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Research Article

On maximum tolerant Radon partitions for all-paths convexity in graphs

Sreekumar Sreedharan[1,2], Manoj Changat[2,*], Kannan Balakrishnan[3]

[1]Department of Mathematics, Government College for Women, Thiruvananthapuram-695014, India
ssreekumar8880@gmail.com

[2]Department of Futures Studies, University of Kerala, Thiruvananthapuram-695581, India
*mchangat@keralauniversity.ac.in

[3]Department of Computer Applications, Cochin University of Science and Technology, Kochi-682022, India
mullayilkannan@gmail.com

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Abstract: In a connected graph G, the all-paths transit function $A(u, v)$, consists of the set of all vertices in the graph G which lies on some path connecting u and v. Convexity obtained by the all-paths transit function is called all-paths convexity. A Radon partition of a set P of vertices of a graph G is a partition of P into two disjoint non-empty subsets such that their convex hulls intersect. A Radon partition (P_t, Q_t) of P is called t-tolerant Radon partition, if for any set $S \subseteq P$ with $|S| \leq t$, the intersection of the convex hulls $\langle P_t \setminus S\rangle \cap \langle Q_t \setminus S\rangle \neq \emptyset$. This paper is devoted to t-tolerant Radon partitions for the all-paths convexity of connected simple undirected graphs. It is proved that the minimum number of vertices needed for t-tolerant Radon partition is $2t + 4$. But, some selection of $2t + 4$ vertices of G has a $(t + 1)$-tolerant Radon partition. In this paper, we discuss the necessary and sufficient condition to the existence of $(t + 1)$-tolerant Radon partition for $2t + 4$ vertices of G. We also develop algorithms to construct the Radon partition, t-tolerant Radon partition, and $(t + 1)$-tolerant Radon partition of a set of $2t + 4$ vertices, if it exists.

Keywords: all-paths convexity, Radon partition, $(t + 1)$-tolerant Radon partition.

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1. Introduction

In Discrete Geometry, Radon's theorem is one of the most useful theorems by its applications. In 1921, Radon [14] proved that every set P of $d + 2$ points in $\mathbb{R}^d$ possesses a partition of P into two non-empty sets (P, Q) such that their convex

* *Corresponding Author*

hulls intersect. Such a partition is called a Radon partition. The classical convexity invariant named as Radon number, which represents the smallest number k of points in the Euclidean space R^d such that any set of k points has a Radon partition. The natural generalisation of Radon's theorem is the Tverberg's theorem. In 1966, Tverberg [21] partitioned $(k-1)(d+1)+1$ points in $\mathbb{R}^d$ into k non-empty subsets such that their convex hulls intersect.

A Radon partition (P_t, Q_t) of a set P in $\mathbb{R}^d$ has tolerance t, if by removing any t points from P, then (P_t, Q_t) still remains as a Radon partition. Generally, zero-tolerant Radon partition refers to the usual Radon partition. Tolerant Radon partitions of $\mathbb{R}^d$ have applications in computational geometry. So many studies are done in the tolerance of Tverberg's partitions in the classical Euclidean convexity. Larman [12], proved that any set of $2d+3$ points in $\mathbb{R}^d$ has one tolerant Radon partition and Garcia-Colin [7] gives the existence of t-tolerant Radon partition for any set of $(t+1)(d+1)+1$ points in $\mathbb{R}^d$. Further, Soberon et al. [19] proved that for any set of $(t+1)(k-1)(d+1)+1$ points in $\mathbb{R}^d$ has t-tolerant Tverberg partition. In 2022, Bereg et al. [2] developed different algorithms to compute the tolerance in Radon partitions in $\mathbb{R}^d$.

Tolerant Radon partitions in abstract convexity spaces, in particular in graph convexity spaces is an interesting area of the Radon partition problem. The theory of abstract convexity spaces was developed with the main intention of generalizing the classical convexity invariants such as Helly, Carathéodory, and Radon numbers in $\mathbb{R}^d$. Later these ideas are used in abstract convexity spaces by Sierksma [16–18], Duchet [10], van de Vel [22], to name a few important references.

In abstract convexity spaces, graph convexity spaces captured more attention and several authors have studied graph convexities in different settings. The most prominent types of graph convexities are defined in terms of paths in the graph. Important studies are done in terms of geodesic, induced path and all-paths convexity [1, 3–6, 9, 15].

The coarsest path convexity in a connected graph is the all-paths convexity [4, 6]. The convexity invariants of the all-paths convexity are determined in [6] for a graph G. It can be proved easily that the Carathéodory number, $c(G)=2$; Helly number, $h(G)=2$ and Radon number $r(G)$ satisfies $3 \leq r(G) \leq 4$, for the all-paths convexity. In [20], it is proved that any set S of $2t+4$ vertices in a graph G guarantees that S has a t-tolerant Radon partition for the all-paths convexity. It may be noted that in some cases, a set of $2t+4$ vertices of G can have a Radon partition with maximum tolerance of $t+1$ for the all-paths convexity. In this paper, we give the necessary and sufficient condition for the existence of maximum tolerance of $t+1$ for a set of $2t+4$ vertices of G with respect to all-paths convexity and discuss algorithms to construct tolerant Radon partitions of a set of vertices and the maximum tolerant Radon partition of a set of vertices, if it exists.

We discuss all the possibilities to form $(t+1)$-tolerant Radon partition to a set of $2t+4$ vertices of G for the all-paths convexity.

The paper is organised in the following way. In Section 2, we present the preliminary concepts, definitions, and results for the remaining sections. The necessary and sufficient condition to the existence of $(t+1)$-tolerant Radon partition for $2t+4$ vertices of G is described in Section 3. In Section 4, we discuss the algorithms.

2. Preliminaries

In this paper, we consider a simple connected undirected graph G with vertex set V. Also, only the tolerant Radon partition with respect to the all-paths convexity in a graph G is taken into account. In this sense, the term convexity that we mention at times needs to be read as all-paths convexity.

A convexity space is a pair $(V, \mathscr{C})$ where V is a non-empty set and $\mathscr{C}$ is a collection of subsets of V such that $\emptyset, V \in \mathscr{C}$, arbitrary intersection of elements of $\mathscr{C}$ is also in $\mathscr{C}$, and every nested union of elements of $\mathscr{C}$ is also in $\mathscr{C}$ [22]. If V is finite, then the nested union condition is redundant and not required.

The elements of $\mathscr{C}$ are called convex sets. The smallest convex set containing a set S is called the convex hull of S and it is represented by $\langle S\rangle_{\mathscr{C}}$. A graph convexity space is a pair $(G, \mathscr{C})$ formed with V, a vertex set and $\mathscr{C}$, a convexity on V.
According to Mulder in [13], a transit function on a finite set V is a function $R : V \times V \to 2^V$ satisfying the following conditions
$u \in R(u, v)$ for any $u, v \in V$, $R(u, v) = R(v, u)$ for all $u, v \in V$ and $R(u, u) = \{u\}$ for all $u \in V$.

Let R be a transit function on V. A set $W \subset V$ is an R-convex set if $R(u, v) \subseteq W$ for all $u, v \in W$. The collection $\mathscr{C}_R$ of all R-convex sets in V is a convexity in the sense that $\mathscr{C}_R$ contains the empty set $\emptyset$ and V itself and $\mathscr{C}_R$ is closed under intersections and nested unions.

