \text{and} \quad \|A\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|, \quad \text{respectively.} \tag{1.4}$$

The Euclidean (Frobenius) and spectral norm ([8], Ch-5) for matrix A are defined as

$$\|A\|_E = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2} \quad \text{and} \quad \|A\|_2 = \sqrt{\max_{1\le i\le n} \mu_i(A^*A)}, \quad \text{respectively,} \tag{1.5}$$

where $\mu_i(A^*A)$ denotes the eigenvalues of A^*A and A^* is the conjugate transpose of A. And for matrix A, these norms are related as

$$\frac{1}{\sqrt{n}}\|A\|_E \le \|A\|_2 \le \|A\|_E. \tag{1.6}$$

Lemma 1. *[12] Let $A = [a_{ij}] \in \mathbb{M}_{m\times n}(\mathbb{C})$, $B = [b_{ij}] \in \mathbb{M}_{m\times n}(\mathbb{C})$ be two matrices and C be the Hadamard product of A and B (i.e $C = A \circ B$), then we have*

$$\|C\|_2 \le u(A)\nu(B), \tag{1.7}$$

where $u(A) = \max_{1\le i\le m} \sqrt{\sum_{j=1}^{n}|a_{ij}|^2}$ and $\nu(B) = \max_{1\le j\le n} \sqrt{\sum_{i=1}^{m}|b_{ij}|^2}$.

This study aims to investigate the properties of Hankel and Toeplitz matrices defined with the entries from Mersenne or Fermat sequences. We present matrix norms, spread and obtain the lower and upper bounds for the spectral norm of these matrices. Further, we use the beautiful property of Mersenne numbers to check about the singularity of these matrices and present rank, eigenvalues, principal minors and the characteristic polynomial of them explicitly.

2. Hankel and Toeplitz matrices with Mersenne sequence

This section starts with partial sum formulae for Mersenne and Fermat numbers that will be used to establish our main results. Further, we discuss different norms and bounds for the spectral norm on these matrices.

2.1. Partial sum formulae

Lemma 2. *[11] The partial sum formulae for the squares of these numbers are*

$$\sum_{j=0}^{n} M_j^2 = \frac{M_{2n+2} - 6M_{n+1} + 3(n+1)}{3} = \frac{4^{n+1}+8}{3} - 2^{n+2} + n$$

$$\text{and} \qquad \sum_{j=0}^{n} R_j^2 = \frac{R_{2n+2} + 6R_{n+1} + 3n - 11}{3} = \frac{2^{2n+2}-4}{3} + 2^{n+2} + n.$$

Proof. These identities can be easily verified using Binet's formula (1.2) and (1.3). □

Lemma 3. *The partial sum of the squared terms of these numbers with negative subscripts are given by*

$$\sum_{j=0}^{n-1} M_{-j}^2 = \frac{2^{2-2n}M_{2n} - 2^{2-n}3M_n + 3n}{3} \quad \text{and}$$
$$\sum_{j=0}^{n-1} R_{-j}^2 = \frac{2^{2-2n}R_{2n} + 2^{2-n}3R_n + 3n - 2^{3-2n}(1+2^n3)}{3}.$$

Proof. The proof follows from the Binet's formulae of Mersenne and Fermat numbers with negative subscripts which are $(1-2^n)/2^n$ and $(1+2^n)/2^n$, respectively. □

Lemma 4. *For fixed $m \in \mathbb{Z}$, the finite sum formulae for terms in arithmetic indices are*

$$\sum_{k=0}^{n-1} M_{m+k}^2 = \frac{2^{2m}M_{2n} - 6M_n2^m + 3n}{3} \quad \text{and}$$
$$\sum_{k=0}^{n-1} R_{m+k}^2 = \frac{2^{2m}R_{2n} + 6R_n2^m - 2^{m+1}(2^m+6) + 3n}{3}.$$

Proof. Using the Binet's formula for Mersenne numbers, we write

$$\begin{aligned}\sum_{k=0}^{n-1} M_{m+k}^2 &= \sum_{k=0}^{n-1}(2^{m+k}-1)^2 = \sum_{k=0}^{n-1}(2^{2(m+k)}+1-2^{m+k+1}) \\ &= 2^{2m}\left(\frac{2^{2n}-1}{2^2-1}\right) + n - 2^{m+1}\left(\frac{2^n-1}{2-1}\right) \\ &= \frac{2^{2m}M_{2n} - 6M_n2^m + 3n}{3}.\end{aligned}$$

The second identity appears by a similar argument using Binet's formula (1.3). □

2.2. Matrix norms

For $n \geq 2$, let $MH_n = (m_{ij})_{i,j=1}^n$ with $m_{ij} = M_{i+j-2}$ and $RH_n = (r_{ij})_{i,j=1}^n$ with $r_{ij} = R_{i+j-2}$ be the $n \times n$ Mersenne and Fermat Hankel matrices, respectively. Then these matrices have the following structure:

$$MH_n = \begin{bmatrix} M_0 & M_1 & M_2 & \cdots & M_{n-2} & M_{n-1} \\ M_1 & M_2 & M_3 & \cdots & M_{n-1} & M_n \\ M_2 & M_3 & M_4 & \cdots & M_n & M_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{n-2} & M_{n-1} & M_n & \cdots & M_{2n-4} & M_{2n-3} \\ M_{n-1} & M_n & M_{n+1} & \cdots & M_{2n-3} & M_{2n-2} \end{bmatrix} \text{ and } RH_n = \begin{bmatrix} R_0 & R_1 & R_2 & \cdots & R_{n-2} & R_{n-1} \\ R_1 & R_2 & R_3 & \cdots & R_{n-1} & R_n \\ R_2 & R_3 & R_4 & \cdots & R_n & R_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ R_{n-2} & R_{n-1} & R_n & \cdots & R_{2n-4} & R_{2n-3} \\ R_{n-1} & R_n & R_{n+1} & \cdots & R_{2n-3} & R_{2n-2} \end{bmatrix}.$$

Now, we give the norms $\|.\|_1$, $\|.\|_\infty$ and the lower and upper bounds for the spectral norm of these matrices.

Observation 1. The maximum absolute column sum norm ($\|.\|_1$) and row sum norm ($\|.\|_\infty$) for Mersenne Hankel matrices MH_n are

$$\|MH_n\|_1 = \|MH_n\|_\infty = M_{2n-1} - M_{n-1} - n.$$

Proof. For matrix MH_n, $\max_j \sum_{i=1}^{n}|m_{ij}| = \max_i \sum_{j=1}^{n}|m_{ij}| = \sum_{k=n-1}^{2n-2} M_k$. And thus using (1.4) and sum identity $\sum_{k=0}^{n-1} M_k = M_n - n$ [4, Prop. 2.5], we have the required result. □

Theorem 2. *The maximum absolute column and row sum norm for the matrix RH_n are given as*

$$\|RH_n\|_1 = \|RH_n\|_\infty = R_{2n-1} - R_{n-1} + n.$$

Proof. The proof follows from (1.4) and using sum identity $\sum_{k=0}^{n-1} R_k = R_n + (n-2)$ [5, Theorem 3.3]. □

Before proceeding to the next theorem, it is worthful to give the Euclidean norm of Mersenne Hankel matrices that will be useful to obtain the lower bound for the spectral norm.

Theorem 3. *The Euclidean norm of the Mersenne Hankel matrices is*

$$\|MH_n\|_E = \Big(\frac{M_{2n}^2 - 18M_n^2 + 9n^2}{9}\Big)^{1/2}. \tag{2.1}$$

Proof. By the definition of the Euclidean norm (1.5), we have

$$\begin{aligned}
\|MH_n\|_E^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}|m_{ij}|^2 = \sum_{k=0}^{n-1} M_k^2 + \sum_{k=1}^{n} M_k^2 + \sum_{k=2}^{n+1} M_k^2 + ... + \sum_{k=n-1}^{2n-2} M_k^2 \\
&= \sum_{k=0}^{n-1} M_k^2 + \sum_{k=0}^{n-1} M_{k+1}^2 + \sum_{k=0}^{n-1} M_{k+2}^2 + ... + \sum_{k=0}^{n-1} M_{k+(n-1)}^2 \\
&= \sum_{k=0}^{n-1}\sum_{s=0}^{n-1} M_{k+s}^2 = \sum_{k=0}^{n-1} \frac{2^{2k}M_{2n} - 2^k 6M_n + 3n}{3} \qquad \text{(using Lemma 4)} \\
&= \frac{M_{2n}}{3}\sum_{k=0}^{n-1} 2^{2k} - \frac{6M_n}{3}\sum_{k=0}^{n-1} 2^k + \sum_{k=0}^{n-1} n = \frac{M_{2n}^2 - 18M_n^2 + 9n^2}{9}.
\end{aligned}$$

Thus,

$$\|MH_n\|_E = \Big(\frac{M_{2n}^2 - 18M_n^2 + 9n^2}{9}\Big)^{1/2}.$$

□

Theorem 4. *The lower and upper bound for the spectral norm of the Mersenne Hankel matrices are*

$$\|MH_n\|_2 \geq \sqrt{\frac{M_{2n}^2 - 18M_n^2 + 9n^2}{9n}},$$
$$\|MH_n\|_2 \leq \frac{1}{3}\sqrt{(2^{2n-2}M_{2n} - 3M_n 2^n + 3n)(2^{2n-2}M_{2n-2} - 2^n 3M_{n-1} + 3n)}.$$

Proof. From Theorem 3 and inequality (1.6), we have

$$\|MH_n\|_2 \geq \sqrt{\frac{M_{2n}^2 - 18M_n^2 + 9n^2}{9n}}.$$

In order to obtain the upper bound for the spectral norm, we use Lemma 1, where the matrix MH_n is written as the Hadamard product of two matrices X and Y, defined as:

$$X = [x_{ij}] = \begin{cases} x_{ij} = 1, & i < j \\ x_{ij} = M_{i+j-2}, & i \geq j \end{cases} \text{ and } Y = [y_{ij}] = \begin{cases} y_{ij} = M_{i+j-2}, & i < j \\ y_{ij} = 1, & i \geq j \end{cases}.$$

Clearly, $MH_n = X \circ Y$. Since

$$u(X) = \max_i \sqrt{\sum_{j=1}^{n} |x_{ij}|^2} = \sqrt{\sum_{k=n-1}^{2n-2} M_k^2} = \sqrt{\sum_{k=0}^{2n-2} M_k^2 - \sum_{k=0}^{n-2} M_k^2},$$

using the sum identity from Lemma 2 and the Binet's formula (1.2), we have

$$\begin{aligned} u(X) &= \sqrt{\frac{M_{2(2n-1)} - 6M_{(2n-1)} + 3(2n-1)}{3} - \frac{M_{2(n-1)} - 6M_{(n-1)} + 3(n-1)}{3}} \\ &= \sqrt{\frac{M_{2(2n-1)} - 6M_{(2n-1)} - M_{2(n-1)} + 6M_{(n-1)} + 3n}{3}} \\ &= \sqrt{\frac{2^{2n-2}M_{2n} - 2^n 3M_n + 3n}{3}} \end{aligned}$$

and

$$\begin{aligned} \nu(Y) &= \max_j \sqrt{\sum_{i=1}^{n} |y_{ij}|^2} = \sqrt{1 + \sum_{k=n-1}^{2n-3} M_k^2} \\ &= \sqrt{1 + \frac{2^{2n-2}M_{2n-2} - 2^n 3M_{n-1} + 3(n-1)}{3}} \\ &= \sqrt{\frac{2^{2n-2}M_{2n-2} - 2^n 3M_{n-1} + 3n}{3}}. \end{aligned}$$

Hence from Lemma 1, we get

$$\|MH_n\|_2 \leq u(X)\nu(Y) = \frac{1}{3}\sqrt{(2^{2n-2}M_{2n} - 3M_n 2^n + 3n)(2^{2n-2}M_{2n-2} - 2^n 3M_{n-1} + 3n)}. \qquad \square$$

Theorem 5. *The Euclidean norm of the Fermat Hankel matrices is*

$$\|RH_n\|_E = \Big(\frac{R_{2n}^2 + 18R_n^2 - 4R_{2n} - 72R_n + 9n^2 + 76}{9}\Big)^{1/2}.$$

Proof. The proof is very similar to Theorem 3 using Lemma 4.