If R is a transit function on the vertex set V of a graph G then we say that R is a transit function on G. Transit function gives a rule to move around from an element $u \in V$ to an element $v \in V$ via the set $R(u, v)$. The geodesic interval function I, the induced path transit function J, and the all-paths transit function A [4, 6, 9, 15] are some examples of transit functions, where

$$I(u, v) = \{w \in V : w \text{ lies on some shortest } u - v \text{ path in } G\},$$

$$J(u, v) = \{w \in V : w \text{ lies on some induced } u - v \text{ path in } G\},$$

and

$$A(u, v) = \{w \in V : w \text{ lies on some } u - v \text{ path in } G\}.$$

The family of R-convex sets in a graph G is called R-convexity on G. Thus the convexities induced by the geodesic, induced path, and all-paths transit functions are called geodesic convexity, induced path convexity and all-paths convexity, respectively. All these graph convexities are extensively studied, for e.g., in [4, 6, 9, 15].

For convenience, we denote the all-paths convex hull of a subset S of V by $\langle S\rangle$. It may be noted that the convex hull $\langle S\rangle$ of subset S always induces a connected subgraph of G.

For the basic graph theoretical terms, we refer to West [23]. A *cut vertex* of a graph G is a vertex whose removal increases the number of components of G. A subgraph H having vertex set $S, S \subseteq V$, is called an *induced subgraph* if for any two vertices $u, v \in S$, u and v are adjacent in H if and only if u and v are adjacent in G. *Pendant vertex (leaf vertex)* of G is a vertex of G having degree one. The *diameter* of G is the maximum distance between any two vertices of G. The graph G is *two-connected* if the removal of any single vertex in G is not sufficient to disconnect the graph. A *block* of a graph G is a maximal two-connected subgraph of G. From the definition of $A(u, v)$, it is clear that $A(u, u) = \{u\}$ for any vertex u of V, and if u, v are two vertices of a block, then $A(u, v)$ is the set of all vertices of that block. For u, v in G, $A(u, v)$ contains all vertices of the blocks in G, in which there exists a $u-v$ path traversing through that block. Thus if a $u-v$ path contains an edge of a particular block, then the all-paths convex hull of $\{u, v\}$ contains all vertices of that particular block. From the definition of A, it follows that, for any $x, y \in V$, $\langle\{x, y\}\rangle = A(x, y)$ where $\langle\{x, y\}\rangle$ is the all-paths convex hull of $\{x, y\}$. The *block cut-vertex tree* denoted as $B(G)$ of G has the blocks and cut vertices of G as its vertices and two vertices of $B(G)$ are adjacent whenever one of them is a cut vertex of a block and the other is a block containing that cut vertex [3]. It is clear that the all-paths transit function has the structure reflecting the block-cut vertex tree structure of the graph. Blocks of a graph G having only one cut vertex are called *end blocks*.

The *Radon number* of a convexity space is the smallest integer r such that every r element subset P of V has a Radon partition. We denote the Radon partition of P concerning the all-paths convexity into sets P_0 and Q_0 as (P_0, Q_0).

A Radon partition of P into two nonempty subsets P_t and Q_t is called *t-tolerant Radon partition* for some non-negative integer t, if for any set $S \subseteq P$ with $|S| \leq t$, we have $\langle P_t \setminus S\rangle \cap \langle Q_t \setminus S\rangle \neq \emptyset$. For t-tolerant Radon partition, each set of a partition must have at least $t+1$ vertices. Clearly, t-tolerant Radon partition of a set is $(t-1)$-tolerant Radon partition. But, the converse need not be true.

3. $(t+1)$-tolerant Radon partition

In [20], we proved that for any set of $2t+4$ vertices of G has a t-tolerant Radon partition for the all-paths convexity. Here t is the greatest lower bound of the tolerance of the Radon partition for any set of $2t+4$ vertices of G. But for some collection of $2t+4$ vertices of G, there exists a $(t+1)$-tolerant Radon partition. In this section we find the different cases for the existence of $(t+1)$-tolerant Radon partition for a set of $2t+4$ vertices of the graph. In [20], we proved that if G has no cut vertices, then any collection of $2t+2$ vertices of P has a t-tolerant Radon partition.

Proposition 1. *[20] If a graph G has no cut vertices, then any set of $2t+2$ vertices of G has a t-tolerant Radon partition, $t \geq 1$.*

Proposition 2. *Let P be a set of $2t+4$ vertices of a graph G and let H be the subgraph induced by $\langle P \rangle$. If P contains only one vertex of some end block of H other than a cut vertex, then P has no $(t+1)$-tolerant Radon partition.*

Proof. To form a $(t+1)$-tolerant Radon partition, each set of a partition must have at least $t+2$ vertices of P. Since P contains only one vertex x of some end block B_i of H other than a cut vertex, it is clear that H contains more than one block. Consider any partition (P_1, Q_1), both having $t+2$ vertices of P. Let $x \in P_1$. Let S be the set of $t+1$ vertices of P_1 such that $x \notin S$. Then $P_1 \setminus S = \{x\}$ and so $\langle P_1 \setminus S \rangle = \{x\}$. Since x is the only vertex from the end block B_i with possible exception of its cut vertex, Q_1 contains no vertex of B_i. Thus $\langle Q_1 \setminus S \rangle$ does not contain x. Hence $\langle P_1 \setminus S \rangle \cap \langle Q_1 \setminus S \rangle = \emptyset$. Thus P has no $(t+1)$-tolerant Radon partition. $\square$

Proposition 3. *Let P be a set of $2t+4$ vertices G other than cut vertices. If each block of G contains an even number (greater than 2) of vertices of P, then P has a $(t+1)$-tolerant Radon partition; $t \geq 0$.*

Proof. Here we form partitions P^* and Q^* such that each partition contains half of the total number of vertices from each block. Then $|P^*| = t+2$ and $|Q^*| = t+2$. We have to show that P^* and Q^* forms a $(t+1)$-tolerant Radon partition.

Case 1. If we remove any $t+1$ vertices from P^* then one vertex of some block B_i remains in P^*. Since no vertices are removed from Q^*, Q^* contains at least one vertex of that block B_i. So the all-paths convex hull of the remaining vertices of the partitions intersects at the remaining vertex of P^* from B_i. Similarly if we remove any $t+1$ vertices from Q^*, then also the convex hull of the remaining vertices of partitions intersects.

Case 2. If we remove any $k, 1 \leq k < t+1$, vertices from P^* and $t+1-k$ vertices from Q^*. After removing k vertices from P^*, $t+2-k$ vertices remain in P^*. Since P^* and Q^* contain the same number of vertices from each block, corresponding to these $t+2-k$ vertices of P^* there exists $t+2-k$ vertices in Q^*. Since $(t+2-k)-(t+1-k) = 1$, after removing any $t+1-k$ vertices from Q^*, at least one vertex of the same block B_i remains in both P^* and Q^*. So the all-paths convex hull of the remaining vertices intersects at block B_i, because there are no cut vertices in P and so (P^*, Q^*) is a $(t+1)$-tolerant Radon partition. $\square$

Thus to form a $(t+1)$-tolerant Radon partition from a set of $2t+4$ vertices of G, P must contain at least two vertices from all the end blocks of the subgraph induced by $\langle P \rangle$ other than cut vertices.

Theorem 1. *Let P be a set of $2t+4$ vertices of a graph G and let H be the subgraph induced by $\langle P\rangle$. Suppose that P contains at least one vertex from every interior block of H and at least two vertices from every end block of H other than cut vertices. Then P has a $(t+1)$-tolerant Radon partition.*

Proof. Given a partition $P=(P_1,Q_1)$, we call (P_1,Q_1) is of the same cardinality, if $|P_1|=|Q_1|$ and of different cardinality if $|P_1|=|Q_1|+1$. We use this terminology in the proof.
Since H is a convex subgraph of G, H is connected. Now, consider the block cut vertex tree, $B(H)$ of the subgraph H. Blocks and cut vertices of H are the vertices of $B(H)$. We represent the vertices of P in $B(H)$ in such a way that, all cut vertices of H in P can be represented in their corresponding places of $B(H)$ and all other vertices of P from the same block can be represented in the corresponding vertex of $B(H)$. Thus, a vertex of $B(H)$ may represent more vertices of P.