□

Theorem 6. *The lower and upper bound for the spectral norm of the Fermat Hankel matrices are*

$$\|RH_n\|_2 \geq \Big(\frac{R_{2n}^2 + 18R_n^2 - 4R_{2n} - 72R_n + 9n^2 + 76}{9n}\Big)^{1/2},$$
$$\|RH_n\|_2 \leq \frac{1}{3}\Big((2^{2n-2}M_{2n} + 2^n 3M_n + 3n)(2^{2n-2}M_{2n-2} + 2^n 3M_{n-1} + 3n)\Big)^{1/2}.$$

Proof. From Theorem 5 and the inequality $\frac{1}{\sqrt{n}}\|RH_n\|_E \leq \|RH_n\|_2$, the lower bound is given as

$$\|RH_n\|_2 \geq \Big(\frac{R_{2n}^2 + 18R_n^2 - 4R_{2n} - 72R_n + 9n^2 + 76}{9n}\Big)^{1/2}.$$

For the upper bound, the argument is very similar to Theorem 4.

□

Example 1. Verify the obtained results of matrix norms for the Mersenne and Fermat Hankel matrices of order 4.

Solution. Here Hankel matrices are

$$MH_4 = \begin{bmatrix} 0 & 1 & 3 & 7 \\ 1 & 3 & 7 & 15 \\ 3 & 7 & 15 & 31 \\ 7 & 15 & 31 & 63 \end{bmatrix} \quad \text{and} \quad RH_4 = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 3 & 5 & 9 & 17 \\ 5 & 9 & 17 & 33 \\ 9 & 17 & 33 & 65 \end{bmatrix}.$$

So

$$\|MH_4\|_1 = 116, \quad \|MH_4\|_\infty = 116 \quad \text{and} \quad \|MH_4\|_E = \sqrt{6791}$$
$$\|RH_4\|_1 = 124, \quad \|RH_4\|_\infty = 124 \quad \text{and} \quad \|RH_4\|_E = \sqrt{7691}$$

which verify the results of $\|.\|_1$ and $\|.\|_\infty$. Now for $\|.\|_E$, Theorem 3 and 5 give

$$\|MH_4\|_E = \Big(\frac{M_8^2 - 18M_4^2 + 9(4)^2}{9}\Big)^{1/2} = \sqrt{6791} = 82.4075 \qquad \text{and}$$
$$\|RH_4\|_E = \Big(\frac{R_8^2 + 18R_4^2 - 4R_8 - 72R_4 + 9(4)^2 + 76}{9}\Big)^{1/2} = \sqrt{7691} = 87.6983.$$

And for $\|.\|_2$, since the largest eigenvalue of MH_4 is $(81 + \sqrt{7021})/2 \sim 82.3957$ which is less than 82.4075 and greater than 41.2037 so it satisfies (1.6). Similarly, the largest eigenvalue of RH_4 is 87.6885 which satisfies (1.6). □

2.3. Toeplitz matrices

For $n \geq 2$, the Toeplitz matrices with Mersenne and Fermat numbers are defined as $MT_n = (m_{ij})_{i,j=1}^n$ with $m_{ij} = M_{i-j}$ and $RT_n = (r_{ij})_{i,j=1}^n$ with $r_{ij} = R_{i-j}$ and it takes the form

$$MT_n = \begin{bmatrix} M_0 & M_{-1} & M_{-2} & \cdots & M_{1-n} \\ M_1 & M_0 & M_{-1} & \cdots & M_{2-n} \\ M_2 & M_1 & M_0 & \cdots & M_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n-2} & M_{n-3} & M_{n-4} & \cdots & M_{-1} \\ M_{n-1} & M_{n-2} & M_{n-3} & \cdots & M_0 \end{bmatrix} \quad \text{and} \quad RT_n = \begin{bmatrix} R_0 & R_{-1} & R_{-2} & \cdots & R_{1-n} \\ R_1 & R_0 & R_{-1} & \cdots & R_{2-n} \\ R_2 & R_1 & R_0 & \cdots & R_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{n-2} & R_{n-3} & R_{n-4} & \cdots & R_{-1} \\ R_{n-1} & R_{n-2} & R_{n-3} & \cdots & R_0 \end{bmatrix},$$

The Toeplitz matrices using generalized Mersenne numbers are discussed by Sevda and Soykan in [2], so here we list some norms properties which we use later to establish other results like Spread, etc. These results can be proved by modifications of results from [2] so we omitted the proofs.
The maximum absolute column sum and row sum norm for the matrices MT_n and RT_n, respectively, are given by

$$\|MT_n\|_1 = \|MT_n\|_\infty = M_n - n.$$
$$\text{and} \qquad \|RT_n\|_1 = \|RT_n\|_\infty = R_n + n - 2.$$

Theorem 7. *The Euclidean norms $\|MT_n\|_E$ and $\|RT_n\|_E$ of Mersenne (Fermat) Toeplitz matrices are given by*

$$\|MT_n\|_E = \Big(\frac{4M_{2n} + M_{2(1-n)} - 36M_n - 18M_{1-n} + 9n^2 + 15}{9}\Big)^{1/2} \tag{2.2}$$

and

$$\|RT_n\|_E = \Big(\frac{32R_{(2n-2)} + 144R_{(n-1)} + 2R_{2(1-n)} + 36R_{1-n} + 18n^2 - 374}{18}\Big)^{1/2}. \tag{2.3}$$