Case 1. $B(H)$ is a path.
Since $2t+4$ is even, the number of vertices of $B(H)$ that corresponds to an odd number of vertices of P is always even. We construct P_1 and Q_1 by partitioning the vertices of P using $B(H)$, starting from a pendant vertex of $B(H)$. Suppose that the pendent vertex of $B(H)$ represents x_1 vertices of P. If x_1 is even, then allocate $\frac{x_1}{2}$ vertices to P_1 and remaining $\frac{x_1}{2}$ vertices to Q_1. If x_1 is odd, then $x_1 \geq 3$ and allocate $\frac{x_1+1}{2}$ vertices to P_1 and the remaining $\frac{x_1-1}{2}$ vertices to Q_1. Continuing the traversal through $B(H)$, we come across the next x_2 vertices of P. If x_2 is even, then as in the previous case, allocate $\frac{x_2}{2}$ vertices to P_1 and remaining $\frac{x_2}{2}$ vertices to Q_1. If x_2 is odd then the allocation of $\frac{x_2+1}{2}$ vertices depends on $|P_1|$ and $|Q_1|$. If $|P_1|=|Q_1|$, then allocate $\frac{x_2+1}{2}$ vertices to P_1 and remaining $\frac{x_2-1}{2}$ vertices to Q_1. If $|P_1|>|Q_1|$, then allocate $\frac{x_2+1}{2}$ vertices to Q_1 and remaining $\frac{x_2-1}{2}$ to P_1. Thus, after the allocation of two sets of odd vertices, we have $|P_1|=|Q_1|$. Continuing this way, we allocate all vertices of P and construct (P_1,Q_1) such that $|P_1|=|Q_1|=t+2$.

Since P_1 and Q_1 contains at least one vertex from all the end blocks, $\langle P_1\rangle=\langle Q_1\rangle=V(H)$. Let S be any set of $t+1$ vertices of P_1. Then it follows that $\langle P_1\setminus S\rangle\cap\langle Q_1\rangle\neq\emptyset$. Similar is the case, when S is any $t+1$ vertices of Q_1. If we remove $k, 1\leq k\leq t$ vertices from P_1 and $t+1-k$ vertices from Q_1, then $t+2-k$ vertices remains in P_1. These $t+2-k$ vertices may be from the same block or from different blocks. If the all-paths convex hull of the remaining $t+2-k$ vertices of P_1 contains $t+2-k$ vertices of Q_1, after removing any $t+1-k$ vertices from Q_1, at least one vertex of Q_1 remains and so the convex hull of the remaining vertices of P_1 and Q_1 intersects.

If the all-paths convex hull of the remaining $t+2-k$ vertices of P_1 contains $t+1-k$ vertices of Q_1, then by the way of construction of partition, Q_1 contains at least one vertex from the block B, which is adjacent to the all-paths convex hull of the remaining vertices of P_1. So after removing any $t+1-k$ vertices from Q_1 the convex hull of the remaining vertices of the partitions intersects at a cut vertex of B or at a block. Thus the partition (P_1,Q_1) is a $(t+1)$-tolerant Radon partition.

Case 2. $B(H)$ is a tree, which is not a path.

We perform a traversal in $B(H)$ to construct the Radon partition. We define a branch T of $B(H)$ as a *sparse branch* if T starts with a leaf u and ends in a vertex w, which is the first vertex such that the neighbor of w as we traverse has a degree at least three in $B(H)$ so that the path connecting u and w contains no vertex having degree ≥ 3. We construct the partition as follows.

We construct P_1 and Q_1 by partitioning the vertices of P using $B(H)$, by following a traversal through the sparse branches of $B(H)$. Let the leaf vertex of a sparse branch, say T of $B(H)$ represent $x_1 \geq 2$ vertices of P. If x_1 is even, $\frac{x_1}{2}$ vertices are allotted to P_1 and remaining $\frac{x_1}{2}$ vertices are allotted to Q_1. If x_1 is odd then $x_1 \geq 3$, $\frac{x_1+1}{2}$ vertices are allotted to P_1 and remaining $\frac{x_1-1}{2}$ vertices are allotted to Q_1. We have that $|P_1| = |Q_1|$ or $|P_1| = |Q_1| + 1$.

We traverse along the vertices of T to find the next vertex, say w of T, which contains vertices of P. Let w contain x_2 vertices of P. If x_2 is even, then as in the previous case, $\frac{x_2}{2}$ vertices are allotted to P_1 and the remaining $\frac{x_2}{2}$ vertices are allotted to Q_1. If x_2 is odd and $|P_1| = |Q_1|$, then allocate $\frac{x_2+1}{2}$ vertices to P_1 and the remaining vertices of x_2 to Q_1, otherwise, if $|P_1| > |Q_1|$, then allocate $\frac{x_2-1}{2}$ vertices to P_1 and the remaining vertices to Q_1. Continue the traversal and the partitioning of the vertices of P through the entire sparse branch T, thus obtaining the partition (P_1, Q_1) of vertices of P lying in T. From this procedure, we obtain that if the number of vertices of P lying in T is even (odd), then $|P_1| = |Q_1|$ ($|P_1| = |Q_1| + 1$). Then delete the branch T and continue the procedure on another sparse branch T_1 to obtain the partition (P_2, Q_2) of vertices of P lying in T_1. Delete the branch and continue until all sparse branches are deleted and obtaining a sequence of partitions (P_1, Q_1), (P_2, Q_2), $\ldots$,(P_m, Q_m) of P. Let there be k partitions having different cardinality (P_i, Q_i) $(1 \leq i \leq k)$ and $m-k$ partitions (P_j, Q_j) $(k+1 \leq j \leq m)$ having same cardinality. Since the cardinality of P is even, k is even. Now, consider the partition (P_t, Q_t) defined as follows.

$$P_t = P_1 \cup P_2 \cup \cdots \cup P_{\frac{k}{2}} \cup Q_{\frac{k}{2}+1} \cup \cdots \cup Q_k \cup P_{k+1} \cup \cdots \cup P_m$$

and

$$Q_t = Q_1 \cup Q_2 \cup \cdots \cup Q_{\frac{k}{2}} \cup P_{\frac{k}{2}+1} \cup \cdots \cup P_k \cup Q_{k+1} \cup \cdots \cup Q_m$$

.

From construction, it follows that $|P_t| = |Q_t| = t + 2$.

We prove that (P_t, Q_t) is a $(t+1)$-tolerant Radon partition of P. From construction, it follows that P_t and Q_t contain at least one vertex from all the end blocks of H different from cut vertices and so $\langle P_t \rangle = \langle Q_t \rangle = V(H)$. Thus, (P_t, Q_t) is a Radon partition of P. Let S be a set of $t+1$ vertices of P_t. Now $P_t \setminus S$ consists of a single vertex, say z. Clearly $Q_t \setminus S = Q_t$ and so $\langle P_t \setminus S \rangle \cap \langle Q_t \setminus S \rangle = \{z\}$. Similarly if S consists of $t+1$ vertices of Q_t, then, also $\langle P_t \setminus S \rangle \cap \langle Q_t \setminus S \rangle \neq \emptyset$.

Now, let K be any set of k $(1 \leq k \leq t)$ vertices of P_t and L be any set of $t+1-k$ vertices of Q_t so that $S = K \cup L$. Then $P_t \setminus K$ contains $t+2-k$ vertices of P_t. Since $B(H)$ is not a path, $\langle P_t \setminus K \rangle$ contains some or no vertices of Q_t. If Q_t contains at least $t+2-k$ vertices from $\langle P_t \setminus K\rangle$, then at least one vertex remains in $Q_t \setminus L$, and so $\langle P_t \setminus K\rangle \cap \langle Q_t \setminus L\rangle \neq \emptyset$. If $\langle P_t \setminus K\rangle$ contains $t+1-k$ vertices of Q_t, then $\langle Q_t \setminus L\rangle$ contains a vertex from a block, which is adjacent to $\langle P_t \setminus K\rangle$ by our construction. Thus $\langle P_t \setminus K\rangle$ and $\langle Q_t \setminus L\rangle$ intersect at a cut vertex. If $\langle P_t \setminus K\rangle$ contains less than $t+1-k$ vertices of Q_t, then there exists a set $T \subset Q_t$ with $|T| \geq t+2-k$ such that $\langle T\rangle$ contains the vertices of $P_t \setminus K$, consisting of vertices from different sparse branches of $B(H)$. Then $T \setminus L \neq \emptyset$ and $T \setminus L \subset Q_t \setminus L$. Then $\langle Q_t \setminus L\rangle$ contains at least one vertex of $P_t \setminus K$. Hence $\langle Q_t \setminus L\rangle$ and $\langle P_t \setminus K\rangle$ intersect, since the all-paths convex hull contains the entire block containing the vertices. Thus in all cases, $\langle P_t \setminus S\rangle \cap \langle Q_t \setminus S\rangle \neq \emptyset$. □

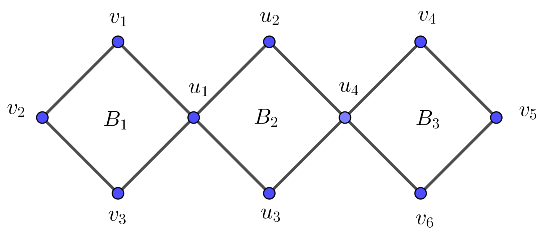

Figure 1. **Graph** H

Example 1. Consider the graph H having three blocks B_1, B_2, B_3 as in Figure 1. Let $P = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Then $|P| = 2t+4$, $t = 1$ and P has no vertex from an inner block B_2. We claim that P has no 2-tolerant Radon partition. Any possible 2-tolerant Radon partition P^* and Q^* should contain at least 3 vertices of P. Since $|P| = 6$, P^* and Q^* should contain exactly 3 vertices of P. The possibilities of P^* and Q^* are as follows. P^* and Q^* consists of one vertex from B_1 or B_3 and two vertices from B_3 or B_1. Let $P^* = \{v_1, v_3, v_5\}$ and $Q^* = \{v_2, v_4, v_6\}$ be one such partition. Let $S = \{v_2, v_5\}$ with $|S| = 2$. Then $P^* \setminus S = \{v_1, v_3\}$ and $\langle P^* \setminus S\rangle = \{v_1, v_2, v_3, u_1\}$. Also, $Q^* \setminus S = \{v_4, v_6\}$ and $\langle Q^* \setminus S\rangle = \{v_4, v_5, v_6, u_3\}$. Thus $\langle P^* \setminus S\rangle \cap \langle Q^* \setminus S\rangle = \emptyset$. For any Radon partition (P^*, Q^*), we can find a set S with $|S| = 2$, such that $\langle P^* \setminus S\rangle \cap \langle Q^* \setminus S\rangle = \emptyset$. Thus P has no 2-tolerant Radon partition.

Now, we see the existence of a $(t+1)$-tolerant Radon partition even if there exists some blocks that contain no interior vertex of P.

For any set P of vertices, a block B of a graph G is called a *cut block with respect to* P if P contains no vertices of B other than cut vertices. If P changes, then the cut blocks of H with respect to P also change. Such a cut block B separates G into two or more components if B contains two or more cut vertices of G. Here, each component contains only one cut vertex of B. The components are formed by removing all vertices of B other than cut vertices from G. In Figure 1, B_2 is a cut block with respect to P. B_2 separates H into two components B_1 and B_3.

Theorem 2. *Let P be a set of $2t+4$ vertices of a graph G, and let H be the subgraph induced by $\langle P\rangle$. Suppose that H has at least one cut block and P contains at least two vertices from every end block of H other than cut vertices. Then P has a $(t+1)$-tolerant Radon partition if and only if every cut block of H that separates H into exactly two components, must contain an even number of vertices of P.*

Proof. Let P be a set of $2t+4$ vertices of a graph G and H be the subgraph induced by $\langle P\rangle$ such that P contains at least two vertices from every end block of H, other than cut vertices. Let G contain a cut block C with respect to P such that the removal of C from H results in exactly two components with each component containing an even number of vertices of P.

Consider the block cut vertex tree $B(H)$ of H. Identify all cut blocks of H that separate H into exactly two components and remove the vertices from $B(H)$ that corresponds to these cut blocks in H. After the removal of these vertices, let $B(H)$ be split into sub trees $T_1, T_2, \cdots, T_n$. From the assumption, we observe that each T_i contains an even number of vertices of P. The sub trees T_i may contain vertices corresponding to some cut blocks of H that separate H into more than two components. Identify these cut blocks of H that separate each T_i into more than two components. Remove the vertices of T_i that correspond to these cut blocks from H so that each T_i can be split into m subtrees $T_{i,1}, T_{i,2}, \cdots, T_{i,m}$ such that each of them contains no cut blocks. Using the method of construction of the Radon partition described in the proof of Theorem 1, we form a partition of vertices of P that lie in each of the subtrees $T_{i,1}, T_{i,2}, \cdots, T_{i,m}$. It may be noted that all these partitions may not be Radon. Let $(P_{i,j}, Q_{i,j})$ be the partition of vertices of P that lies in $T_{i,j}$, for $1 \le j \le m$. Among these m partitions, let k partitions have different cardinality and the remaining $m-k$ partitions have same cardinality. From the construction of the partitions as described in the proof of Theorem 1, it follows that for partitions with different cardinality, $|P_{i,r}| = |Q_{i,r}| + 1$ for $1 \le r \le k$ and for partitions with same cardinality, $|P_{i,r}| = |Q_{i,r}|$ for $k+1 \le r \le m$. Since each T_i contains an even number of vertices of P, k must be even. Now we form a partition (P_i, Q_i) of T_i such as

$$P_i = P_{i,1} \cup P_{i,2} \cup \cdots \cup P_{i,\frac{k}{2}} \cup Q_{i,\frac{k}{2}+1} \cup \cdots \cup Q_{i,k} \cup P_{i,k+1} \cup \cdots \cup P_{i,m}$$

and

$$Q_i = Q_{i,1} \cup Q_{i,2} \cup \cdots \cup Q_{i,\frac{k}{2}} \cup P_{i,\frac{k}{2}+1} \cup \cdots \cup P_{i,k} \cup Q_{i,k+1} \cup \cdots \cup Q_{i,m}$$

for each i. Also, P_i and Q_i contain the same number of vertices of P.

Now we construct the partition, (P^*, Q^*) as

$$P^* = P_1 \cup P_2 \cup \cdots \cup P_n$$

and

$$Q^* = Q_1 \cup Q_2 \cup \cdots \cup Q_n.$$

It is clear that (P^*, Q^*) forms a Radon partition. Since $|P_i| = |Q_i|$ for each i, we have $|P^*| = |Q^*| = t + 2$. We have to show that (P^*, Q^*) forms a $(t+1)$-tolerant Radon partition.