Theorem 8. *The lower and upper bounds for spectral norm of the Mersenne Toeplitz and Fermat Toeplitz matrices are*

$$\|MT_n\|_2 \geq \sqrt{\frac{4M_{2n} + M_{2(1-n)} - 36M_n - 18M_{1-n} + 9n^2 + 15}{9n}},$$
$$\|MT_n\|_2 \leq \frac{1}{3}\sqrt{(M_{2(n-1)} - 6M_{(n-1)} + 3n)(M_{2n} - 6M_n + 3n)},$$
$$\|RT_n\|_2 \geq \sqrt{\frac{32R_{(2n-2)} + 144R_{(n-1)} + 2R_{2(1-n)} + 36R_{1-n} + 18n^2 - 374}{18n}},$$
$$\|RT_n\|_2 \leq \frac{1}{3}\sqrt{(R_{2(n-1)} + 6R_{(n-1)} + 3n - 14)(R_{2n} + 6R_n + 3n - 14)}.$$

Example 2. The Mersenne and Fermat Toeplitz matrices of order 3 are as follows:

$$MT_3 = \begin{bmatrix} 0 & -1/2 & -3/4 \\ 1 & 0 & -1/2 \\ 3 & 1 & 0 \end{bmatrix} \quad \text{and} \quad RT_3 = \begin{bmatrix} 2 & 3/2 & 5/4 \\ 3 & 2 & 3/2 \\ 5 & 3 & 2 \end{bmatrix}.$$

Solution. The matrix norms for these matrices are

$$\|MT_3\|_1 = 4, \quad \|MT_3\|_\infty = 4 \quad \text{and} \quad \|MT_3\|_E = \sqrt{193/16}$$
$$\|RT_3\|_1 = 10, \quad \|RT_3\|_\infty = 10 \quad \text{and} \quad \|RT_3\|_E = \sqrt{977/16},$$

which verifies the results of $\|.\|_1$ and $\|.\|_\infty$ for the above matrices. Also from (2.2) and (2.3),

$$\|MT_3\|_E = \Big(\frac{4M_6 + M_{2(-2)} - 36M_3 - 18M_{-2} + 9(3)^2 + 15}{9}\Big)^{1/2} = \sqrt{193/16} \quad \text{and}$$
$$\|RT_3\|_E = \Big(\frac{32R_4 + 144R_2 + 2R_{2(-2)} + 36R_{-2} + 18(3)^2 - 374}{18}\Big)^{1/2} = \sqrt{977/16}$$

which confirms the given results. □

3. Spread, Determinant and Characteristics Polynomials

For a given matrix, the problem of estimation of maximum distance between two eigenvalues was first noticed by L. Mirsky, who introduced [13] the spread for a complex matrix (of order n) to solve this problem. The spread of a matrix $A \in \mathbb{M}_{n\times n}(\mathbb{C})$ where $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues of A is defined as

$$s(A) = \max_{i,j} |\lambda_i - \lambda_j|.$$

The upper bound for the spread is

$$s(A) \leq \sqrt{2\|A\|_E^2 - \frac{2}{n}|tr(A)|^2}, \tag{3.1}$$

where $tr(A)$ represents the trace of matrix A.

Lemma 5. *For matrices MH_n, RH_n, MT_n and RT_n, we have*

$$tr(MH_n) = \frac{4^n - 3n - 1}{3} = \frac{M_{2n}}{3} - n,$$
$$tr(RH_n) = \frac{4^n + 3n - 1}{3} = \frac{M_{2n}}{3} + n,$$
$$tr(MT_n) = 0,$$
$$\textit{and} \quad tr(RT_n) = 2n.$$

Proof. The results follow from the definition of trace of a matrix. □

Theorem 9. *The upper bound for the spread of the Mersenne Hankel matrix MH_n and Fermat Hankel matrix RH_n are given, respectively, by*

$$s(MH_n) \leq \frac{\sqrt{2/n}}{3}\sqrt{(n-1)M_{2n}^2 + 6nM_{2n} - 18nM_n^2 + 9(n^3-n^2)} \quad \textit{and}$$
$$s(RH_n) \leq \frac{1}{3}\sqrt{2(R_{2n}^2 + 18R_n^2 - 4R_{2n} - 72R_n + 9n^2 + 76) - \frac{2}{n}(M_{2n}+3n)^2}.$$

Proof. Using the Frobenius norm $\|MH_n\|_E$ from Theorem 3 and the trace formula from Lemma 5 in Eqn. (3.1), we have

$$\begin{aligned} s(MH_n) &\leq \sqrt{2\|MH_n\|_E^2 - \frac{2}{n}|tr(MH_n)|^2} \\ &= \sqrt{\frac{2(M_{2n}^2 - 18M_n^2 + 9n^2)}{9} - \frac{2}{n}\left(\frac{M_{2n}-3n}{3}\right)^2} \\ &= \frac{\sqrt{2/n}}{3}\sqrt{(n-1)M_{2n}^2 + 6nM_{2n} - 18nM_n^2 + 9(n^3-n^2)}. \end{aligned}$$

Similarly, the second inequality can be easily proved using the fact that $tr(RH_n) = \frac{M_{2n}}{3} + n$ and

$$\|RH_n\|_E = \left(\frac{R_{2n}^2 + 18R_n^2 - 4R_{2n} - 72R_n + 9n^2 + 76}{9}\right)^{1/2}. \qquad \square$$