Consider a cut block that separates $B(H)$ into two components C_1 and C_2. By construction of the partition, vertices of C_1 has a partition (P_1, Q_1) and vertices of C_2 has a partition (P_2, Q_2). Since each component contains an even number of vertices of P, clearly $|P_1| = |Q_1|$ and $|P_2| = |Q_2|$. Form $P^* = P_1 \cup P_2$ and $Q^* = Q_1 \cup Q_2$. Let $S \subset P$, has cardinality $t + 1$. Clearly $\langle P^* \rangle = \langle Q^* \rangle = V(H)$. If $S \subset P^*$ (or $S \subset Q^*$), then $\langle P^* \setminus S \rangle \cap \langle Q^* \setminus S \rangle \neq \emptyset$, since $\langle P^* \setminus S \rangle$ (or $\langle Q^* \setminus S \rangle$) contains one vertex of P^* (or Q^*). Now, consider the case when S contains k, $(1 < k < t + 1)$ vertices of P^* and $t + 1 - k$ vertices of Q^*. If $P^* \setminus S$ and $Q^* \setminus S$ contain vertices from both components C_1 and C_2, then $\langle P^* \setminus S \rangle$ and $\langle Q^* \setminus S \rangle$ intersect at the cut block. If $P^* \setminus S$ or $Q^* \setminus S$ contains the vertices from only one component,
then $\langle P^* \setminus S \rangle$ and $\langle Q^* \setminus S \rangle$ intersect in a cut vertex of the cut block or at a block other than the cut block. Thus for any set S having $t+1$ vertices, $\langle P^* \setminus S \rangle \cap \langle Q^* \setminus S \rangle \neq \emptyset$. This shows that (P^*, Q^*) forms a $(t + 1)$-tolerant Radon partition.

Conversely, let P be a set of $2t + 4$ vertices of G such that P has a $(t + 1)$-tolerant Radon partition. We have to prove that for all cut blocks that separate H into two components, each of the components must contain an even number of vertices of P. Suppose there exists a cut block B that separates H into two components C_1 and C_2 such that C_1 and C_2 contain odd number of vertices of P. Let C_1 contains p vertices of P and C_2 contains q vertices of P. We have $p + q = 2t + 4$. Using $B(H)$, we can construct partitions P^* and Q^* such that $|P^*| = |Q^*| = t + 2$. Since p and q are odd, P^* contains $\frac{p+1}{2}$ vertices of P from C_1 and $\frac{q-1}{2}$ vertices of P from C_2. Similarly Q^* contains $\frac{p-1}{2}$ vertices of P from C_1 and $\frac{q+1}{2}$ vertices of P from C_2. We can form a set S of $t + 1$ vertices of P such that S contains all the $\frac{p-1}{2}$ vertices of Q^* from C_1 and all the $\frac{q-1}{2}$ vertices of P^* from C_2. Thus $P^* \setminus S$ contains $\frac{p+1}{2}$ vertices of C_1 and $Q^* \setminus S$ contains $\frac{q+1}{2}$ vertices of C_2. Clearly $|S| = \frac{p+q-2}{2} = t + 1$.
Since there is a cut block B between C_1 and C_2, we have $\langle P^* \setminus S \rangle \cap \langle Q^* \setminus S \rangle = \emptyset$. Thus (P^*, Q^*) is not a $(t+1)$-tolerant Radon partition, which completes the converse part of the theorem. □

We observe the following remark, which can be established in the same procedure as we have described in Theorem 2.

Remark 1. For a set P of vertices of a graph G, the cut blocks that separate G into more than two components will not reduce the tolerance of the Radon partition even if the components contain an odd number of vertices of P.

An illustration of the Remark 1 is given in Figure 2, where B is a cut block with respect to $P = \{v_1, v_2, \ldots, v_{12}\}$ and C_1, C_2, C_3, C_4 are the components. Here $P_1 = \{v_1, v_3, v_5, v_7, v_9, v_{11}\}$ and $Q_1 = \{v_2, v_4, v_6, v_8, v_{10}, v_{12}\}$ form a 5-tolerant Radon partition of P.

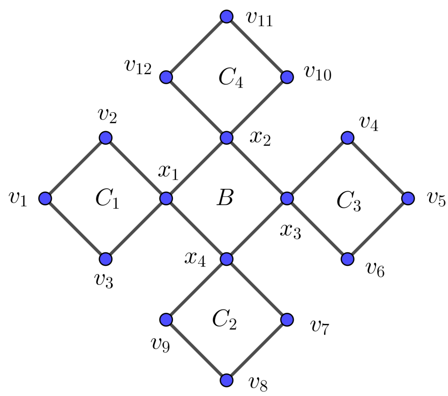

Figure 2. **A 5-tolerant Radon partition as an illustration for Remark 1**

4. Algorithms for tolerant Radon partitions

In this section, we develop five algorithms related to tolerant Radon partitions for a given set of vertices of a graph. We assume that G is a connected graph having n vertices and m edges. The following Proposition (Corollary 3.1 from [20]) guarantees that a given set of $2t + 3$ vertices of a path has a t-tolerant Radon partition.

Proposition 4 (Corollary 3.1[20]). *Any set of $2t+3$ vertices of a path has a t-tolerant Radon partition.*

Algorithm 1. Algorithm for t-tolerant Radon partition on paths.

Input : A path X represented by a list of vertices $(v_1, v_2, \ldots, v_n)$ and a set P of $2t + 3$ vertices of X.
Output: A t-tolerant Radon partition (P_t, Q_t) of P.
Initially, set $P_t = \emptyset$ and $Q_t = \emptyset$.

Starting from the first vertex, v_1 of X and make a traversal from v_1 and add each vertex of X encountered in the traversal alternatively to P_t and Q_t respectively. The resulting partition (P_t, Q_t) is a t-tolerant Radon partition.

Running time analysis: Each vertex and edge is considered only once during the traversal and so the time complexity of the traversal is $O(n+m)$ where m is the number of edges. For a path, $m = n-1$ and hence the Algorithm 1 has time complexity $O(n)$.

Theorem 3. *The partition (P_t, Q_t) formed in Algorithm 1 is a t-tolerant Radon partition of a set P of $2t+3$ vertices of a path.*

Proof. Suppose that during the traversal through X, we renamed the vertices of P as $v_1, v_2, v_3, \ldots, v_{2t+3}$ in the order in which they are encountered. From the algorithm, we formed a partition to the set P as $P_t = \{v_1, v_3, \ldots, v_{2t+3}\}$ and $Q_t = \{v_2, v_4, \ldots, v_{2t+2}\}$ with $|P_t| = t+2$ and $|Q_t| = t+1$.

Case 1. Removing any t vertices from P_t.
Then two vertices remain in P_t having an odd label. In this case no vertices are removed from Q_t. Between the two odd labeled vertices of P_t, there exists at least one even labeled vertex and that vertex is in Q_t. Thus the convex hull of the remaining vertices of the partitions intersects.

Case 2. Removing any t vertices from Q_t.
Then only one vertex remains in Q_t. Clearly the remaining vertex of Q_t has even label. In this case no vertex is removed from P_t and so there exists at least two vertices in P_t having odd label, between them the vertex of Q_t lies. Thus the convex hull of the remaining vertices intersects.