Theorem 10. *The upper bound for the spread of the Mersenne Toeplitz matrix MT_n and Fermat Toeplitz matrix RT_n are*

$$s(MT_n) \leq \frac{\sqrt{2}}{3}\sqrt{4M_{2n} + M_{2(1-n)} - 36M_n - 18M_{1-n} + 9n^2 + 15} \quad \textit{and}$$
$$s(RT_n) \leq \sqrt{\frac{32R_{(2n-2)} + 144R_{(n-1)} + 2R_{2(1-n)} + 36R_{1-n} + 18n^2 - 72n - 374}{9}}.$$

Proof. From Eqn. (2.2) on Frobenius norm and the trace formula from Lemma 5, we have

$$tr(MT_n) = 0 \quad \text{and} \quad \|MT_n\|_E^2 = \frac{4M_{2n} + M_{2(1-n)} - 36M_n - 18M_{1-n} + 9n^2 + 15}{9}.$$

So, the first inequality follows from Eqn. (3.1).
Similarly, the second inequality follows using Eqn. (2.3) and Lemma 5 i.e

$$tr(RT_n) = 2n$$
$$\text{and} \qquad \|RT_n\|_E^2 = \frac{32R_{(2n-2)} + 144R_{(n-1)} + 2R_{2(1-n)} + 36R_{1-n} + 18n^2 - 374}{18}. \square$$

3.1. Rank, Determinant and Characteristics polynomial

Theorem 11. *The rank of Hankel matrices with Mersenne (or Fermat) numbers is 2.*

Proof. Since for $n = 2$, the determinant is a non zero number, i.e.

$$\det(MH_2) = \begin{vmatrix} M_0 & M_1 \\ M_1 & M_2 \end{vmatrix} = M_2M_0 - M_1^2 = -1. \tag{3.2}$$

So in this case rank is 2.
Now we prove for $n \geq 3$. To see the rank we reduce the matrix MH_n into echelon form by performing the elementary row operations on MH_n and then substituting the entries using the identity $M_{n+1} - M_n = 2^n$. The matrix MH_n is given as

$$MH_n = \begin{bmatrix} M_0 & M_1 & M_2 & \cdots & M_{n-2} & M_{n-1} \\ M_1 & M_2 & M_3 & \cdots & M_{n-1} & M_n \\ M_2 & M_3 & M_4 & \cdots & M_n & M_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{n-2} & M_{n-1} & M_n & \cdots & M_{2n-4} & M_{2n-3} \\ M_{n-1} & M_n & M_{n+1} & \cdots & M_{2n-3} & M_{2n-2} \end{bmatrix}.$$

Applying $R_i \leftarrow R_i - R_{i-1},\ \ 2 \leq i \leq n$ on MH_n, where R_i denotes ith-row, we get

$$MH_n \sim \begin{bmatrix} M_0 & M_1 & M_2 & \cdots & M_{n-2} & M_{n-1} \\ 2^0 & 2^1 & 2^2 & \cdots & 2^{n-2} & 2^{n-2} \\ 2^1 & 2^2 & 2^3 & \cdots & 2^{n-1} & 2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2^{n-3} & 2^{n-2} & 2^{n-1} & \cdots & 2^{2n-5} & 2^{2n-4} \\ 2^{n-2} & 2^{n-1} & 2^n & \cdots & 2^{2n-4} & 2^{2n-3} \end{bmatrix}.$$

Now, applying $R_i \leftrightarrow 2^{n-i}R_i$ for $2 \leq i \leq n-1$ and substituting $M_r = 2^r - 1$ for $i = 1$, we get

$$MH_n \sim \begin{bmatrix} 0 & 1 & 3 & \cdots & 2^{n-2}-1 & 2^{n-1}-1 \\ 2^{n-2} & 2^{n-1} & 2^n & \cdots & 2^{2n-4} & 2^{2n-3} \\ 2^{n-2} & 2^{n-1} & 2^n & \cdots & 2^{2n-4} & 2^{2n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2^{n-2} & 2^{n-1} & 2^n & \cdots & 2^{2n-4} & 2^{2n-3} \\ 2^{n-2} & 2^{n-1} & 2^n & \cdots & 2^{2n-4} & 2^{2n-3} \end{bmatrix}.$$

In the above matrix, rows $R_2, R_3, \ldots, R_{n-1}$ are identical and rows R_1 and R_2 are linearly independent. Thus, conclusively only two rows of MH_n are linearly independent and hence the rank is 2.
Similarly, the rank of Fermat Hankel matrices is 2 where nth term of the Fermat numbers is given by $2^n + 1$. □

Theorem 12. *The rank of Toeplitz matrices with Mersenne (Fermat) numbers is 2.*

Proof. The argument is very similar to Theorem 11. □

Corollary 1. *For Mersenne-Hankel matrices MH_n and Fermat-Hankel matrices RH_n, 0 is an eigenvalue with an algebraic multiplicity $n-2$ and the other two eigenvalues are non-zero.*

Theorem 13. *For Hankel matrices MH_n and RH_n and Toeplitz matrices MT_n and RT_n, we have*

$$\det(MH_n) = \begin{cases} -1, & n = 2 \\ 0, & n \geq 3 \end{cases} \quad \textit{and} \quad \det(RH_n) = \begin{cases} 1, & n = 2 \\ 0, & n \geq 3 \end{cases},$$

$$\det(MT_n) = \begin{cases} \frac{1}{2}, & n = 2 \\ 0, & n \geq 3 \end{cases} \quad \textit{and} \quad \det(RT_n) = \begin{cases} -\frac{1}{2}, & n = 2 \\ 0, & n \geq 3. \end{cases}$$

Proof. For $n = 2$,

$$\det(MH_2) = \begin{vmatrix} M_0 & M_1 \\ M_1 & M_2 \end{vmatrix} = M_2 M_0 - M_1^2 = -1.$$

Similarly, $\det(MH_2) = 1$, $\det(MT_2) = 1/2$ and $\det(RT_2) = -1/2$.
And, by using Theorems 11 and 12, it is rapidly follows that determinant of the matrices MH_n, RH_n, MT_n, RT_n is zero for $n \geq 3$. □

Corollary 2. *The Hankel and Toeplitz matrices with Mersenne and Fermat numbers are nonsingular for $n = 2$ and singular for $n \geq 3$.*

Theorem 14. *The sum of principal minors of order two of MH_n is given by*

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + ... + \lambda_{n-1}\lambda_n = \frac{(3-n)4^n - (6)2^n + (n+3)}{3},$$

where $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ are eigenvalues of MH_n.

Proof. We should note that the sum of principal minors of order two is equal to the sum of products of distinct eigenvalues taking two at a time i.e. $\lambda_1\lambda_2 + \lambda_1\lambda_3 + ... + \lambda_{n-1}\lambda_n$ [8]. From Corollary 1, without loss of generality we assume that λ_1 and λ_2 are non zero, thus we have

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + ... + \lambda_{n-1}\lambda_n = \lambda_1\lambda_2 + 0 + 0 + ... + 0 = \lambda_1\lambda_2.$$

So,

$$\lambda_1\lambda_2 = \begin{vmatrix} M_0 & M_1 \\ M_1 & M_2 \end{vmatrix} + \begin{vmatrix} M_0 & M_2 \\ M_2 & M_4 \end{vmatrix} + ... + \begin{vmatrix} M_2 & M_3 \\ M_3 & M_4 \end{vmatrix} + \begin{vmatrix} M_2 & M_4 \\ M_4 & M_6 \end{vmatrix} + ... + \begin{vmatrix} M_{2n-4} & M_{2n-3} \\ M_{2n-3} & M_{2n-2} \end{vmatrix}$$
$$= (M_0M_2 - M_1^2) + (M_0M_4 - M_2^2) + ... + (M_0M_{2n-2} - M_{n-1}^2) + (M_2M_4 - M_3^2) +$$
$$... + \ (M_2M_{2n-2} - M_n^2) + ... + (M_{2n-6}M_{2n-2} - M_{2n-4}^2) + (M_{2n-4}M_{2n-2} - M_{2n-3}^2)$$
$$= \sum_{j=0}^{n-2}\sum_{i=j+1}^{n-1}\Big(M_{2j}M_{2i} - M_{i+j}^2\Big).$$

Since by using the Binet's formula $M_n = 2^n - 1$, we have

$$M_{2j}M_{2i} - M_{i+j}^2 = 2^{i+j+1} - 2^{2i} - 2^{2j}.$$

Hence,

$$\lambda_1\lambda_2 = \sum_{j=0}^{n-2}\sum_{i=j+1}^{n-1}(2^{i+j+1} - 2^{2i} - 2^{2j}) = \sum_{j=0}^{n-2}\Big(2^{j+1}\sum_{i=j+1}^{n-1}2^i - \sum_{i=j+1}^{n-1}2^{2i} - \sum_{i=j+1}^{n-1}2^{2j}\Big)$$
$$= \sum_{j=0}^{n-2}\Big(2^{2(j+1)}(2^{n-j-1} - 1) - \frac{2^{2(j+1)}}{3}(2^{2(n-j-1)} - 1) - 2^{2j}(n - j - 1)\Big)$$
$$= 2^{n+1}\sum_{j=0}^{n-2}2^j - 2^2\sum_{j=0}^{n-2}2^{2j} - \sum_{j=0}^{n-2}\frac{2^{2n}}{3} + \frac{2^2}{3}\sum_{j=0}^{n-2}2^{2j} - (n-1)\sum_{j=0}^{n-2}2^{2j} + \sum_{j=0}^{n-2}j2^{2j}$$
$$= 2^{n+1}(2^{n-1} - 1) - \frac{4^n(n-1)}{3} - \frac{(5+3n)}{3}\frac{(4^{n-1} - 1)}{3} + \Big(\frac{4(1 - 4^{n-2})}{(1-4)^2}$$
$$+ \frac{(n-2)4^{n-1}}{3}\Big) = \frac{(3-n)4^n - (6)2^n + (n+3)}{3}.$$

Thus, for Mersenne Hankel matrices

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + ... + \lambda_{n-1}\lambda_n = \lambda_1\lambda_2 = \frac{(3-n)4^n - (6)2^n + (n+3)}{3}. \qquad \square$$

By a similar argument, we have the following theorem for Fermat Hankel matrices.

Theorem 15. *If $\lambda_1', \lambda_2', \lambda_3', \ldots, \lambda_n'$ are n eigenvalues of Fermat Hankel matrices, then we have*

$$\lambda_1'\lambda_2' + \lambda_1'\lambda_3' + \cdots + \lambda_{n-1}'\lambda_n' = \frac{(n-3)4^n + (6)2^n - (n+3)}{3}.$$

Remark 1. Since rank of the Mersenne (Fermat) Hankel matrices is 2, so principal minors of order $n \geq 3$ are zero.

Theorem 16. *The sum and product of non-zero eigenvalues of Mersenne-Hankel matrices MH_n are $(4^n - 3n - 1)/3$ and $((3-n)4^n - (6)2^n + (n+3))/3$, respectively.*

Proof. From Corollary 1, let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ be eigenvalues of MH_n such that $\lambda_1, \lambda_2 \neq 0$. We should note that the sum of all eigenvalues of a matrix is equal to the trace. Hence from Lemma 5, we have

$$\lambda_1 + \lambda_2 + \lambda_3 + ... + \lambda_{n-1} = \lambda_1 + \lambda_2 = \frac{4^n - 3n - 1}{3}.$$