Case 3. Removing any k vertices from P_t and $t-k$ vertices from Q_t for $1 \leq k \leq t-1$. Then $t+2-k$ vertices remain in P_t. So, the convex hull of these $t-k+2$ vertices of P_t contains at least $t-k+1$ vertices of Q_t. After removing $t-k$ vertices from Q_t, at least one vertex of Q_t remains and that vertex is in the convex hull of the remaining vertices of P_t. Thus, the convex hull of the remaining vertices intersects. Hence the partition (P_t, Q_t) is a t-tolerant Radon partition to the given set of $2t+3$ vertices of a path. □

Next, we discuss the algorithm for determining t-tolerant Radon partition for a given input of $2t+4$ vertices of a tree. Before discussing the algorithm on trees, we need an algorithm for constructing the Radon partition. It follows from Theorem 11 in [6] that every set of four points in a tree has a Radon partition. In this algorithm, at first, we construct a subtree W of the given tree T such that all pendant vertices of W are from the given input set P of four vertices. Then it follows that $W = \langle P \rangle$. The algorithm determines the Radon partition of P depending upon the number of pendant vertices of W.
We use the adjacency list for representing trees and graphs in all the algorithms discussed from here onwards.

Algorithm 2. Algorithm for Radon partition of a set of four vertices of a tree.

Input : A tree T and a set P of four vertices $\{a, b, c, d\}$ of T.
Output: A Radon partition (P_0, Q_0) of P.

Initially set $P_0 = \emptyset$ and $Q_0 = \emptyset$.

Step 1: Start a Breadth First Search traversal from any one vertex of P to identify the remaining three vertices and find the path connecting the three pairs of vertices. Combination of these three paths represents $\langle P \rangle$. Let W be the newly obtained tree.

Step 2: Make a Breadth First Search traversal through W to identify the degree of all vertices of W.

Step 3: If W contains only two pendant vertices a and d, we form $P_0 = \{a, d\}$ and $Q_0 = \{b, c\}$. Return (P_0, Q_0).

Step 4: If W contains three pendant vertices a, b, c; then make a Depth First Search traversal from the unique vertex x of degree 3 in W towards a. If we find d during this traversal then construct $P_0 = \{a, b\}$ and $Q_0 = \{c, d\}$. Otherwise construct $P_0 = \{b, c\}$ and $Q_0 = \{a, d\}$. Return (P_0, Q_0).

Step 5: In this case all four vertices of P are pendant vertices of W. If W contains a unique vertex of degree 4, construct $P_0 = \{a, b\}$ and $Q_0 = \{c, d\}$. Return (P_0, Q_0). Otherwise go to Step 6.

Step 6: If W contains two vertices u_1 and u_2 having degree 3, then make a Depth First Search traversal from a to b. If u_1 and u_2 are encountered, then $P_0 = \{a, b\}$ and $Q_0 = \{c, d\}$. Otherwise form $P_0 = \{a, c\}$ and $Q_0 = \{b, d\}$. Return (P_0, Q_0).

Running time analysis: Time complexity to do the BFS traversal is $O(n + m)$. Since $m = n - 1$, time taken to complete the first Step of the algorithm is $O(n)$. In Step 2, each vertex and each edge is listed at most once and so time complexity is $O(n + m)$ and hence it is $O(n)$. Time complexity of Step 3 is $O(1)$. In the Step 4 and Step 6, one more traversal is needed in each Step and so the time complexity of each Step is $O(n + m)$ and it is reduced to $O(n)$. Time complexity of Step 5 is $O(1)$. Thus the total time complexity of the Algorithm 2 is $O(n)$.

Theorem 4. *The partition P_0 and Q_0 formed in Algorithm 2 is a Radon partition of a set P of four vertices of a tree.*

Proof. If W contains only two pendant vertices a and d, then $P_0 = \{a, d\}$ and $Q_0 = \{c, d\}$. Here $\langle P_0 \rangle$ contains all vertices of $\langle Q_0 \rangle$. If W contains three pendant vertices a, b, c; then there exists the unique vertex x in W having degree 3. If d lies in the path connecting a and x, then $P_0 = \{a, b\}$ and $Q_0 = \{c, d\}$. Otherwise d lies in the path connecting b and c. In this case $P_0 = \{b, c\}$ and $Q_0 = \{a, d\}$. So $\langle P_0 \rangle$ and

$\langle Q_0 \rangle$ intersect at d. If all vertices of W are pendant vertices, then W contains either a unique vertex of degree 4 or two vertices of degree 3. If W contains the vertex of degree 4, then $P_0 = \{a, b\}$ and $Q_0 = \{c.d\}$. If W contains two vertices of degree 3, then we can form $\langle P_0 \rangle$ and $\langle Q_0 \rangle$ are paths containing these two vertices. Hence in all cases, $\langle P_0 \rangle$ and $\langle Q_0 \rangle$ intersect and hence the partition is a Radon partition. □

Now, we find t-tolerant Radon partitions on trees by using the Radon partition obtained from Algorithm 2. The following Theorem from [20] guarantees that for a given input of $2t + 4$ vertices of the tree T has a t-tolerant Radon partition. We continuously remove two special vertices from P in the tree T and form a partition (P_t, Q_t) by putting one of the vertices in P_t and the other in Q_t. After t steps, the Radon partition of the remaining four vertices is obtained and the t-tolerant Radon partition is developed by using it.

Theorem 5. *[20] Any set of $2t + 4$ vertices of a tree has a t-tolerant Radon partition.*

Algorithm 3. Algorithm for t-tolerant Radon partitions on trees.

Input : A tree T and a set Y of $2t + 4$ vertices of T.
Output : A t-tolerant Radon partition (P_t, Q_t) of Y.
Initially set $P_t = \emptyset$ and $Q_t = \emptyset$.

Step 1: Compute the distance from each vertex of Y to all other vertices of the tree T and store the distances in a matrix M. Let $Y_1 = Y$

For each $j, 1 \leq j \leq t$, repeat Steps 2 to 6

Step 2: Start a Breadth First Search traversal from any one vertex of P to identify the remaining $2t+3$ vertices and find the path connecting the $2t+3$ pairs of vertices. Combination of these paths represents $\langle Y \rangle$. Let T_j be the newly obtained tree.

Step 3: We start a Depth First Search traversal in T_j from one pendant vertex u_j, to find another vertex v_j of Y_j or to find a vertex x_j of T_j having degree ≥ 3. If we met v_j first, then form the set $S_j = \{u_j, v_j\}$. Then $P_t = P_t \cup \{u_j\}$ and $Q_t = Q_t \cup \{v_j\}$.

Step 4: If we met $x_j \notin Y_j$ first, then from the distance matrix M, find the pendant vertex w_j of T_j having minimum distance with x_j, other than u_j. Repeat Step 4 starting from w_j.

Step 5: If all the intermediate vertices of the $w_j - x_j$ path are not from Y_j and if there is no intermediate vertex having degree ≥ 3 in the $w_j - x_j$ path, then we form $S_j = \{u_j, w_j\}$. Rename w_j as v_j and form $S_j = \{u_j, v_j\}$. Construct

$P_t = P_t \cup \{u_j\}$ and $Q_t = Q_t \cup \{v_j\}$.

Step 6: Form $Y_{j+1} = Y_j \setminus S_j$.

Step 7: Use Algorithm 2 to form a Radon partition (P_0, Q_0) to Y_{t+1}.

Step 8: Form $P_t = P_t \cup P_0$ and $Q_t = Q_t \cup Q_0$ and Stop.

Running time analysis: Time complexity to find the distance from a vertex of Y to all other vertices of the tree is $O(n)$ where n is the number of vertices of T. This can be done for $2t + 4$ vertices. So, the time complexity of the first Step is $O(nt)$. Time complexity to do the BFS traversal is $O(n+m)$. Since $m = n - 1$, it is reduced to $O(n)$. The process can be repeated for t iterations and so the time complexity of Step 2 of the algorithm is $O(nt)$. For each Step 3 to 5, the time complexity of the traversal to find the set S_j for $1 \leq j \leq t$ is $O(n)$. The process is continued for a maximum of t iterations. So the total time complexity of the algorithm from Step 3 to Step 5 is $O(nt)$. The total time complexity of Step 6 is $O(t)$. For the Step 7, the time complexity of constructing a Radon partition is $O(n)$. Hence the total time complexity of the Algorithm 3 is $O(nt)$.

Theorem 6. *Algorithm 3 correctly computes the t-tolerant Radon partition for vertices in a tree T, for the convexity defined by the paths in T.*

Proof. We have constructed the sets $S_j = \{u_j, v_j\}$, for $1 \leq j \leq t$, either in Step 4 or in Step 6. It is to be shown that the path connecting every vertex of S_j to a vertex of Y_{j+1}, as defined in Step 7 of the Algorithm, must passes through a common vertex x_j of T_j. Clearly S_j contains one pendant vertex or two pendant vertices. Also $\langle S_j \rangle$ does not contain other vertices of Y_{j+1}. If u_j is a pendant vertex and v_j is not a pendant vertex, then the path from a vertex of S_j to a vertex of Y_{j+1} must passes through the vertex v_j. Here, with out loss of generality, we take $x_j = v_j$. If u_j and v_j are pendant vertices, then there is no other vertex of Y_j in the interior of the $u_j - v_j$ path. So the path connecting a vertex of S_j to a vertex of Y_{j+1} must passes through an interior vertex x_j that lies in the $u_j - v_j$ path such that degree$(x_j) \geq 3$. Thus in any case, for each $j, 1 \leq j \leq t$, both paths connecting a vertex of S_j to any vertex of Y_{j+1} will intersect at a vertex x_j of the tree T_j.

From the algorithm described above we constructed partitions as $P_t = P_0 \cup \{u_1, u_2, \ldots, u_t\}$ and $Q_t = Q_0 \cup \{v_1, v_2, \ldots, v_t\}$. Then $|P_t| = t + 2$ and $|Q_t| = t + 2$. The following cases are considered for proving the theorem.

Case 1. Removing t vertices from P_t.
In this case, no vertex is removed from Q_t and hence two vertices will remain in P_t. If the remaining two vertices are from P_0, since (P_0, Q_0) is a Radon partition, the convex hull of the remaining vertices intersects.

Suppose the remaining vertices of P_t are u_i, u_j where $i < j$, where $u_i \in S_i$ and $u_j \in S_j$. Then by the method of selection of S_i, the path connecting every vertex of S_i to every vertex of S_j must passes through a common vertex x_i. Since v_i and v_j are in Q_t, the convex hull of the remaining vertices of P_t and Q_t intersect at x_i.
Suppose that one vertex u_i and one vertex of P_0 remains in P_t. Then also the path connecting u_i to a vertex of P_0 intersects with the path connecting v_i to Q_0 and so the convex hull of the remaining vertices intersects.

Case 2. Removing t vertices from Q_t.
By using the symmetric arguments used in the previous case, we conclude that the convex hull of the remaining vertices intersects.

Case 3. Removing any k vertices from P_t and $t-k$ vertices from Q_t for $1 \leq k \leq t-1$. Consider the partitions $P_t = \{u_1, u_2, ..., u_t\} \cup P_0$ and $Q_t = \{v_1, v_2, .., v_t\} \cup Q_0$. Suppose that (P_0, Q_0) intersects at x_0 and suppose that two paths from vertices of S_j to a vertex of $P_0 \cup Q_0 \cup S_1 \cup S_2 \cup \cdots \cup S_{j-1}$ must passes through a common vertex x_j for $1 \leq j \leq t$. Thus $\langle P_t \rangle$ and $\langle Q_t \rangle$ intersect at least $t+1$ vertices of the tree (intersecting points), counted with repetition. Removing of one vertex from one partition may reduce the intersection points by one. If we remove any $t-k$ vertices from Q_t, then at least $k+2$ vertices remains in Q_t. The convex hull of the remaining $k+2$ vertices of Q_t and $\langle P_t \rangle$ intersect with at least $k+1$ intersecting points, counted with repetition. After removing any k vertices from P_t, then again the convex hull of the remaining vertices of P_t and the convex hull of the remaining vertices of Q_t intersect with at least one intersecting point. Thus the partition (P_t, Q_t) is a t-tolerant Radon partition to the set of $2t+4$ vertices of a tree. □

The all-paths convexity reflects the block-cut vertex tree structure of graphs. Now we design the algorithm to construct t-tolerant Radon partition on an arbitrary connected graph G. The idea is that first we construct a spanning tree of G using the simple standard Breadth First Search algorithm. Then form t-tolerant Radon partition to the vertices of the spanning tree using Algorithm 3. We prove that the Radon partition of the spanning tree obtained using Algorithm 3 is the same that of the given graph G.

Algorithm 4. Algorithm for t-tolerant Radon partitions of an arbitrary connected graph.

Input: A connected graph $G = G(V, E)$ and a set P of $2t+4$ vertices of a graph G
Output: A t-tolerant Radon partition (P_t, Q_t) of P.
Initially set $P_t = \emptyset$ and $Q_t = \emptyset$.

Step 1: Construct a spanning tree $T = G(V, E')$ of G by using Breadth First Search algorithm, where $E' \subseteq E$.

Step 2: Use Algorithm 3 to construct a t-tolerant Radon partition (P_t, Q_t) to the set of $2t + 4$ vertices of the spanning tree T. Return (P_t, Q_t) and Stop.

Running time analysis: The construction of spanning tree has complexity $O(m + n)$ by a Breadth First Search algorithm, where m is the number of edges and n is the number of vertices of the graph. The time complexity of constructing t-tolerant Radon partition of vertices of a tree is $O(nt)$. Hence the time complexity of the Algorithm 4 is $O(m + nt)$.

We proved the following Theorem in [20], which guarantees that any set of $2t + 4$ vertices in a graph G has a t-tolerant Radon partition on all-paths convexity.

Theorem 7. *[20] Any set of $2t + 4$ vertices of a graph G has t-tolerant Radon partition with respect to all-paths convexity if the block cut vertex tree of G is not a path.*

Theorem 8. *The Algorithm 4 correctly computes the t-tolerant Radon partition of vertices of a given connected graph G.*

Proof. In the Algorithm 4, a spanning tree T of the graph is constructed at first and a t-tolerant Radon partition (P_t, Q_t) to a set P of vertices of the tree T is formed. Thus for any set $C \subset P$ with $|C| \leq t$, $\langle P_t \setminus C\rangle \cap \langle Q_t \setminus C\rangle \neq \emptyset$. In the spanning tree T of the graph G, we are considering only one path between any two vertices of G. Since the all-paths transit function considers all paths connecting any two vertices of G, we have $\langle P_t \setminus C\rangle \subseteq A\langle P_t \setminus C\rangle$ and $\langle Q_t \setminus C\rangle \subseteq A\langle Q_t \setminus C\rangle$, where $\langle P_t \setminus C\rangle$ represents the convex hull of $P_t \setminus C$ in the tree T and $A\langle P_t \setminus C\rangle$ represents the all-paths convex hull of $P_t \setminus C$ in the graph G. So for any set $C \subset P$ with $|C| \leq t$, $A\langle P_t \setminus C\rangle \cap A\langle Q_t \setminus C\rangle \neq \emptyset$. Thus the same partition (P_t, Q_t) is a t-tolerant Radon partition of P with respect to all-paths convexity. □