And from Theorem 14, we have

$$\lambda_1 \lambda_2 = \frac{(3-n)4^n - (6)2^n + (n+3)}{3}.$$

□

Theorem 17. *The characteristic polynomials $ch_{MH}(t)$ for Mersenne-Hankel matrices MH_n and $ch_{RH}(t)$ for Fermat-Hankel matrices RH_n are, respectively, given by*

$$ch_{MH}(t) = t^n - \Big(\frac{4^n - 3n - 1}{3}\Big)t^{n-1} + \Big(\frac{(3-n)4^n - (6)2^n + (n+3)}{3}\Big)t^{n-2} \quad \textit{and}$$
$$ch_{RH}(t) = t^n - \Big(\frac{4^n + 3n - 1}{3}\Big)t^{n-1} - \Big(\frac{(3-n)4^n - (6)2^n + (n+3)}{3}\Big)t^{n-2}.$$

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of MH_n, then the characteristics polynomial $ch_{MH}(t)$ is

$$ch_{MH}(t) = (t - \lambda_1)(t - \lambda_2) \ldots (t - \lambda_n). \tag{3.3}$$

Since $\lambda_1, \lambda_2 \neq 0$ and rest $n-2$ eigenvalues are zero, so (3.3) reduced to

$$\begin{aligned} ch_{MH}(t) &= t^{n-2}(t - \lambda_1)(t - \lambda_2) \\ &= t^{n-2}(t^2 - (\lambda_1 + \lambda_2)t - \lambda_1\lambda_2) \\ &= t^n - \Big(\frac{4^n - 3n - 1}{3}\Big)t^{n-1} \\ &- \Big(\frac{(3-n)4^n - (6)2^n + (n+3)}{3}\Big)t^{n-2} \quad \text{(by Theorem 16).} \end{aligned}$$

Similarly, the second identity can be proved. □

Example 3. For $n = 2, 3, 4, 5$, the characteristic polynomials for Mersenne Hankel matrices MH_n are
$x^2 - 3x - 1, x^3 - 18x^2 - 14x, x^4 - 81x^3 - 115x^2$ and $x^5 - 336x^4 - 744x^3$, respectively.

Example 4. For $n = 2, 3, 4, 5$, the characteristic polynomial for Fermat Hankel matrices RH_n are
$x^2 - 7x + 1, x^3 - 24x^2 + 14x, x^4 - 89x^3 + 115x^2$ and $x^5 - 346x^4 + 744x^3$, respectively.

Theorem 18. *The sum of principal minors of order two of Toeplitz matrices MT_n and RT_n are*

$$\alpha_1\alpha_2 + \alpha_1\alpha_3 + ... + \alpha_{n-1}\alpha_n = 2^{(n+1)} + 2^{(-n+1)} - n^2 - 4$$
$$\text{and} \qquad \gamma_1\gamma_2 + \gamma_1\gamma_3 + ... + \gamma_{n-1}\gamma_n = n^2 - 2^{(n+1)} - 2^{(-n+1)} + 4,$$

respectively, where $\alpha_1, \alpha_2, ..., \alpha_n$ are eigenvalues of MT_n and $\gamma_1, \gamma_2, ..., \gamma_n$ are eigenvalues of RT_n.

Proof. The argument is very similar to Theorem 14. □

Since from Theorem 12, the rank of Toeplitz matrices with Mersenne or Fermat numbers is 2. So total $n-2$ eigenvalues will be zero for these matrices. Hence without loss of generality we can assume that $\alpha_1, \alpha_2 \neq 0$ for MT_n and $\gamma_1, \gamma_2 \neq 0$ for RT_n. Thus from Theorem 18, we have

$$\alpha_1\alpha_2 = 2^{(n+1)} + 2^{(-n+1)} - n^2 - 4 \quad \text{and} \quad \gamma_1\gamma_2 = n^2 - 2^{(n+1)} - 2^{(-n+1)} + 4. \qquad (3.4)$$

Because $n-2$ eigenvalues of Mersenne (Fermat) Toeplitz matrices are zero so principal minors of order $n \geq 3$ are zero.
From Lemma 5 we have $trace(MT_n) = 0$ and $trace(RT_n) = 2n$, Thus by following a similar argument to Theorem 17, the characteristic polynomials $ch_{MT}(t)$ for Mersenne-Toeplitz matrices and $ch_{RT}(t)$ for Fermat-Toeplitz matrices are given by

$$ch_{MT}(t) = t^n + \Big(2^{(n+1)} + 2^{(-n+1)} - n^2 - 4\Big)t^{n-2}$$

and

$$ch_{RT}(t) = t^n - (2n)t^{n-1} + \big(n^2 - 2^{(n+1)} - 2^{(-n+1)} + 4\big)t^{n-2},$$

respectively.

Example 5. For $n = 2, 3, 4, 5$, the characteristic polynomials for Mersenne Toeplitz matrices MT_n are
$x^2 + 1/2,\ x^3 + (13/4)x,\ x^4 + (97/8)x^2$ and $x^5 + (561/16)x^3$, respectively.

Example 6. For $n = 2, 3, 4, 5$, the characteristic polynomials for Fermat Toeplitz matrices RT_n are
$x^2 - 4x - 1/2,\ x^3 - 6x^2 - (13/4)x,\ x^4 - 8x^3 - (97/8)x^2$ and $x^5 - 10x^4 - (561/16)x^3$, respectively.

4. Conclusion

This study is about some new properties of the Hankel matrices $MH_n = (m_{ij})_{i,j=1}^n$ with $m_{ij} = M_{i+j-2}$, $RH_n = (r_{ij})_{i,j=1}^n$ with $r_{ij} = R_{i+j-2}$ and Toeplitz matrices $MT_n = (m_{ij})_{i,j=1}^n$ with $m_{ij} = M_{i-j}$ and $RT_n = (r_{ij})_{i,j=1}^n$ with $r_{ij} = R_{i-j}$, where M_n and R_n are Mersenne and Fermat numbers, respectively. Here, we developed efficient formulas for the matrix norms like $\|.\|_1, \|.\|_\infty, \|.\|_E$ and bounds for spectral norm $\|.\|_2$ and spread of these matrices. Furthermore, we evaluated the rank, determinant, principal minors, and characteristic polynomials for these matrices explicitly in closed form. The results are supported by numerical examples.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] M. Akbulak and D. Bozkurt, *On the norms of Toeplitz matrices involving Fibonacci and Lucas numbers*, Hacet. J. Math. Stat. **37** (2008), no. 2, 89–95.