4.1. Algorithm for $(t + 1)$-tolerant Radon partition

Here we design a decision algorithm to check the existence of $(t + 1)$-tolerant Radon partition for any input of $2t + 4$ vertices and the algorithm outputs a $(t + 1)$-tolerant Radon partition, if there is such a partition. If P has no $(t + 1)$-tolerant Radon partition, then the algorithm outputs "False". Such an algorithm is called *a decision algorithm*. The main idea of the algorithm can be described as follows. At first, we determine all the cut vertices of G, which is available in [8]. Then we determine all blocks of G. For this we make use of an algorithm by Hopcroft et al. in [11]. Then we construct the block cut vertex tree of G, $B(G)$, with blocks and cut vertices of G as its vertices. From $B(G)$, we construct a subtree of $B(G)$, which is precisely the block cut vertex tree $B(H)$ of the subgraph H, where H is the subgraph induced by $\langle P\rangle$ in G. A vertex of $B(H)$ represents either a cut vertex of H or a block of H. Vertex of $B(H)$ corresponding to a block will represent all vertices of P which are from a particular block of H other than cut vertices. We count the number of vertices of P

in each block. We can construct $(t+1)$-tolerant Radon partition, if it exists using the results that we have presented in the previous Section (Section 3) with the help of $B(H)$. A block which has only one cut vertex is an end block. Now, we are ready to describe the algorithm.

Algorithm 5. A decision algorithm to determine a $(t+1)$-tolerant Radon partition on a set of $2t+4$ vertices of a graph G.

Input : A graph G represented as adjacency list and a set P of $2t+4$ vertices of G.

Output : A $(t+1)$-tolerant Radon partition of $P = (X, Y)$ or False, that returns that P has no such partition.

Initially set $X = \emptyset$, $Y = \emptyset$.

Step 1: Determine the set C of cut vertices of G by using an application of the standard DFS algorithm and the set $\{B_1, B_2, \ldots, B_k\}$ of blocks of G by the algorithm of Hopcroft and Tarjan from [11].

Step 2: Identify the sets $P_C = P \cap C$, the set of cut vertices of P; and $P_{B_i} = P \cap (B_i \setminus C)$, the set of vertices of P which are not cut vertices from the blocks B_i. Let b_i is the cardinality of P_{B_i} for $1 \leq i \leq k$.

Step 3: Identify the sets $C_{B_i} = B_i \cap C$ of cut vertices of a particular block B_i for each i.

Step 4: Construct block cut vertex tree $B(G)$ of G by using the adjacency of blocks and cut vertices.

Step 5: If $b_i = 0$ for any pendant vertex B_i of $B(G)$, remove it from $B(G)$. Repeat this process until $b_i > 0$ for all pendant vertices. The resulting subtree is precisely the block-cut vertex tree $B(H)$ of the subgraph H induced by $\langle P \rangle$.

Step 6: Check whether $b_i = 1$ for any pendant vertex B_i of $B(H)$, then go to Step 17, otherwise continue.

Step 7: Identify the vertices of $B(H)$, which are B_i's with degree two such that $b_i = 0$. Rename these vertices as $u_1, u_2, \ldots, u_q$. If there is no such vertex then go to Step 10.

For each $r, 1 \leq r \leq q$, repeat Step 8 to Step 9.

Step 8: Find the components C_1, C_2 of $B(H) \setminus \{u_r\}$ and find the total number of vertices of P in C_1 (say x).

Step 9: If x is odd, then go to Step 17.

Step 10: If $B(H)$ contains only one vertex, (say B_s) then $\frac{b_s}{2}$ vertices of P_{B_s} are added to X and remaining vertices of P_{B_s} are added to Y and go to Step 16. Otherwise start partitioning from one pendant vertex B_i of $B(H)$ for some i.

Step 11: If b_i is even, $\frac{b_i}{2}$ vertices of P_{B_i} are added to X and remaining vertices of P_{B_i} are added to Y. Go to Step 13.

Step 12: If b_i is odd and $|X| = |Y|$, then $\frac{b_i+1}{2}$ vertices of P_{B_i} are added to X and remaining vertices of P_{B_i} are added to Y. If b_i is odd and $|X| > |Y|$ then $\frac{b_i+1}{2}$ vertices of P_{B_i} are added to Y and remaining vertices of P_{B_i} are added to X.

Step 13: If a new vertex adjacent to the already partitioned vertex has degree > 2 during the DFS (completed one sparse branch), select another pendant vertex B_l of $B(H)$. Remove the already partitioned vertex from $B(H)$ and continue partitioning from B_l. put $i = l$ and go to Step 11.

Step 14: If a new vertex adjacent to the already partitioned vertex has degree ≤ 2 (still in the same sparse branch), select that new vertex for partitioning and remove the already partitioned vertex from $B(H)$. If the newly selected vertex is a block B_j then $i = j$ and go to Step 11.

Step 15: If the vertex is a cut vertex of P_C, then that vertex is added to X if $|X| = |Y|$ or that vertex is added to Y if $|X| > |Y|$.

Step 16: If $B(H)$ contains only one vertex, then return (X, Y) and Stop.

Step 17: Return 'False' and Stop.

Running time analysis: At first, all the cut vertices and blocks of the given graph G are identified. Each part of Step 1 is a Depth First Search algorithm with time complexity $O(n + m)$. Thus the time complexity of this Step of the algorithm is $O(n+m)$. In Step 2, by visiting only the vertices of $B_i, 1 \leq i \leq k$ we get the vertices of P which are cut vertices and which are non cut vertices in blocks. In Step 3, the cut vertices of a particular block are identified. The time complexity of Steps 2 and 3 is $O(n)$. To construct block cut vertex tree $B(G)$ of G, we have to check whether which of the blocks contains a particular cut vertex and which of the cut vertices are connected to a particular block. Represent $B(G)$ as adjacency list of blocks and cut vertices and the time complexity of Step 4 is $O(n)$. The time complexity of the construction of $B(H)$ is also $O(n)$. Cut blocks of H which separates H into two components are identified in Step 7 with time complexity $O(n + m)$. In Step 8, all blocks in one component can be identified by using the adjacency list of vertices of $B(H)$. Here we find one cut vertex of the cut block. Then find all other blocks which contains this cut vertex. Then find the cut vertices of these selected blocks. Continuing this way we can select the blocks of one component. Since all blocks and cut vertices are already listed, time complexity of this Step is $O(n)$. The process is repeated at most $r, r < n$ steps and so the total time complexity is $O(n^2)$. Time complexity of Steps 11 to 16 is $O(n + m)$. Thus the total time complexity of the algorithm is $O(n^2)$.

Theorem 9. *The Algorithm 5 is correct.*

Proof. In Step 1, we find all the cut vertices of the graph [8] and the blocks of the graph using the Hopcroft-Tarjan Algorithm described in [11]. The proof of the algorithm completely depends on the theorems we proved in Section 3. In Proposition 2 it is proved that if an end block of G contains only one vertex of P other than cut vertex then the partition we obtained is not $(t+1)$-tolerant Radon partition. From Theorem 1, it is clear that if G contains no cut blocks and end blocks contains at least two vertices of P other than cut vertices, then G has a $(t+1)$-tolerant Radon partition. Theorem 2 proves that if G contains cut blocks that separate G into two components such that each component contains an even number of vertices of P, then P has a $(t+1)$-tolerant Radon partition. Here, each end block must contain at least two vertices of P other than the cut vertex. Also, Theorem 2 states that if G contains a cut block that separates G into two components such that each component contains an odd number of vertices of P, then P has no $(t+1)$-tolerant Radon partition. The proof of all these theorems completes the proof of the algorithm. □

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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