[2] S. Aktas and Y. Soykan, *A study on the norms of Toeplitz matrices with the generalized mersenne numbers*, Arch. Curr. Res. Int. **23** (2023), no. 7, 143–157. https://doi.org/10.9734/acri/2023/v23i7600.

[3] Y. Alp and E.G. Kocer, *The Gram and Hankel matrices via special number sequences*, Honam Math. J. **45** (2023), no. 3, 418–432. https://doi.org/10.5831/HMJ.2023.45.3.418.

[4] P. Catarino, H. Campos, and P. Vasco, *On the Mersenne sequence*, Ann. Math. Inform. **46** (2016), 37–53.

[5] M. Chelgham and A. Boussayoud, *On the k-Mersenne–Lucas numbers*, Notes Number Theory Discrete Math. **27** (2021), no. 1, 7–13. https://doi.org/10.7546/nntdm.2021.27.1.7-13.

[6] A. Daşdemir, *On the norms of Toeplitz matrices with the Pell, Pell-Lucas and modified Pell numbers*, J. Eng. Technol. Appl. Sci. **1** (2016), no. 2, 51–57. https://doi.org/10.30931/jetas.283838.

[7] A.F. Horadam, *Chebyshev and Fermat polynomials for diagonal functions*, Fibonacci Q. **17** (1979), no. 4, 328–333. https://doi.org/10.1080/00150517.1979.12430205.

[8] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, 2012.

[9] G.Ö. Kızılırmak, *On some identities and Hankel matrices norms involving new defined generalized modified Pell numbers*, J. New Results Sci. **10** (2021), no. 3, 60–66. https://doi.org/10.54187/jnrs.989508.

[10] M. Kumari, K. Prasad, and H. Mahato, *On the norms of Toeplitz and Hankel matrices with balancing and Lucas-balancing numbers*, Advanced Mathematical Techniques in Computational and Intelligent Systems, CRC Press, 2023, pp. 196–210.

[11] M. Kumari, K. Prasad, J. Tanti, and E. Özkan, *On the properties of r-circulant matrices involving Mersenne and Fermat numbers*, Int. J. Nonlinear Anal. Appl. **14** (2023), no. 5, 121–131.
https://doi.org/10.22075/ijnaa.2023.27875.3742.

[12] R. Mathias, *The spectral norm of a nonnegative matrix*, Linear Algebra Appl. **139** (1990), 269–284.
https://doi.org/10.1016/0024-3795(90)90403-Y.

[13] L. Mirsky, *The spread of a matrix*, Mathematika **3** (1956), no. 2, 127–130.
https://doi.org/10.1112/S0025579300001790.

[14] B.J. Olson, S.W. Shaw, C. Shi, C. Pierre, and R.G. Parker, *Circulant matrices and their application to vibration analysis*, Appl. Mech. Rev. **66** (2014), no. 4, 040803.
https://doi.org/10.1115/1.4027722.

[15] K. Prasad and M. Kumari, *Some new properties of Frank matrices with entries Mersenne numbers*, Natl. Acad. Sci. Lett. (2024), In press.
https://doi.org/10.1007/s40009-024-01538-6.

[16] K. Prasad, M. Kumari, and H. Mahato, *A modified public key cryptography based on generalized Lucas matrices*, Commun. Comb. Optim. (2024), In press.
https://doi.org/10.22049/cco.2024.28022.1419.

[17] R.M. Robinson, *Mersenne and Fermat numbers*, Proc. Amer. Math. Soc. **5** (1954), no. 5, 842–846.
https://doi.org/10.2307/2031878.

[18] N. Saba, A. Boussayoud, and K.V. Kanuri, *Mersenne-Lucas numbers and complete homogeneous symmetric functions*, J. Math. Comput. Sci. **24** (2021), no. 2, 127–139.
http://dx.doi.org/10.22436/jmcs.024.02.04.

[19] S. Shen, *On the norms of Toeplitz matrices involving k-Fibonacci and k-Lucas numbers*, Int. J. Contemp. Math. Sciences **7** (2012), no. 8, 363–368.

[20] S. Solak and M. Bahşi, *On the spectral norms of Hankel matrices with Fibonacci and Lucas numbers*, Selcuk J. Appl. Math. **12** (2011), no. 1, 71–76.
http://dx.doi.org/10.12988/ijma.2015.411370.

[21] Ş. Uygun, *On the bounds for the norms of Toeplitz matrices with the Jacobsthal and Jacobsthal Lucas numbers*, J. Eng. Technol. Appl. Sci. **4** (2019), no. 3, 105–114.
https://doi.org/10.30931/jetas.569742.

[22] P. Vasco, P. Catarino, H. Campos, A.P. Aires, and A. Borges, *k-Pell, k-Pell-Lucas and modified k-Pell numbers: Some identities and norms of Hankel matrices*, Int. J. Math. Anal. **9** (2015), no. 1, 31–37.
http://dx.doi.org/10.12988/ijma.2015.411370.

[23] A.C. Wilde, *Differential equations involving circulant matrices*, Rocky Mt. J.

Math. **13** (1983), no. 1, 1–13.

[24] Y. Yazlik, N. Yilmaz, and N. Taskara, *On the norms of Hankel matrices with the k-Jacobsthal and k-Jacobsthal Lucas numbers*, J. Selcuk Uni. Natural Appl. Sci. **3** (2014), no. 2, 35–42.

[25] G. Zielke, *Some remarks on matrix norms, condition numbers, and error estimates for linear equations*, Linear Algebra Appl. **110** (1988), 29–41. https://doi.org/10.1016/0024-3795(83)90130-1.